Известия НАН Армении. Математика, том 50, н. 2, 2015, стр. 80-84.

THE PROBLEM OF ADDITIONAL SAMPLES FOR SPATIO-TEMPORAL SAMPLING IN $l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$

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Abstract. The problem of recovering a time-varying signals from their spatio-temporal samples, often referred by dynamical sampling problem, has been well-studied for one-variable signals. Many examples coming from real-world applications (sampling of air pollution, wireless networks etc.) involve spatial coordinates. We state the problem of spatio-temporal sampling for two-variable functions and consider the problem of finding additional sampling locations for one specific family of kernels.

MSC2010 numbers: 94A20, 40C05.
Keywords: sampling and reconstruction.

1. Introduction

Let V and V', $V \subseteq V'$ be spaces of functions defined on a set X. We assume the initial state of a (linear time-invariant dynamical) system $f_n = A^n f_{n-1}$, is given by an unknown function $f \in V$, i.e. $f_0 = f$, and $A : V' \to V'$ is a known linear operator. At each time instance n ($n = 0, \dots, L-1$) the values (samples) of the evolved function $A^n f$ are measured on some subset $\Omega_n \subseteq X$:

$$y_0 = f \restriction_{\Omega_0},\ y_1 = (Af) \restriction_{\Omega_1},\ \dots\ y_{L-1} = (A^{L-1} f) \restriction_{\Omega_{L-1}}.$$

The main problem in dynamical sampling is to uniquely reconstruct the function $f \in V$ from these samples.

In [1] – [4] the dynamical sampling problem for a single variable function f on domains $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$, $\mathbb{Z}$ or $\mathbb{R}$ is treated. The assumption is that the evolution operator is given as a repeated convolution with a kernel a: $A_n(f) = a * a * \dots * a * f = a^n f$ for $n = 0, 1, \dots, L-1$ and, at each time n, the evolved state $A_n(f)$ is undersampled at fixed positions Ω:

$$\{f(\Omega),\ a * f(\Omega),\ \cdots,\ (a^{L-1} * f)(\Omega)\},\quad \text{for } \Omega \subset X.$$

In [5] the case when the positions of sampling points are allowed to change at different time levels is considered. If the number of sampling points at any time is constant, a necessary and sufficient condition is found for the existence of positions that allow full recovery of any function by samples taken at those positions. Also, for single measurement per time level, a lower bound on the number of such sampling configurations is computed.

2. Dynamical sampling in $l^2(\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2})$

Let the domain be the direct sum of two cyclic groups $X=\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}$, $d_1,d_2\in\mathbb{N}^+$ and the evolution operator be given as a convolution with a kernel $a=(a_{k,l})_{(k,l)\in X}$:

$$Af(k,l)=a*f(k,l)=\sum_{(s,p)\in X}a_{s,p}f(k-s,l-p)\quad\text{for all}\quad(k,l)\in X$$

where $(k-s,l-p)$ is understood in terms of summation operations in cyclic groups $\mathbb{Z}_{d_1}$ and $\mathbb{Z}_{d_2}$. We assume that $d_1=J_1m_1$, $d_2=J_2m_2$, where d_1, d_2 are odd numbers and the initial state f and its temporally evolved states Af, $A^2f,\dots,A^{L-1}f$ are sampled on a uniform grid $\Omega=m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}$. Let $S_{m_1,m_2}=\mathbf{1}_{m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}}f$ be the subsampling operator on $m_1\mathbb{Z}_{d_1}\times m_2\mathbb{Z}_{d_2}$. Our objective is to reconstruct f from the samples set

$$\begin{cases}y_0=S_{m_1,m_2}f\\ y_1=S_{m_1,m_2}Af\\ \vdots\\ y_{L-1}=S_{m_1,m_2}A^{L-1}f.\end{cases}\tag{2.1}$$

Denote by $\hat{g}$ the discrete Fourier transform (DFT) of $g\in l^2(\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2})$:

$$\hat{g}(s,p)=\sum_{k=0}^{d_1-1}\sum_{l=0}^{d_2-1}g(s,p)e^{\frac{-i2\pi sk}{d_1}}e^{\frac{-i2\pi pl}{d_2}}\quad\text{for all}\quad(s,p)\in\mathbb{Z}_{d_1}\times\mathbb{Z}_{d_2}.$$

After applying the DFT to both sides of (2.1), and using the fact that the Fourier transform of downsampled signal $S_{m_1,m_2}g$ is

$$(S_{m_1,m_2}g)\hat{}\,(s,p)=\frac{1}{m_1m_2}\sum_{k=0}^{m_1-1}\sum_{l=0}^{m_2-1}\hat{g}\left(s+kJ_1,p+lJ_2\right),$$

also $(a*f)\hat{}\,(s,p)=\hat{a}(s,p)\hat{f}(s,p)$, we get

$$\hat{y}_n(i,j)=\frac{1}{m_1m_2}\sum_{k=0}^{m_1-1}\sum_{l=0}^{m_2-1}\hat{a}^n\left(i+kJ_1,j+lJ_2\right)\hat{f}\left(i+kJ_1,j+lJ_2\right)\tag{2.2}$$

for $(i,j)\in I=\{0,\cdots,J_1-1\}\times\{0,\cdots,J_2-1\}$ and $n=0,1,\dots,L-1$.

We use the block-matrices

$$A_{l,m_1,m_2}(i,j) = \begin{pmatrix} \hat{a}^{1}(i,j+lJ_2) & \hat{a}^{1}(i+J_1,j+lJ_2) & \cdots & \hat{a}^{1}(i+(m_1-1)J_1,j+lJ_2) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{L-1}(i,j+lJ_2) & \hat{a}^{L-1}(i+J_1,j+lJ_2) & \cdots & \hat{a}^{L-1}(i+(m_1-1)J_1,j+lJ_2) \end{pmatrix},$$

where $l = 0, 1, ..., m_2 - 1$, and for all $(i,j) \in I$ we define

$$A_{m_1,m_2}(i,j) = [A_{0,m_1m_2}(i,j)\ A_{1,m_1m_2}(i,j) ... A_{m_2-1,m_1m_2}(i,j))] \tag{2.3}$$

For every $(i,j) \in I$ put $\bar{\mathbf{y}}(i,j) = [\hat{y}_0(i,j)\ \ \hat{y}_1(i,j)\ \ ...\ \ \hat{y}_{L-1}(i,j)]^T$, and let

$$\bar{\mathbf{f}}(i,j) = \begin{pmatrix} \hat{f}(i,j) \\ \hat{f}(i+J_1,j) \\ \cdots \\ \hat{f}(i+(m_1-1)J_1,j) \\ \hat{f}(i,j+J_2) \\ \cdots \\ \hat{f}(i+(m_1-1)J_1,j+J_2) \\ \cdots \\ \cdots \\ \hat{f}(i,j+(m_2-1)J_2) \\ \cdots \\ \hat{f}(i+(m_1-1)J_1,j+(m_2-1)J_2) \end{pmatrix}.$$

Then the equations (2.2) can be written as

$$\bar{\mathbf{y}}(i,j) = \frac{1}{m_1m_2}\mathcal{A}_{m_1,m_2}(i,j)\bar{\mathbf{f}}(i,j). \tag{2.4}$$

Note that, to be able to recover the vector f from (2.1), we need to take samples at least m_1m_2 times and, when $L = m_1m_2$, $\mathcal{A}_{m_1,m_2}(i,j)$ becomes a square matrix.

Proposition 2.1. *For $L = m_1m_2$, any $f \in l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$ can be uniquely recovered from its samples (2.1) if and only if for every $(i,j) \in I$ we have*

$$\det \mathcal{A}_{m_1,m_2}(i,j) \neq 0. \tag{2.5}$$

If we put

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{m_1,m_2}(0,0) & 0 & \cdots & 0 \\ 0 & \mathcal{A}_{m_1,m_2}(1,0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_{m_1,m_2}(J_1-1,J_2-1) \end{pmatrix}$$

then (2.4) is equivalent to

$$\frac{1}{m_1m_2}\mathcal{A}\bar{\mathbf{f}} = \bar{\mathbf{y}}, \tag{2.6}$$

where

$$\bar{\mathbf{f}} = \begin{pmatrix} \bar{f}(0,0) \\ \bar{f}(1,0) \\ \vdots \\ \bar{f}(J_1-1,J_2-1) \end{pmatrix}, \quad \bar{\mathbf{y}} = \begin{pmatrix} \bar{y}(0,0) \\ \bar{y}(1,0) \\ \vdots \\ \bar{y}(J_1-1,J_2-1) \end{pmatrix}.$$

Notice that, $\bar{\mathbf{f}}$ is the column with rearranged Fourier coefficients of f such that they match the order of columns in matrix $\mathcal{A}$.

Proposition 2.2. *Any $f \in l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$ can be uniquely recovered from its samples (2.1), if an only if the matrix $\mathcal{A}$ is non-singular.*

3. The set of additional sampling points

Because $\mathcal{A}_{m_1,m_2}(i,j)$ is a Vandermonde matrix, it is singular at an $(i,j) \in I$ if and only if

$$\hat{a}\,(i + kJ_1, j + lJ_2) = \hat{a}\,(i + k'J_1, j + l'J_2) \tag{3.1}$$

for some $(k,l),(k',l') \in \{0,\cdots,m_1-1\} \times \{0,\cdots,m_2-1\}$. Hence, taking samples after the first m_1m_2 measurements is not going to add anything new in terms of recovery. In that case, we need to consider adding extra sampling points to overcome the singularities of $\mathcal{A}_{m_1,m_2}(i,j)$. For functions of one variable the problem of additional samples has been discussed in [1]. If the operator $\mathcal{A}$ in (2.6) is singular, we want to be able to find a set $\Omega_{add} \subset X \setminus (m_1\mathbb{Z}_{d_1} \times m_2\mathbb{Z}_{d_2})$ such that, for the related sampling operator $S_{\Omega_{add}}$, any function can be uniquely recovered from the samples

$$\{S_{\Omega_{add}}f, S_{m_1,m_2}f, \cdots, S_{m_1,m_2}\mathcal{A}^{m_1m_2-1}f\}. \tag{3.2}$$

Let $\dim(\ker(\mathcal{A})) = n$. Note that $ker(\mathcal{A}) = \bigoplus_{(i,j)\in I} ker(\mathcal{A}_{m_1,m_2})(i,j)$, hence, if the nullity of matrix $\mathcal{A}_{m_1,m_2}(i,j)$ is $w_{i,j}$, then $n = \sum_{i,j} w_{i,j}$.

The kernel of A is generated by linearly independent vectors $\bar{v}_s$, $s = 1,\ldots,n$ where every $\bar{v}_s$ has exactly two non-zero components, 1 and -1, corresponding to a pair of coinciding columns in $\mathcal{A}_{m_1,m_2}(i,j)$, for an $(i,j) \in I$.

Let $R_{\Omega_{add}}$ be $|\Omega_{add}| \times n$ matrix with rows corresponding to $\{(v_1(k,l),\cdots,v_n(k,l)) : (k,l) \in \Omega_{add}\}$, where v_s is the vector whose rearranged DFT is $\bar{v}_s$. With this notation, the following result holds:

Theorem 3.1. *Every $f \in l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$ can be uniquely reconstructed from its spatio-temporal samples (3.2) if and only if $rank(R_{\Omega_{add}}) = n$.*

Corollary 3.1. *If for the set Ω_{add} any $f \in l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$ can be uniquely determined by its samples (3.2), then $|\Omega_{add}| \geq dim(\ker(\mathcal{A}))$.*

4. Quadrantally symmetric kernels

We consider one special class of filters for which we are able to explicitly construct an additional sampling set of possible minimal size.

Definition 4.1. *The matrix $\hat{a}$ is quadrantally symmetric, if*

$$\hat{a}(s,p) = \hat{a}(d_1 - s, p) = \hat{a}(s, d_2 - p) = \hat{a}(d_1 - s, d_2 - p)$$

for all $(s,p) \in \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2}$, and $\hat{a}(s,p) \neq \hat{a}(k,l)$ for any other pairs (k,l) .

If for the kernel a, $\hat{a}$ is quadrantally symmetric, then it can be easily verified that (in particular) $\mathcal{A}_{m_1,m_2}(0,0)$ is singular. In fact, the following lemma holds

Lemma 4.1. *For quandrantally symmetric $\hat{a}$,*

$$dim(ker(\mathcal{A})) = \frac{d_1(m_2-1)}{2} + \frac{d_2(m_1-1)}{2} - \frac{(m_1-1)(m_2-1)}{4}.$$

Theorem 4.1. *Let the DFT $\hat{a}$ of the kernel a be quadrantally symmetric and let*

$$\begin{aligned} \Omega_{add} \ &= \ \left\{(k,l) : k = 1, \cdots, \frac{m_1-1}{2},\ l \in \mathbb{Z}_{d_2}\right\} \\ &\bigcup \ \left\{(k,l) : k \in \mathbb{Z}_{d_1},\ l = 1, \cdots, \frac{m_2-1}{2}\right\}. \end{aligned}$$

Then, any $f \in l^2(\mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2})$ can be uniquely recovered from the expanded set of samples

$$\left\{S_{\Omega_{add}} f, S_{m_1,m_2} f, \cdots, S_{m_1,m_2} A^{m_1 m_2 - 1} f\right\}. \tag{4.1}$$

Note that, from Lemma 4.1, the cardinality of Ω_{add} in the previous theorem is equal to $dim(\ker(\mathcal{A}))$ which from Corollary 3.1 is the possible minimal size among the sets of additional sampling points which allow unique recovery of every f from (4.1).

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Поступила 20 января 2015