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SECOND ORDER ASYMPTOTICAL EFFICIENCY FOR A POISSON PROCESS

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Abstract. We consider the problem of non-parametric estimation of the mean function of an inhomogeneous Poisson process when its intensity function is periodic. For integral-type quadratic loss functions there is a classical lower bound for all estimators and the empirical mean function attains that lower bound, thus it is asymptotically efficient. Following the ideas of the work by Golubev and Levit, we compare asymptotically efficient estimators and propose an estimator which is second order asymptotically efficient. Second order efficiency is done over Sobolev ellipsoids, following the ideas of Pinsker.

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1. Introduction

We consider the problem of non-parametric estimation of the mean function of an inhomogeneous Poisson process. We suppose that the unknown intensity function is periodic. It is known that empirical mean function is an asymptotically efficient (in several senses, see e.g. Kutoyants [7],[8]) estimator. Particularly, we are interested in asymptotic efficiency with respect to the integral-type quadratic loss function. Note that there are many estimators that are asymptotically efficient in this sense. The goal of present work is to choose in this class of asymptotically efficient estimators the estimators which are asymptotically efficient of the second order. Such a statement of the problem was considered by Golubev and Levit [6] in the problem of distribution function estimation for the model of independent and identically distributed random variables. Then applying the ideas of this work to the second order asymptotically efficient estimation for different models, Dalalyan and Kutoyants [1] proved second order asymptotic efficiency in the estimation problem of the invariant density of an ergodic diffusion process, in partial linear models the second order asymptotic efficiency was proved by Golubev, Härdle [5]. In this paper we prove second order asymptotic efficiency result for the mean function of a Poisson process. The main idea that led to development of these type problems was proposed by Pinsker in [10] (more details on the Pinsker bound can be found in [9], [11]).

2. Auxiliary Results

Let a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a stochastic process $\mathbf{X}^T = \{X_t,\ t \in [0,T]\}$ be given. Recall that $\mathbf{X}^T$ is an inhomogeneous Poisson process if 1. $X_0 = 0$ a.s. 2. The increments of the process $\mathbf{X}^T$ on the disjoints intervals are independent random variables. 3. We have

$$\mathbf{P}(X_t - X_s = k) = \frac{[\Lambda(t) - \Lambda(s)]^k}{k!} e^{-[\Lambda(t)-\Lambda(s)]},\ 0 \leq s < t \leq T, k \in \mathbb{Z}_+,$$

where $\mathbb{Z}_+$ is the set of all nonnegative integers. Here $\Lambda(t)$, $t \in [0,T]$ is a non-decreasing function, and is called the mean function of the Poisson process, because $\mathbf{E}X(t) = \Lambda(t)$. If the mean function is absolutely continuous

$$\Lambda(t) = \int_0^t \lambda(s) \mathrm{d}s,$$

then $\lambda(t)$, $0 \leq t \leq \tau$ is called the intensity function.

Let us consider the problem of estimation $\Lambda(t)$, when its intensity function is a τ-periodic function. For simplicity we suppose that $T = T_n = \tau n$. Then the observations $\mathbf{X}^T = \{X_t\,, t \in [0, \tau n]\}$, can be written in the form

$$\mathbf{X}^n = (X_1, X_2, \cdots, X_n), \tag{2.1}$$

where

$$X_j = (X_j(t), 0 \leq t \leq \tau),\ \ X_j(t) = X_{j\tau+t} - X_{j\tau}.$$

It is well known that *the empirical estimator*

$$\hat{\Lambda}_n(t) = \frac{1}{n}\sum_{j=1}^{n} X_j(t),\ t \in [0,\tau]$$

is consistent and asymptotically normal: for all $t \in [0,\tau]$

$$\sqrt{n}(\hat{\Lambda}(t) - \Lambda(t)) \Longrightarrow \mathcal{N}(0, \Lambda(t)).$$

Moreover, this estimator is asymptotically efficient in the sense of the following lower bound: for all estimators $\bar{\Lambda}(t)$, $t \in [0,\tau]$ and all $t^* \in (0,\tau]$ we have

$$\varliminf_{\delta \to 0} \varliminf_{n \to +\infty} \sup_{\Lambda \in V_\delta} n \mathbf{E}_\Lambda (\bar{\Lambda}_n(t^*) - \Lambda(t^*))^2 \geq \Lambda^*(t^*),$$

where $V_\delta = \{\Lambda(\cdot) :\ \sup_{t \in [0,\tau]} |\Lambda(t) - \Lambda^*(t)| \leq \delta\}$ and for the empirical mean function one has equality. This is a particular case of a general lower bound given in [7]. Similar inequality holds for integral-type quadratic loss function ([8])

$$\lim_{\delta\to 0}\lim_{n\to+\infty}\sup_{\Lambda\in V_\delta} n\int_0^\tau \mathbf{E}_\Lambda(\bar\Lambda_n(s)-\Lambda(s))^2\mathrm{d}s \geq \int_0^\tau \Lambda^*(s)\mathrm{d}s. \tag{2.2}$$

Definition 2.1. *The estimators $\Lambda_n^*(\cdot)$ for which we have equality in* (2.2), *i.e.,*

$$\lim_{\delta\to 0}\lim_{n\to+\infty}\sup_{\Lambda\in V_\delta} n\int_0^\tau \mathbf{E}_\Lambda(\Lambda_n^*(s)-\Lambda(s))^2\mathrm{d}s = \int_0^\tau \Lambda^*(s)\mathrm{d}s,$$

are called (first order) asymptotically efficient.

The empirical mean function is asymptotically efficient estimator also in this sense ([8]).

The goal of the present work is to find in the class of first order asymptotically efficient estimators an estimator which is second order asymptotically efficient. We follow the mains steps of the proof of Golubev, Levit [6].

3. Main Result

Denote by $\tilde{\mathcal{C}}_m(\mathbf{R}_+)$, $m\in\mathcal{N}$ the class of all $f:\mathbf{R}_+\to\mathbf{R}_+$ τ-periodic functions so that their $(m-1)$-th derivative $f^{(m-1)}$ exists and is absolutely continuous. Let us consider the following class of functions

$$\mathcal{F}_m(R,S)=\left\{\Lambda(t)=\int_0^t\lambda(s)\mathrm{d}s : \lambda\in\tilde{\mathcal{C}}_{m-1}(\mathbf{R}_+),\right.$$

$$\left.\int_0^\tau[\Lambda^{(m)}(t)]^2\mathrm{d}t\leq R,\ \frac{2}{\tau}\Lambda(\tau)=S\right\}, \tag{3.1}$$

where $R>0$, $S>0$, $m>1$, $m\in\mathcal{N}$ are given constants. Introduce as well

$$\Pi=\Pi_m(R,S)=(2m-1)R\left(\frac{2S}{R}\frac{\tau}{2\pi}\frac{m}{(2m-1)(m-1)}\right)^{\frac{2m}{2m-1}}. \tag{3.2}$$

Proposition 3.1. *Suppose we have observations of the model* (2.1)*. Then, for all estimators $\bar\Lambda_n(t)$ of the mean function $\Lambda(t)$, following lower bound holds*

$$\lim_{n\to+\infty}\sup_{\Lambda\in\mathcal{F}_m(R,S)} n^{\frac{2m}{2m-1}}\left(\int_0^\tau \mathbf{E}_\Lambda(\bar\Lambda_n(t)-\Lambda(t))^2\mathrm{d}t-\frac{1}{n}\int_0^\tau\Lambda(t)\mathrm{d}t\right)\geq -\Pi.$$

This proposition is proved in the forthcoming work [3]. In this work we propose an estimator which attains this lower bound, thus we will prove that this lower bound is sharp. Introduce

$$\Lambda_n^*(t)=\hat\Lambda_{1,n}\phi_1(t)+\sum_{l=1}^{+\infty}K_{2l,n}\hat\Lambda_{2l,n}\phi_{2l}(t)+$$

$$+\sum_{l=1}^{+\infty}\left[K_{2l,n}(\hat\Lambda_{2l+1,n}-a_{2l+1})+a_{2l+1}\right]\phi_{2l+1}(t),$$

where $\{\phi_l\}_{l=1}^{+\infty}$ is the trigonometric basis on $L_2[0,\tau]$, $\hat{\Lambda}_{l,n}$ are the Fourier coefficients of the empirical mean function with respect to this basis and

$$K_{2l,n} = \left(1 - \left|\frac{2\pi l}{\tau}\right|^m \alpha_n\right)_+,$$

$$\alpha_n = \left[\frac{1}{n}\frac{\tau}{2\pi}\frac{2S}{R}\frac{m}{(2m-1)(m-1)}\right]^{\frac{m}{2m-1}}, \quad a_{2l+1} = \sqrt{\frac{\tau}{2}}\frac{\tau}{2\pi l}S.$$

Here $x_+ = \max(x, 0)$. The main result of this work is the following theorem.

Theorem 3.1. *The estimator $\Lambda_n^*(t)$ attains the lower bound described above, that is,*

$$\lim_{n\to+\infty} \sup_{\Lambda\in\mathcal{F}_m(R,S)} n^{\frac{2m}{2m-1}} \left(\int_0^\tau \mathbf{E}_\Lambda(\Lambda_n^*(t) - \Lambda(t))^2 dt - \frac{1}{n}\int_0^\tau \Lambda(t)dt\right) = -\Pi.$$

4. The proof

Consider the normed linear space

$$L_2[0,\tau] = \left\{f : \int_0^\tau |f(t)|^2 dt < +\infty\right\},$$

with the norm

$$||f|| = \left(\int_0^\tau |f(t)|^2 dt\right)^{\frac{1}{2}}.$$

Evidently, $\mathcal{F}_m(R,S) \subset L_2[0,\tau]$. The main idea of the proof is to replace the estimation problem of the infinite-dimensional (continuum) mean function by the estimation problem of infinite-dimensional (countable) vector of its Fourier coefficients. Recall that the space $L_2[0,\tau]$ is isomorphic to the space

$$\ell_2 = \left\{\theta = (\theta_k)_{k=1}^{+\infty} : \sum_{k=1}^{+\infty}\theta_k^2 < +\infty\right\},$$

with the norm

$$||\theta|| = \left(\sum_{k=1}^{+\infty}\theta_k^2\right)^{\frac{1}{2}}.$$

Consider a complete, orthonormal system in the space $L_2[0,\tau]$,

$$\phi_1(t) = \sqrt{\frac{1}{\tau}}, \quad \phi_{2k}(t) = \sqrt{\frac{2}{\tau}}\cos\frac{2\pi k}{\tau}t, \quad \phi_{2k+1}(t) = \sqrt{\frac{2}{\tau}}\sin\frac{2\pi k}{\tau}t, \quad k \in \mathbb{N}.$$

Each function $f \in L_2[0,\tau]$ is a L_2−limit of its Fourier series

$$f(t) = \sum_{k=1}^{+\infty}\theta_k\phi_k(t), \quad \theta_k = \int_0^\tau f(t)\phi_k(t)dt.$$

Our first goal is to describe the set $\Theta \subset \ell_2$ of Fourier coefficients of the functions from the set $\mathcal{F}_m(R,S)$: Introduce following subset of $L_2[0,\tau]$

$$\Xi_m(R) = \{f : f^{(m-1)} \in AC[0,\tau], f^{(i)}(0) = f^{(i)}(\tau), i = \overline{0, m-1}, \int_0^\tau [f^{(m)}(t)]^2 dt \le R\},$$

where $AC[0,\tau]$ is the class of all absolutely continuous functions on the interval $[0,\tau]$. The proof of the next lemma can be found in [11], lemma A.3.

Lemma 4.1. *The function f belongs to the set $\Xi_m(R)$ if and only if its Fourier coefficients with respect to the trigonometric basis belong to the set*

$$\Theta_m = \left\{\theta \in \ell_2, \ \sum_{k=2}^{+\infty} A_k^2 \theta_k^2 \le R\right\}, \tag{4.1}$$

where $A_{2k} = A_{2k+1} = \left(\frac{2\pi k}{\tau}\right)^m$, $k \in \mathcal{N}$.

Denote

$$\Lambda_k = \int_0^\tau \Lambda(t)\phi_k(t)dt, \quad \lambda_k = \int_0^\tau \lambda(t)\phi_k(t)dt.$$

Since $\Lambda \in \mathcal{F}_m(R,S)$ is equivalent to $\lambda \in \Xi_{m-1}(R)$ for its intensity function and, by the Lemma 4.1, the later is equivalent to $(\lambda_k)_{k\ge1} \in \Theta_{m-1}$, then, calculating $(\Lambda(0) = 0)$

$$\frac{\tau}{2\pi k}\lambda_{2k+1} = \Lambda_{2k}, \quad \frac{\tau}{2\pi k}\lambda_{2k} = \sqrt{\frac{2}{\tau}}\frac{\tau}{2\pi k}\Lambda(\tau) - \Lambda_{2k+1},$$

we obtain necessary and sufficient condition for $\Lambda \in \mathcal{F}_m(R,S)$ that is $(\Lambda_k)_{k\ge1}$ satisfies

$$\sum_{k=1}^{+\infty}\left(\frac{2\pi k}{\tau}\right)^{2m}\left[\Lambda_{2k}^2 + \left(\sqrt{\frac{\tau}{2}}\frac{\tau}{2\pi k}S - \Lambda_{2k+1}\right)^2\right] \le R. \tag{4.2}$$

Let us write the empirical mean function as a stochastic integral

$$\hat{\Lambda}_n(t) = \frac{1}{n}\sum_{j=1}^n X_j(t) = \frac{1}{n}\sum_{j=1}^n \int_0^\tau I\{s \le t\} dX_j(s).$$

We consider generalization of this estimator

$$\tilde{\Lambda}_n(t) = \frac{1}{n}\sum_{j=1}^n \int_0^\tau K_n(s-t)X_j(s)ds,$$

where $K_n(u)$ for each $n \in \mathcal{N}$ is such a function that

$$K_n(u+\tau) = K_n(u), \quad K_n(u) = K_n(-u), \ u \in [0,\tau].$$

We show that there are functions $K_n(u)$ for which the estimator described above is asymptotically efficient. Introduce

$$\hat{\Lambda}_{l,n} = \int_0^\tau \hat{\Lambda}_n(t)\phi_l(t)dt, \quad K_{l,n} = \int_0^\tau K_n(t)\phi_l(t)dt.$$

Let us first study

$$\tilde{\Lambda}_{l,n} = \int_0^\tau \tilde{\Lambda}_n(t)\phi_l(t)\mathrm{d}t = \frac{1}{n}\sum_{j=1}^{n}\int_0^\tau X_j(s)\left(\int_0^\tau K_n(s-t)\phi_l(t)\mathrm{d}t\right)\mathrm{d}s$$
$$= \frac{1}{n}\sum_{j=1}^{n}\int_0^\tau X_j(s)\left(\int_{s-\tau}^{s} K_n(u)\phi_l(s-u)\mathrm{d}u\right)\mathrm{d}s.$$

We calculate separately the even and odd Fourier coefficients

$$\tilde{\Lambda}_{2l+1,n} = \sqrt{\frac{2}{\tau}}\frac{1}{n}\sum_{j=1}^{n}\int_0^\tau X_j(s)\left(\int_{s-\tau}^{s} K_n(u)\sin\frac{2\pi l}{\tau}(s-u)\mathrm{d}u\right)\mathrm{d}s$$
$$= \sqrt{\frac{2}{\tau}}\frac{1}{n}\sum_{j=1}^{n}\int_0^\tau X_j(s)\sin\frac{2\pi l}{\tau}s\left(\int_{s-\tau}^{s} K_n(u)\cos\frac{2\pi l}{\tau}u\mathrm{d}u\right)\mathrm{d}s$$
$$- \sqrt{\frac{2}{\tau}}\frac{1}{n}\sum_{j=1}^{n}\int_0^\tau X_j(s)\cos\frac{2\pi l}{\tau}s\left(\int_{s-\tau}^{s} K_n(u)\sin\frac{2\pi l}{\tau}u\mathrm{d}u\right)\mathrm{d}s.$$

Since

$$\int_{s-\tau}^{s} K_n(u)\sin\frac{2\pi l}{\tau}u\mathrm{d}u = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} K_n(u)\sin\frac{2\pi l}{\tau}u\mathrm{d}u = 0,$$
$$\int_{s-\tau}^{s} K_n(u)\cos\frac{2\pi l}{\tau}u\mathrm{d}u = \int_0^\tau K_n(u)\cos\frac{2\pi l}{\tau}u\mathrm{d}u = K_{2l,n},$$

then (the second one can be proved in the same way)

$$\tilde{\Lambda}_{2l+1,n} = \sqrt{\frac{\tau}{2}}K_{2l,n}\cdot\hat{\Lambda}_{2l+1,n},\quad \tilde{\Lambda}_{2l,n} = \sqrt{\frac{\tau}{2}}K_{2l,n}\cdot\hat{\Lambda}_{2l,n}.$$

Instead of this we consider the estimator which Fourier coefficients have the form ([2])

$$\tilde{\Lambda}_{1,n} = \hat{\Lambda}_{1,n},\quad \tilde{\Lambda}_{2l,n} = K_{2l,n}\cdot\hat{\Lambda}_{2l,n},\quad \tilde{\Lambda}_{2l+1,n} = K_{2l,n}(\hat{\Lambda}_{2l+1,n} - a_{2l+1}) + a_{2l+1},$$

where

$$a_{2l+1} = \sqrt{\frac{\tau}{2}}\frac{\tau}{2\pi l}S.$$

Now we are ready to evaluate the risk described in the theorem. First,

$$\mathbf{E}_\Lambda||\tilde{\Lambda}_n - \Lambda||^2 - \frac{1}{n}\int_0^\tau \Lambda(s)\mathrm{d}s = \mathbf{E}_\Lambda||\tilde{\Lambda}_n - \Lambda||^2 - \mathbf{E}_\Lambda||\hat{\Lambda}_n - \Lambda||^2.$$

Using the fact that $\mathbf{E}_\Lambda\hat{\Lambda}_{l,n} = \Lambda_l$, and denoting $\sigma_{l,n}^2 = \mathbf{E}_\Lambda|\hat{\Lambda}_{l,n} - \Lambda_l|^2$, by the Parseval's equality we get

$$\mathbf{E}_\Lambda\|\tilde{\Lambda}_n - \Lambda\|^2 - \mathbf{E}_\Lambda\|\hat{\Lambda}_n - \Lambda\|^2 = \sum_{l=1}^{+\infty}(|K_{2l,n}|^2 - 1)(\sigma_{2l,n}^2 + \sigma_{2l+1,n}^2)$$
$$+\sum_{l=1}^{+\infty}|K_{2l,n} - 1|^2\left[|\Lambda_{2l+1} - a_{2l+1}|^2 + |\Lambda_{2l}|^2\right]. \tag{4.3}$$

To compute the variance $\sigma^2_{2l,n} + \sigma^2_{2l+1,n}$, introduce the notation

$$\pi_j(t) = X_j(t) - \Lambda(t).$$

In the sequel, we are going to use the following property of stochastic integral

$$\mathbf{E}_\Lambda\left[\int_0^\tau f(t)\mathrm{d}\pi_j(t)\int_0^\tau g(t)\mathrm{d}\pi_j(t)\right] = \int_0^\tau f(t)g(t)\mathrm{d}\Lambda(t), \quad f, g \in \mathrm{L}_2[0,\tau]. \tag{4.4}$$

Further, integrating by parts, we get

$$\hat{\Lambda}_{l,n} - \Lambda_l = \frac{1}{n}\sum_{j=1}^{n}\int_0^\tau \pi_j(t)\phi_l(t)\mathrm{d}t = \frac{1}{n}\sum_{j=1}^{n}\int_0^\tau\left(\int_t^\tau \phi_l(s)\mathrm{d}s\right)\mathrm{d}\pi_j(t),$$

which entails that

$$\sigma^2_{l,n} = \mathbf{E}_\Lambda|\hat{\Lambda}_{l,n} - \Lambda_l|^2 = \frac{1}{n}\int_0^\tau\left(\int_t^\tau \phi_l(s)\mathrm{d}s\right)^2\mathrm{d}\Lambda(t).$$

Simple algebra yields

$$\sigma^2_{2l,n} + \sigma^2_{2l+1,n} = \frac{2}{n}\sqrt{\frac{2}{\tau}}\left(\frac{\tau}{2\pi l}\right)^2\left[\sqrt{\frac{2}{\tau}}\Lambda(\tau) - \lambda_{2l}\right].$$

Combining with (4.3), this leads to

$$\mathbf{E}_\Lambda\|\tilde{\Lambda}_n - \Lambda\|^2 - \mathbf{E}_\Lambda\|\hat{\Lambda}_n - \Lambda\|^2 = \sum_{l=1}^{+\infty}\frac{2}{n}S\left(\frac{\tau}{2\pi l}\right)^2(|K_{2l,n}|^2 - 1)$$

(4.5)

$$+\sum_{l=1}^{+\infty}|K_{2l,n} - 1|^2\left[|\Lambda_{2l+1} - a_{2l+1}|^2 + |\Lambda_{2l}|^2\right] + \sum_{l=1}^{+\infty}\frac{2}{n}\sqrt{\frac{2}{\tau}}\left(\frac{\tau}{2\pi l}\right)^2(1 - |K_{2l,n}|^2)\lambda_{2l}.$$

For the third term in the right-hand side we have

$$\left|\sum_{l=1}^{+\infty}\frac{2}{n}\sqrt{\frac{2}{\tau}}\left(\frac{\tau}{2\pi l}\right)^2(1 - |K_{2l,n}|^2)\lambda_{2l}\right| \le$$

$$\le \frac{2}{n}\sqrt{\frac{2}{\tau}}\max_l\frac{|1 - |K_{2l,n}|^2|}{\left(\frac{2\pi l}{\tau}\right)^m}\sum_{l=1}^{+\infty}\left(\frac{2\pi l}{\tau}\right)^{m-1}\lambda_{2l}\left(\frac{2\pi l}{\tau}\right)^{-1}$$

$$\le \frac{2}{n}\sqrt{\frac{2}{\tau}}\max_l\frac{|1 - |K_{2l,n}|^2|}{\left(\frac{2\pi l}{\tau}\right)^m}\left(\sum_{l=1}^{+\infty}\left(\frac{2\pi l}{\tau}\right)^{2(m-1)}\lambda^2_{2l}\right)^{\frac{1}{2}}\left(\sum_{l=1}^{+\infty}\left(\frac{2\pi l}{\tau}\right)^{-2}\right)^{\frac{1}{2}}.$$

Since $(\lambda_l)_{l\ge1} \in \Theta_{m-1}$, then from (4.1) we obtain

$$\left(\sum_{l=1}^{+\infty}\left(\frac{2\pi l}{\tau}\right)^{2(m-1)}\lambda^2_{2l}\right)^{\frac{1}{2}} \le \sqrt{R}.$$

Hence

$$\left|\sum_{l=1}^{+\infty}\frac{2}{n}\sqrt{\frac{2}{\tau}}\left(\frac{\tau}{2\pi l}\right)^2(1 - |K_{2l,n}|^2)\lambda_{2l}\right| \le \frac{\mathrm{C}}{n}\max_l\frac{|1 - |K_{2l,n}|^2|}{\left(\frac{2\pi l}{\tau}\right)^m}$$

Now, consider the first two terms of the right-hand side of the equation (4.5). Introduce a set of possible kernels (for all $c_n > 0$)

$$\mathfrak{C}_n = \left\{ K_{2l,n} : |K_{2l,n} - 1| \le \left|\frac{2\pi l}{\tau}\right|^m c_n \right\}.$$

It follows from (4.2)

$$\sum_{l=1}^{+\infty} \frac{2}{n} S \left(\frac{\tau}{2\pi l}\right)^2 (|K_{2l,n}|^2 - 1) + + \sum_{l=1}^{+\infty} |K_{2l,n} - 1|^2 \left[|\Lambda_{2l+1} - a_{2l+1}|^2 + |\Lambda_{2l}|^2\right]$$
$$= \sum_{l=1}^{+\infty} \frac{2}{n} S \left(\frac{\tau}{2\pi l}\right)^2 (|K_{2l,n}|^2 - 1) + \sum_{l=1}^{+\infty} \frac{|K_{2l,n} - 1|^2}{\left(\frac{2\pi l}{\tau}\right)^{2m}} \left(\frac{2\pi l}{\tau}\right)^{2m} \left[|\Lambda_{2l+1} - a_{2l+1}|^2 + |\Lambda_{2l}|^2\right]$$
$$\le \sum_{l=1}^{+\infty} \frac{2}{n} S \left(\frac{\tau}{2\pi l}\right)^2 (|K_{2l,n}|^2 - 1) + c_n^2 R.$$

Hence, minimizing the later over the set $\mathfrak{C}_n$

$$\tilde{K}_{2l,n} = \arg\min_{\mathfrak{C}_n} |K_{2l,n}| = \left(1 - \left|\frac{2\pi l}{\tau}\right|^m c_n\right)_+, \tag{4.6}$$

we obtain

$$\sup_{\Lambda \in \mathcal{F}_m(R,S)} \left(\mathbf{E}_\Lambda \|\tilde{\Lambda}_n - \Lambda\|^2 - \mathbf{E}_\Lambda \|\hat{\Lambda}_n - \Lambda\|^2\right) \le$$
$$\frac{2}{n} S \sum_{l=1}^{+\infty} \left(\frac{\tau}{2\pi l}\right)^2 (|\tilde{K}_{2l,n}|^2 - 1) + c_n^2 R + \frac{C}{n} \max_l \frac{|1 - |\tilde{K}_{2l,n}|^2|}{\left(\frac{2\pi l}{\tau}\right)^m}. \tag{4.7}$$

Here $\tilde{\Lambda}_n(t)$ is the estimator corresponding to the kernel $\tilde{K}(u)$. In fact, we have not yet constructed the estimator. We have to specify the sequence of positive numbers c_n in the definition (4.6). Consider the function

$$H(c_n) = \frac{2}{n} S \sum_{l=1}^{+\infty} \left(\frac{\tau}{2\pi l}\right)^2 (|\tilde{K}_{2l,n}|^2 - 1) + c_n^2 R$$

and minimize it with respect to the positive sequence c_n. Introduce as well $N_n = \frac{\tau}{2\pi} c_n^{-\frac{1}{m}}$. Then

$$H(c_n) = \frac{2}{n} S \sum_{l \le N_n} \left(\frac{\tau}{2\pi l}\right)^2 \left[c_n^2 \left(\frac{2\pi l}{\tau}\right)^{2m} - 2c_n \left(\frac{2\pi l}{\tau}\right)^m\right] + \sum_{l > N_n} \left(\frac{\tau}{2\pi l}\right)^2 + c_n^2 R.$$

To minimize this function consider its derivative

$$H'(c_n) = \frac{2}{n} S \sum_{l \le N_n} \left(\frac{\tau}{2\pi l}\right)^2 \left[2c_n \left(\frac{2\pi l}{\tau}\right)^{2m} - 2\left(\frac{2\pi l}{\tau}\right)^m\right] + 2c_n R = 0. \tag{4.8}$$

Consider such sums ($\beta \in \mathcal{N}$)

$$\sum_{l \le N_n} l^\beta = \sum_{l=1}^{[N_n]} \left(\frac{l}{[N_n]}\right)^\beta [N_n]^\beta = [N_n]^{\beta+1} \sum_{l=1}^{[N_n]} \left(\frac{l}{[N_n]}\right)^\beta \frac{1}{[N_n]},$$

hence, if $c_n \longrightarrow 0$, as $n \longrightarrow +\infty$,

$$\frac{1}{[N_n]^{\beta+1}} \sum_{l \le N_n} l^\beta \longrightarrow \int_0^1 x^\beta \mathrm{d}x,$$

that is,

$$\sum_{l \le N_n} l^\beta = \frac{[N_n]^{\beta+1}}{\beta+1}(1+o(1)), \quad n \longrightarrow +\infty.$$

Using this identity we can transform (4.7) (remembering that $N_n = \frac{\tau}{2\pi} c_n^{-\frac{1}{m}}$)

$$\begin{aligned}
&\frac{2}{n} S \left(c_n \left(\frac{2\pi}{\tau}\right)^{2(m-1)} \sum_{l \le N_n} l^{2(m-1)} - \left(\frac{2\pi}{\tau}\right)^{m-2} \sum_{l \le N_n} l^{m-2} \right) = -c_n R, \\
&\frac{2}{n} S \left(c_n \left(\frac{2\pi}{\tau}\right)^{2(m-1)} \frac{N_n^{2m-1}}{2m-1} - \left(\frac{2\pi}{\tau}\right)^{m-2} \frac{N_n^{m-1}}{m-1} \right) = -c_n R(1+o(1)), \\
&\frac{2}{n} S \frac{\tau}{2\pi} c_n^{-\frac{m-1}{m}} \left(\frac{1}{2m-1} - \frac{1}{m-1} \right) = -c_n R(1+o(1)).
\end{aligned}$$

Finally, for the solution of (4.8), we can write

$$c_n^* = \alpha_n^*(1+o(1)),$$

$$\alpha_n^* = \left[\frac{1}{n} \frac{\tau}{2\pi} \frac{2S}{R} \frac{m}{(2m-1)(m-1)} \right]^{\frac{m}{2m-1}}. \tag{4.9}$$

Now, using the identity ($\beta \in \mathcal{N}$)

$$\sum_{l > N_n} \frac{1}{l^\beta} = \frac{1}{N_n^{\beta-1}} \int_1^{+\infty} \frac{1}{x^\beta} \mathrm{d}x \cdot (1+o(1)), \, n \longrightarrow +\infty,$$

for $\beta = 2$

$$\sum_{l > N_n} \frac{1}{l^2} = \frac{1}{N_n} \cdot (1+o(1)), \, n \longrightarrow +\infty,$$

calculate

$$H(c_n^*) = \frac{2}{n} S \left[(c_n^*)^2 \left(\frac{2\pi}{\tau} \right)^{2(m-1)} \frac{N_n^{2m-1}}{2m-1} - \right.$$

$$\left. - 2c_n^* \left(\frac{2\pi}{\tau} \right)^{m-2} \frac{N_n^{m-1}}{m-1} - \frac{1}{N_n} \right] (1 + o(1)) + (c_n^*)^2 R =$$

$$= \frac{2}{n} S \left[(c_n^*)^2 \frac{(c_n^*)^{-\frac{2m-1}{m}}}{2m-1} - 2c_n^* \frac{(c_n^*)^{-\frac{m-1}{m}}}{m-1} \right] (1 + o(1)) + (c_n^*)^2 R =$$

$$= \frac{2}{n} S (c_n^*)^{-\frac{1}{m}} \frac{-3m+1}{(2m-1)(m-1)} (1 + o(1)) + (c_n^*)^2 R =$$

$$= (c_n^*)^{-\frac{1}{m}} (c_n^*)^{-\frac{2m-1}{m}} R \frac{-3m+1}{m} (1 + o(1)) + (c_n^*)^2 R = -(2m-1)(\alpha_n^*)^2 R(1 + o(1)),$$

where we have used the relation (4.9). Now, choosing the sequence $c_n = \alpha_n^*$ for the definition of the estimator in (4.6), we obtain from (4.7)

$$\sup_{\Lambda \in \mathcal{F}_m(R,S)} \left(\mathbf{E}_\Lambda \|\tilde{\Lambda}_n - \Lambda\|^2 - \mathbf{E}_\Lambda \|\hat{\Lambda}_n - \Lambda\|^2 \right) \leq$$

$$\leq -(2m-1)(\alpha_n^*)^2 R(1 + o(1)) + \frac{C}{n} \max_l \frac{|1 - |\tilde{K}_{2l,n}|^2|}{\left(\frac{2\pi l}{\tau}\right)^m}. \tag{4.10}$$

If we show that

$$\frac{1}{n} \max_l \frac{|1 - |\tilde{K}_{2l,n}|^2|}{\left(\frac{2\pi l}{\tau}\right)^m} = o(n^{-\frac{2m}{2m-1}}), \tag{4.11}$$

then, since

$$\Pi = (2m-1)(\alpha_n^*)^2 R n^{\frac{2m}{2m-1}},$$

we get from (4.10)

$$\varlimsup_{n \to +\infty} n^{\frac{2m}{2m-1}} \sup_{\Lambda \in \mathcal{F}_m(R,S)} \left(\mathbf{E}_\Lambda \|\tilde{\Lambda}_n - \Lambda\|^2 - \mathbf{E}_\Lambda \|\hat{\Lambda}_n - \Lambda\|^2 \right) \leq -\Pi.$$

This combined with the proposition will end the proof. To prove (4.11) recall that

$$\tilde{K}_{2l,n} = \left(1 - \left| \frac{2\pi l}{\tau} \right|^m \alpha_n^* \right)_+, \quad \alpha_n^* = \left[\frac{1}{n} \frac{\tau}{2\pi} \frac{2S}{R} \frac{m}{(2m-1)(m-1)} \right]^{\frac{m}{2m-1}}.$$

Therefore,

$$\frac{1}{n} \max_l \frac{|1 - |\tilde{K}_{2l,n}|^2|}{\left(\frac{2\pi l}{\tau}\right)^m} \leq \frac{2}{n} \max_l \frac{1 - \tilde{K}_{2l,n}}{\left(\frac{2\pi l}{\tau}\right)^m} = \frac{2}{n} \alpha_n^* = \frac{C}{n^{\frac{3m-1}{2m-1}}} = o(n^{-\frac{2m}{2m-1}}), \ m > 1.$$

Theorem 3.1 is proved.

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