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ZEROS AND SHARED ONE VALUE OF q-SHIFT DIFFERENCE POLYNOMIALS

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Abstract. [1] In this paper, we investigate uniqueness problems and zero distributions of q-shift difference polynomials of meromorphic functions with zero order in the complex plane. The obtained results extend some previous known results.

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1. Introduction and main results

In this paper, a meromorphic function always means a nonconstant analytic function in the whole complex plane except at possible poles. If no poles occur, it reduces to an entire function. Let q and c be non-zero complex constants, the q-shift of a function $f(z)$ is defined by $f(qz+c)$. We assume that the reader is familiar with the elementary Nevanlinna theory (see, e.g., [2, 3, 12]).

We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of logarithmic density 0. For a meromorphic function $f(z)$ in complex plane, denote by $S(r,f)$ the family of all meromorphic functions $\alpha(z)$ that satisfy $T(r,\alpha) = o(T(r,f))$ as $r \to \infty$ outside a possible exceptional set of logarithmic density 0.

We say that the functions f and g are meromorphic and share a small function α IM (ignoring multiplicities) if $f-\alpha$ and $g-\alpha$ have the same zeros. If $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities, then we say that f and g share α CM (counting multiplicities). Let f be a nonconstant meromorphic function, p be a positive integer and a be a complex constant. By $N_p(r, \frac{1}{f-a})$ we denote the counting function of the zeros of $f-a$, where an m-fold zero is counted m times if $m \le p$ and p times if $m > p$.

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Let f be a transcendental meromorphic function. In 1959, Hayman[1] proved that $f^n f'$ takes every non-zero complex value infinitely often if $n \geq 3$. Yang and Hua [11], obtained some results about the uniqueness problems for entire functions. Since then the difference has become a subject of great interest (see, e.g., [6, 8, 14, 15], and references therein). Among them Liu and Cao [6], have obtained results on the uniqueness and value distributions of q-shift difference polynomials. Some of them are stated below.

Theorem A. ([6, Theorem 1.1]). *Let $f(z)$ be a transcendental meromorphic (resp. entire) function with zero order, and let m, n be positive integers and a, q be non-zero complex constants. If $n \geq 6$ (resp. $n \geq 2$), then $f(z)^n(f(z)^m - a)f(qz+c) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a non-zero small function with respect to f. In particular, if $f(z)$ is a transcendental entire function and $\alpha(z)$ is a non-zero rational function, then m and n can be any positive integers.*

Theorem B. ([6, Theorem 1.5]). *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order. If $n \geq m+5$, and $f(z)^n(f(z)^m - a)f(qz+c))$ and $g(z)^n(g(z)^m - a)g(qz+c))$ share a non-zero polynomial $p(z)$ CM, then $f(z) \equiv g(z)$.*

In this paper, on the basis of Theorems A and B, we study the k-th derivative of q-shift difference polynomials and prove the following results.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function with zero order, and let n, k be positive integers. If $n > k+5$, then $(f(z)^n f(qz+c))^{(k)} - 1$ has infinitely many zeros.*

Theorem 1.2. *Let $f(z)$ be a transcendental entire function with zero order, and let n, k be positive integers, then $(f(z)^n f(qz+c))^{(k)} - 1$ has infinitely many zeros.*

Theorem 1.3. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integers. If $n > 2k+5$, and $(f(z)^n f(qz+c))^{(k)}$ and $(g(z)^n g(qz+c))^{(k)}$ share z CM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

Theorem 1.4. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integers. If $n > 2k+5$, and $(f(z)^n f(qz+c))^{(k)}$ and $(g(z)^n g(qz+c))^{(k)}$ share 1 CM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

When sharing a single value IM, we can prove the following two results.

Theorem 1.5. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integers. If $n > 5k+11$, and $(f(z)^n f(qz+c))^{(k)}$ and $(g(z)^n g(qz+c))^{(k)}$ share a value z IM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

Theorem 1.6. *Let $f(z)$ and $g(z)$ be transcendental entire functions with zero order, and let n, k be positive integers. If $n > 5k+11$, and $(f(z)^n f(qz+c))^{(k)}$ and $(g(z)^n g(qz+c))^{(k)}$ share 1 IM, then $f = tg$ for a constant t with $t^{n+1} = 1$.*

2. Lemmas

In this section, we present some lemmas which play an important role in the proofs of the main results. The following q-shift difference analogue of the logarithmic derivative lemma is very important when considering q-shift difference polynomials.

Lemma 2.1 ([7, Theorem 2.1]). *Let $f(z)$ be a meromorphic function of zero order. Then on a set of logarithmic density 1*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = o(T(r,f)).$$

The next two lemmas are essential in our proofs, they allow to estimate the characteristic function and the counting function of $f(qz+c)$ (see Lemmas 3.4 and 3.6 in [10]).

Lemma 2.2. *If $f(z)$ is a nonconstant zero order meromorphic function, then on a set of lower logarithmic density 1*

$$T(r, f(qz+c)) = (1+o(1))T(r, f(z)) + O(\log r).$$

Lemma 2.3. *If $f(z)$ is a nonconstant zero order meromorphic function, then on a set of lower logarithmic density 1*

$$N(r, f(qz+c)) = (1+o(1))N(r, f(z)) + O(\log r).$$

When considering two nonconstant meromorphic functions F and G that share at least one finite value CM, the following lemma plays a key role. In the original paper, [11], $S(r,F)$ denotes any quantity satisfying $S(r,F) = o(T(r,F))$ as $r \to \infty$ possibly outside a set of finite linear measure. So it holds when $S(r,F) = o(T(r,F))$ as $r \to \infty$ possibly outside a set of logarithmic density 0.

Lemma 2.4 ([11, Lemma 3]). *Let F and G be two nonconstant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*

$$\begin{aligned}(1)\qquad & max\{T(r,F),T(r,G)\} \le N_2(r,1/F)+N_2(r,1/G)+N_2(r,F)\\ & +N_2(r,G)+S(r,F)+S(r,G),\\ (2)\qquad & FG=1,\\ (3)\qquad & F=G,\end{aligned}$$

where $N_2(r,1/F)$ denotes the counting function of zeros of F such that the simple zeros are counted once and multiple zeros twice.

When two nonconstant meromorphic functions share at least one finite value IM, then the following lemma is needed.

Lemma 2.5 ([9, Lemma 2.3]). *Let F and G be two nonconstant meromorphic functions such that F and G share 1 IM, and let*

$$H := \left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{2.1}$$

If $H \not\equiv 0$, then

$$\begin{aligned}T(r,F)+T(r,G) \le\ & 2\left(N_2(r,1/F)+N_2(r,1/G)+N_2(r,F)+N_2(r,G)\right)+3(\overline{N}(r,F)\\ & +\overline{N}(r,G)+\overline{N}(r,1/F)+\overline{N}(r,1/G))+S(r,F)+S(r,G).\end{aligned}$$

Lemma 2.6 ([4]). *Let $f(z)$ be a nonconstant meromorphic function, and let s, k be two positive integers. Then*

$$\begin{aligned}N_s\left(r,\frac{1}{f^{(k)}}\right) &\le T(r,f^{(k)})-T(r,f)+N_{s+k}\left(r,\frac{1}{f}\right)+S(r,f),\\ N_s\left(r,\frac{1}{f^{(k)}}\right) &\le k\overline{N}(r,f)+N_{s+k}\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned}$$

Clearly, $\overline{N}\left(r,\frac{1}{f^{(k)}}\right)=N_1\left(r,\frac{1}{f^{(k)}}\right)$.

3. Proofs of the Theorems

In this section we prove our main results.

Proof of Theorem 1.1. Let $F(z)=f(z)^n f(qz+c)$. Using the second main theorem, we obtain

$$T(r,F^{(k)}) \le \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{F^{(k)}}\right)+\overline{N}\left(r,F^{(k)}\right)+S(r,F).$$

From Lemma 2.6, we get

$$\begin{aligned} T(r,F^{(k)}) &\leq \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right)+T(r,F^{(k)})-T(r,F)+N_{k+1}\left(r,\frac{1}{F}\right) \\ &\quad +\overline{N}\left(r,F^{(k)}\right)+S(r,F). \end{aligned}$$

Since $T(r,F)\leq (n+1)T(r,f)$, we have $S(r,F)=S(r,f)$. Thus the above inequality and Lemma 2.3 imply

$$\begin{aligned} T(r,F) &\leq \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right)+N_{k+1}\left(r,\frac{1}{F}\right)+\overline{N}(r,F^{(k)})+S(r,f) \\ &\leq \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right)+N_{k+1}\left(r,\frac{1}{f^n}\right)+N\left(r,\frac{1}{f(qz+c)}\right) \\ &\leq \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right)+(k+1)T(r,f)+T(r,f)+2\overline{N}(r,f)+S(r,f) \\ &\leq \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right)+(k+4)T(r,f)+S(r,f). \qquad (3.1) \end{aligned}$$

On the other hand, from Lemma 2.1, we get

$$\begin{aligned} (n+1)T(r,f) &= T(r,f^{n+1})=m(r,f^{n+1})+N(r,f^{n+1}) \\ &\leq m\left(r,F(z)\cdot\frac{f(z)}{f(qz+c)}\right)+N\left(r,F(z)\cdot\frac{f(z)}{f(qz+c)}\right)+S(r,f) \\ &\leq T(r,F(z))+m\left(r,\frac{f(z)}{f(qz+c)}\right)+N\left(r,\frac{f(z)}{f(qz+c)}\right)+S(r,f) \\ &\leq T(r,F(z))+2T(r,f)+S(r,f). \qquad (3.2) \end{aligned}$$

According to (3.1) and (3.2), we obtain

$$(n-k-5)T(r,f)\leq \overline{N}\left(r,\frac{1}{F^{(k)}-1}\right)+S(r,f).$$

Note that $n>k+5$, we conclude that $F^{(k)}(z)-1$ has infinitely many zeros. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let the function $F(z)$ be as in the proof of Theorem 1.1. Assume the opposite, that $F^{(k)}(z)-1$ has only a finite number of zeros. Since by assumption, f is a transcendental entire function with zero order, there exists a polynomial $P(z)$ such that

$$F^{(k)}(z)-1=P(z).$$

By integrating k times, we get from the above equation that $F(z)=Q(z)$, where $Q(z)$ is a polynomial, given by $Q(z)=f(z)^n f(qz+c)$. Obviously, $Q(z)\not\equiv 0$. Hence we can write

$$
\begin{aligned}
(n+1)T(r,f) &= T(r,f^{n+1}) = m(r,f^{n+1}) \\
&\le m\left(r, F(z)\cdot\frac{f(z)}{f(qz+c)}\right) + S(r,f) \\
&\le T(r,F(z)) + m\left(r,\frac{f(z)}{f(qz+c)}\right) + S(r,f) \\
&\le T(r,F(z)) + S(r,f) = T(r,Q(z)) + S(r,f), \qquad (3.3)
\end{aligned}
$$

which is impossible. Therefore $F^{(k)}(z)-1$ has infinitely many zeros. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.5. Let $F(z)$ be as in the proof of Theorem 1.1, $G(z)=g(z)^n g(qz+c)$, and H be as in Lemma 2.5. Define

$$\Phi(z)=\frac{F^{(k)}(z)}{z},\quad \Psi(z)=\frac{G^{(k)}(z)}{z}.$$

Then $\Phi(z)$ and $\Psi(z)$ share 1 IM by the conditions. Since f is a transcendental entire function, from the definition of $\Phi(z)$ we deduce that $N_2(r,\Phi)=O(\log r)=S(r,f)$. Using Lemmas 2.6 and 2.3, we can write

$$
\begin{aligned}
N_2\left(r,\frac{1}{\Phi}\right) &\le N_2\left(r,\frac{1}{F^{(k)}}\right) + S(r,f) \\
&\le k\overline{N}(r,F) + N_{k+2}\left(r,\frac{1}{F}\right) + S(r,f) \\
&\le N_{k+2}\left(r,\frac{1}{f^n}\right) + N_{k+2}\left(r,\frac{1}{f(qz+c)}\right) + S(r,f) \\
&\le (k+2)\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f(qz+c)}\right) + S(r,f) \\
&\le (k+3)T(r,f) + S(r,f).
\end{aligned}
$$

In the same manner, we get

$$\overline{N}\left(r,\frac{1}{\Phi}\right) \le (k+2)T(r,f) + S(r,f). \qquad (3.4)$$

Therefore

$$N_2\left(r,\frac{1}{\Phi}\right) + N_2(r,\Phi) \le (k+3)T(r,f) + S(r,f). \qquad (3.5)$$

Similarly, we obtain

$$N_2\left(r,\frac{1}{\Psi}\right) + N_2(r,\Psi) \le (k+3)T(r,g) + S(r,g). \qquad (3.6)$$

$$\overline{N}\left(r,\frac{1}{\Psi}\right) \le (k+2)T(r,g) + S(r,g). \qquad (3.7)$$

Next, by Lemmas 2.6 and 2.3, we get

$$\begin{aligned} N_2\left(r,\frac{1}{\Phi}\right) &\leq N_2\left(r,\frac{1}{F^{(k)}}\right)+S(r,f) \\ &\leq T(r,F^{(k)})-T(r,F)+N_{k+2}\left(r,\frac{1}{F}\right)+S(r,f) \\ &\leq T(r,F^{(k)})-T(r,F)+N_{k+2}\left(r,\frac{1}{f^n}\right)+N_{k+2}\left(r,\frac{1}{f(qz+c)}\right)+S(r,f) \\ &\leq T(r,F^{(k)})-T(r,F)+(k+2)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f(qz+c)}\right)+S(r,f) \\ &\leq T(r,\Phi)-T(r,F)+(k+3)T(r,f)+S(r,f). \qquad (3.8) \end{aligned}$$

By (3.3) we have

$$(n+1)T(r,f)\leq T(r,F)+S(r,f). \qquad (3.9)$$

Combining (3.8) and (3.9), we get

$$(n+1)T(r,f)\leq T(r,\Phi)-N_2\left(r,\frac{1}{\Phi}\right)+(k+3)T(r,f)+S(r,f). \qquad (3.10)$$

Similarly, we can obtain

$$(n+1)T(r,g)\leq T(r,\Psi)-N_2\left(r,\frac{1}{\Psi}\right)+(k+3)T(r,g)+S(r,g). \qquad (3.11)$$

It follows from Lemma 2.5 that if $H\not\equiv 0$, then

$$\begin{aligned} T(r,\Phi)+T(r,\Psi) &\leq 2\left(N_2\left(r,\frac{1}{\Phi}\right)+N_2\left(r,\frac{1}{\Psi}\right)\right)+3\left(\overline{N}\left(r,\frac{1}{\Phi}\right)+\overline{N}\left(r,\frac{1}{\Psi}\right)\right) \\ &\quad +S(r,\Phi)+S(r,\Psi). \end{aligned}$$

Substituting (3.4)-(3.7), (3.10) and (3.11) into the above inequality, we obtain

$$(n-5k-11)[T(r,f)+T(r,g)]\leq S(r,f)+S(r,g),$$

which is a contradiction, because by assumption we have $n>5k+11$. Hence, we have $H\equiv 0$. By integrating (2.1) two times, we get

$$\frac{1}{\Phi-1}=\frac{A}{\Psi-1}+B,$$

where $A\neq 0$ and B are constants. The above equation implies

$$\Psi=\frac{(B-A)\Phi+(A-B-1)}{B\Phi-(B+1)}. \qquad (3.12)$$

Hence, we easily get

$$T(r,\Phi)=T(r,\Psi)+O(1).$$

Thus, we have $S(r,f)=S(r,g)$.

In the following, we discuss three cases.

Case 1. Suppose that $B \neq 0, -1$. In this case, from (3.12) we obtain

$$\overline{N}(r, 1/(\Phi - \frac{B+1}{B})) = \overline{N}(r, \Psi).$$

Next, from the second fundamental theorem and (3.4), we have

$$\begin{aligned} T(r,\Phi) &\leq \overline{N}(r,\Phi) + \overline{N}\left(r, \frac{1}{\Phi}\right) + \overline{N}\left(r, 1/(\Phi - \frac{B+1}{B})\right) + S(r,\Phi) \\ &\leq (k+2)T(r,f) + S(r,f). \end{aligned}$$

In view of (3.8) and (3.9), we have $(n-k-2)T(r,f) \leq T(r,\Phi)$, implying that $(n-2k-4)T(r,f) \leq S(r,f)$. This contradicts the assumption $n > 5k+11$.

Case 2. Suppose that $B = 0$. From (3.12) we have

$$\Psi = A\Phi - (A-1). \tag{3.13}$$

If $A \neq 1$, then from (3.13) we can deduce $\overline{N}(r, 1/(\Phi - \frac{A-1}{A})) = \overline{N}(r, \frac{1}{\Psi})$. Then, by the second fundamental theorem and (3.7), we obtain

$$\begin{aligned} T(r,\Phi) &\leq \overline{N}(r,\Phi) + \overline{N}\left(r, \frac{1}{\Phi}\right) + \overline{N}\left(r, 1/(\Phi - \frac{A-1}{A})\right) + S(r,\Phi) \\ &\leq (k+2)T(r,g) + (k+2)T(r,f) + S(r,f). \end{aligned} \tag{3.14}$$

Similarly, we have

$$T(r,\Psi) \leq (k+2)T(r,g) + (k+2)T(r,f) + S(r,g). \tag{3.15}$$

By (3.10), (3.11), (3.14) and (3.15), we obtain

$$(n-3k-6)[T(r,f) + T(r,g)] \leq S(r,f) + S(r,g),$$

which is a contradiction since by assumption $n > 5k+11$. Thus, we have $A = 1$, and from (3.13), we obtain $\Phi = \Psi$, implying that

$$(f(z)^n f(qz+c))^{(k)} = (g(z)^n g(qz+c))^{(k)}.$$

Integrating the last equality, we get

$$f(z)^n f(qz+c) = g(z)^n g(qz+c) + p(z),$$

where $p(z)$ is a polynomial of degree at most $k-1$. If $p(z)\not\equiv 0$, then from the second main theorem for the small function case, we get

$$\begin{aligned}(n+1)T(r,f) &\le T(r,F)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f(qz+c)}\right)+\overline{N}\left(r,\frac{1}{g}\right)\\ &\quad+\overline{N}\left(r,\frac{1}{g(qz+c)}\right)+S(r,f)\\ &\le 2T(r,f)+2T(r,g)+S(r,f).\end{aligned}$$

Similarly, we have

$$(n+1)T(r,g) \ \le\ 2T(r,g)+2T(r,f)+S(r,f).$$

Therefore

$$(n+1)[T(r,f)+T(r,g)] \ \le\ 4[T(r,f)+T(r,g)]+S(r,f)+S(r,g),$$

which is a contradiction since by assumption $n>5k+11$. Thus, $p(z)\equiv 0$, which implies that

$$f(z)^n f(qz+c)=g(z)^n g(qz+c).$$

Let $\frac{f}{g}=h$. If h is not a constant, then the above equation implies

$$h(z)^n=\frac{1}{h(qz+c)}. \tag{3.16}$$

Thus, from the first main theorem, we obtain

$$\begin{aligned}nT(r,h(z)) &= T(r,h(z)^n)=T(r,h(qz+c)+O(1)\\ &\le T(r,h(z))+S(r,h).\end{aligned}$$

Since $n\ge 2$, we know that h is a constant. Then by (3.16), we have $h^{n+1}=1$. Hence $f(z)=tg(z)$, where t is a constant and $t^{n+1}=1$.

Case 3. Suppose that $B=-1$. From (3.12) we have

$$\Psi=\frac{(A+1)\Phi-A}{\Phi}. \tag{3.17}$$

If $A\ne -1$, then from (3.17) we can deduce $\overline{N}(r,1/(\Phi-\frac{A}{A+1}))=\overline{N}(r,\frac{1}{\Psi})$. By the same reasoning, discussed in the Case 2, we obtain a contradiction. Hence, $A=-1$. From (3.17), we have $\Phi\cdot\Psi=1$, that is,

$$(f(z)^n f(qz+c))^{(k)}\cdot(g(z)^n g(qz+c))^{(k)}=z^2. \tag{3.18}$$

Notice that $n > 5k+11$, hence if z_0 is a zero of $f(z)$ with multiplicity p, then z_0 is a zero of $(f(z)^n f(qz+c))^{(k)}$ with multiplicity at least $np-k > 4k+11$, which is impossible by checking the right-hand side of (3.18). Hence, zero is a Picard exceptional value of $f(z)$, and thus $f(z)$ is a constant, which is impossible. This completes the proof of Theorem 1.5. □

Proof of Theorem 1.3. Let $\Phi(z)$ and $\Psi(z)$ be as in Theorem 1.5. Then $\Phi(z)$ and $\Psi(z)$ share 1 CM, and from (3.8) we obtain

$$N_2\left(r,\frac{1}{\Phi}\right) \le T(r,\Phi) - T(r,F) + (k+3)T(r,f) + S(r,f). \tag{3.19}$$

Similarly, we get

$$N_2\left(r,\frac{1}{\Psi}\right) \le T(r,\Psi) - T(r,G) + (k+3)T(r,g) + S(r,g).$$

Assume that the Case 1 of Lemma 2.4 holds. Then, in view of Lemma 2.5 and (3.19), we can write

$$\begin{aligned} T(r,\Phi) &\le N_2\left(r,\frac{1}{\Phi}\right) + N_2\left(r,\frac{1}{\Psi}\right) + N_2(r,\Phi) + N_2(r,\Psi) + S(r,\Phi) + S(r,\Psi) \\ &\le T(r,\Phi) - T(r,F) + (k+3)T(r,f) + N_2\left(r,\frac{1}{G^{(k)}}\right) + S(r,f) + S(r,g) \\ &\le T(r,\Phi) - T(r,F) + (k+3)T(r,f) + k\overline{N}(r,G) + N_{k+2}\left(r,\frac{1}{G}\right) + S(r,f) + S(r,g) \\ &\le T(r,\Phi) - T(r,F) + (k+3)T(r,f) + (k+3)T(r,g) + S(r,f) + S(r,g). \end{aligned}$$

From the above inequality we get

$$T(r,F) \le (k+3)T(r,f) + (k+3)T(r,g) + S(r,f) + S(r,g).$$

On the other hand, from (3.3) we have

$$(n+1)T(r,f) \le T(r,F) + S(r,f).$$

Combining the last two inequalities we conclude that

$$(n-k-2)T(r,f) \le (k+3)T(r,g) + S(r,f) + S(r,g).$$

Similarly, we obtain

$$(n-k-2)T(r,g) \le (k+3)T(r,f) + S(r,f) + S(r,g).$$

Therefore

$$(n-2k-5)[T(r,f) + T(r,g)] \le S(r,f) + S(r,g),$$

which contradicts the assumption $n > 2k+5$. Hence $\Phi(z)\cdot\Psi(z) \equiv 1$ or $\Phi(z) \equiv \Psi(z)$ by Lemma 2.4.

The rest of the proof repeats the lines of the proof of Theorem 1.5. This completes the proof of Theorem 1.3. □

The proofs of Theorems 1.4 and 1.6 are similar to that of Theorems 1.3 and 1.5, and we omit them here.

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Список литературы

[1] W. K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. of Math., **70**, 9 – 42 (1959).

[2] W. K. Hayman, Meromorphic Functions, *Clarendon Press, Oxford*, 1964.

[3] I. Laine, Nevanlinna Theory and Complex Differential Equations, *Walter de Gruyter, Berlin-New York*, 1993.

[4] I. Lahiri, A. Sarkar, Uniqueness of a Meromorphic Function and its derivative, *J. Inequal. Pure Appl. Math.*, 5(1) (2004), Art 20.

[5] P. Li, C. C. Yang, Some further results on the unique range sets of meromorphic functions, *Kodai. Math. J.* **18** (1995), 437–450.

[6] K. Liu, X. L. Liu and T. B. Cao, Uniqueness and zeros of q-shift difference polynomials, *Proc. Indian Acad. Sci (Math. Sci), Vol.121, No.3, August* 2011, 301–310.

[7] K. Liu and X. G. Qi, Meromorphic solutions of q-shift difference equations, *Ann. Pol. Math.* **355** 101(3)(2011), 215–225.

[8] Li. S, Z. S. Gao, Finite order meromorphic solutions of linear difference equations, *Proc. Jpn. Acad. Ser. A. Math. Sci.* **87**, 73–76 (1959).

[9] J. F. Xu and H. X. Yi, Uniqueness of entire functions and differental polynomials, *Bulletin of the Korem Mathmatical Society, Vol.44, no.4* (2007), 623–629.

[10] J. F. Xu and X. B. Zhang, The zeros of q-shift difference polynomials of meromorphic functions, *Advances in Difference Equations*(2012), 2012:200.

[11] C. C. Yang and X. H. Hua, Uniqueness and value-sharing of meromorphic functions, *Ann. Acad. Sci. Fenn. Math.* **22**(2) (1997), 395–406.

[12] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, *Kluwer Academic Publishers, Dordrecht*, 2003.

[13] H. X. Yi, Uniqueness Theory of Meromorphic Functions and a question of C.C. Yang, *Complex Var.* **14** (1990), 169–176.

[14] J. L. Zhang and Risto Korhonen, On the Nevanlinna characteristic of $f(qz)$ and its applications, *J. Math. Anal. Appl.* **369** (2010) 537–544.

[15] J. L. Zhang and L. Z. Yang, Entire solutions of q-difference equations and value distribution of q-difference polynomials, *Annales Polonici Mathematici.* **109.1** (2013) 39–46.

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