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ASYMPTOTIC PROPERTIES OF MAXIMUM LIKELIHOOD ESTIMATORS FOR A GENERALIZED PARETO-TYPE DISTRIBUTION

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Abstract. Astola and Danielian [1], using stochastic birth-death process, have proposed a regular four-parameter discrete probability distribution, called *generalized Pareto-type model*, which is an appealing distribution for modeling phenomena in Bioinformatics. Farbod and Gasparian [5], fitted this distribution to the two sets of real data, and have derived conditions under which a solution for the system of likelihood equations exists and coincides with the maximum likelihood estimators (MLE) for the model unknown parameters. Also, in [5], an accumulation method for approximate computation of the MLE has been considered with simulation studies. In this paper we show that for sufficiently large sample size the system of likelihood equations has a solution, which according to [5], coincides with the MLE of vector-valued parameter for the underlying model. Besides, we establish asymptotic unbiasedness, weak consistency, asymptotic normality, asymptotic efficiency, and convergence of arbitrary moments of the MLE, by verifying the so-called regularity conditions.

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1. Introduction

The mechanism of biomolecular large-scale systems dynamic often can be explained with the help of standard stochastic birth-death process with various specific constraints on its coefficients. The stationary solutions of the process, which always are right-skewed, can be used as frequency distributions of different events, occurring in large-scale biomolecular systems. For details we refer to [1, 4].

Based on the standard *birth-death models*, several frequency distributions have been considered for biomolecular applications (see, for instance, Bornholdt and Ebel [2], Kuznetsov [6, 7], Kuznetsov et al. [8], and Danielian and Astola [4]). Since then Astola and Danielian [1], based on data sets, have introduced the following "four-parameter"regular frequency distribution, called *generalized Pareto-type frequency* *distribution* (see also [5]):

$$
(1.1)\quad \begin{cases} p_\alpha(k) = \mathbb{P}_\alpha(\xi = k) = [g(\alpha)]^{-1} \cdot \frac{\theta^k}{(k+b)^\rho} \cdot \prod_{m=0}^{k-1}(1 + \frac{c-1}{(m+b)^\rho}), \quad k = 1, 2..., \\ \\ p_\alpha(0) = [g(\alpha)]^{-1} = \left[1 + \sum_{n=1}^{\infty} \frac{\theta^n}{(n+b)^\rho} \cdot \prod_{m=0}^{n-1}(1 + \frac{c-1}{(m+b)^\rho})\right]^{-1}, \end{cases}
$$

where $\alpha = (\theta, c, b, \rho)$ is an unknown parameter, such that $0 < \theta < 1,\ 0 < c < \infty,\ 0 < b < \infty,\ 1 < \rho < \infty,\ b^\rho > 1 - c$.

The model (1.1) is described by a four-component vector parameter: $\alpha = (\theta, c, b, \rho)$, in which c is the *non-linear scale* parameter (or *exponential scale* parameter), b is the *location* parameter, the parameter ρ describes the *shape* of the probability distribution, as for the parameter θ, its role is explained in [1], Ch. 4, Theorem 4.2.

The problem of interest is to investigate the statistical properties of the parameters for the *generalized Pareto-type frequency distribution*, given by (1.1). However, the model (1.1) suffers from two major drawbacks. First, it lacks a simple closed form expression for probability mass function. The second disadvantage is that the r-th ($r \in \mathbb{N}$) absolute moment of this distribution exists only for $\rho > r+1$ (see [5], Lemma 1). This leads to a serious difficulties in making statistical inferences about the model unknown parameters.

Some aspects of this problem has been considered in [5]. For instance, conditions under which a solution of the system of likelihood equations exists and coincides with the MLE for the unknown parameters of the model (1.1), were obtained; an approximate method (with *simulation studies*) for estimating the model parameters was proposed, as well as, two real data sets on the number of proteins and the number of residues have been proposed for fitting the model (1.1).

The purpose of the present paper is to continue the investigations conducted in [5]. Specifically, in this paper we prove that for sufficiently large sample size the system of likelihood equations has a solution, which according to the results from [5], coincides with the Maximum Likelihood Estimator (MLE) of a vector parameter for the underlying model. Besides, we establish asymptotic unbiasedness, weak consistency, asymptotic normality, asymptotic efficiency, and convergence of arbitrary moments of the MLE to the corresponding moments of the limiting normal distribution. To this end, we make use the well-known results on asymptotic behavior of the MLE (see [3], [9]), by verifying the corresponding regularity conditions, called RR-conditions.

The rest of the paper is organized as follows. Section 2 contains the RR conditions (in the general case), and the asymptotic properties of the MLE. In Section 3 we introduce some notation and prove an auxiliary result (Lemma 3.1). The main results of the paper are given in Section 4.

2. The RR–conditions and the asymptotic properties of the MLE

Let $X^n = (X_1, ..., X_n)$ be a random sample drawn from the distribution $\mathbb{P}_\alpha$ belonging to the parametric family of distributions $\mathcal{P} = \{\mathbb{P}_\alpha,\ \alpha = (\alpha_1, \ldots, \alpha_k) \in A \subset \mathbb{R}^k\}$, and let $p_\alpha(x)$ be the density function of $\mathbb{P}_\alpha$.

We say that the parametric family of distributions $\mathcal{P}$ satisfies the RR–conditions if the following are satisfied (see [3]):

1. There exists a *compact* subset Ω of the parametric set $A = \{\alpha\}$ containing an open neighborhood of the true value α^0 of the parameter α.
2. The distributions $\mathbb{P}_\alpha$ are *distinct*, that is, $p_{\alpha^1}(x) \neq p_{\alpha^2}(x)$ for all $\alpha^1 \neq \alpha^2$ $(\alpha^1, \alpha^2 \in \Omega)$ and all $x \in Supp\, \mathbb{P}_\alpha = \{x \in \mathbb{R} : p_\alpha(x) > 0\}$.
3. The distributions $\mathbb{P}_\alpha$ have a *common support*, that is, the set $Supp\, \mathbb{P}_\alpha$ does not depend on α.
4. For all $x \in Supp\, \mathbb{P}_\alpha$ the functions $l_\alpha(x) = \ln p_\alpha(x)$ are twice continuously differentiable in α, and there exists a function $M(x)$ satisfying $\int_{\mathbb{R}} |M(x)| p_\alpha(x) dx < \infty$ and
$$\lim_{N \longrightarrow \infty} \sup_{\alpha \in \Omega} \int_{|M(x)|>N} M(x) p_\alpha(x) dx = 0,$$
such that for all $\alpha = (\alpha_1, ..., \alpha_n) \in \Omega$ and $x \in Supp\, \mathbb{P}_\alpha$
$$| l_\alpha^{ij}(x) | \leq M(x),$$
where $l_\alpha^{ij}(x) = \frac{\partial^2 l_\alpha(x)}{\partial \alpha_i \partial \alpha_j}$.
5. The Fisher information matrix
$$I(\alpha) = \| I_{ij}(\alpha) \|_{1 \leq i,j \leq 4},$$
where $I_{ij}(\alpha) = E_\alpha\left[\frac{\partial}{\partial \alpha_i} l_\alpha(X_1) \cdot \frac{\partial}{\partial \alpha_j} l_\alpha(X_1)\right] = -E_\alpha[l_\alpha^{ij}(X_1)]$ is a positive definite continuous function for all $\alpha \in \Omega$ such that $| I(\alpha) | = \det I(\alpha) > 0$.

Here and in what follows $E_\alpha[\cdot]$ stands for the expectation by the distribution $\mathbb{P}_\alpha$ of the random variable in brackets.

Theorem 2.1. *(see [3]). Let the RR-conditions be satisfied. Then with probability tending to one as n tends to infinity, there exists a solution $\widehat{\alpha}_n = \widehat{\alpha}(X^n)$ of the system of likelihood equations*

$$\frac{\partial L_\alpha(X^n)}{\partial \alpha_i} = 0, \; 1 \leq i \leq k, \tag{2.1}$$

where $L_\alpha(X^n) = \sum_{i=1}^n l_\alpha(X_i)$ is the logarithm of the likelihood function $f_\alpha(X^n) = \prod_{i=1}^n p_\alpha(X_i)$, possessing the following properties:

(i) *$\widehat{\alpha}_n$ is an asymptotically normal and asymptotically efficient estimator for α, that is,*

$$u_n = \sqrt{n}(\widehat{\alpha}_n - \alpha) \xrightarrow{d} u \sim N(0, I^{-1}(\alpha)),$$

where $u \sim N(0, I^{-1}(\alpha))$ is a k-dimensional normally distributed random variable with mean vector 0 and covariance matrix $I^{-1}(\alpha)$, and $\xrightarrow{d}$ means convergence in distribution.

(ii) *$\widehat{\alpha}_n$ is a consistent estimator for α, that is, $\widehat{\alpha}_n \xrightarrow{P_\alpha} \alpha$ as $n \longrightarrow \infty$, where $\xrightarrow{P_\alpha}$ means convergence in probability.*

(iii) *for all $k \geq 1$*

$$E_\alpha[u_n^k] \longrightarrow E_\alpha[u^k].$$

(iv) *$\widehat{\alpha}_n$ is an asymptotically unbiased estimator for α, that is,*

$$E_\alpha[\widehat{\alpha}_n] = \alpha + o\left(\frac{1}{\sqrt{n}}\right) \quad as \quad n \to \infty,$$

and also

$$E_\alpha\left[(\widehat{\alpha}_n - \alpha)^T(\widehat{\alpha}_n - \alpha)\right] = \frac{1}{n} \cdot I^{-1}(\alpha) + o\left(\frac{1}{n}\right) \quad as \quad n \to \infty,$$

where m^T stands for the transpose of a vector $m \in R^k$.

Remark 2.1. It follows from the results of [5] that the solution $\widehat{\alpha}_n$ of the system (2.1) coincides with the MLE of the parameter α (recall that the statistic $\widehat{\alpha}_n = \widehat{\alpha}(X^n)$ on which the function $L_\alpha(X^n)$ attains its (local) maximal value is called the MLE for parameter α).

3. Notation and Preliminaries

Let ξ be a discrete random variable with probability distribution given by (1.1), and let the parametric space Ω be defined as follows:

$$\Omega = \Big\{\alpha:\ 0 < \theta_0 \le \theta \le \Theta_0 < 1,\ 1 < c_0 \le c \le C_0 < \infty,\ 0 < b_0 \le b \le B_0 < \infty$$

$$(b_0 < 1), 3 < \rho_0 \le \rho \le R_0 < \infty\Big\}.$$

For $x \in \mathbb{N}$, $\alpha \in \Omega$, $j, k \in \mathbb{N}$ and $\gamma \in \mathbb{R}$, we denote (see [5]):

$$h_{\gamma,j}(x,\alpha) = \textstyle\sum_{m=0}^{x-1}(m+b)^{-\gamma}[(m+b)^{\rho}+c-1]^{-j};$$

$$l_{\gamma,j,k}(x,\alpha) = \textstyle\sum_{m=0}^{x-1}(m+b)^{\gamma}\ [(m+b)^{\rho}+c-1]^{-j}\cdot[\ln(m+b)]^{k};$$

$$H(x,\alpha) = (c-1)h_{1,1}(x,\alpha) + (x+b)^{-1};$$
$$\Lambda(x,\alpha) = (c-1)\ l_{0,1,1}(x,\alpha) + \ln(x+b).$$

According to [5], the first and second order partial derivatives of function $l_\alpha(x)$ with respect to α_i, where $\alpha_1 = \theta$, $\alpha_2 = c$, $\alpha_3 = b$, $\alpha_4 = \rho$, are finite if $\rho_0 > 3$, and can be represented as follows.

For the first order partial derivatives we have

$$\frac{\partial l_\alpha(x)}{\partial\theta} = -\frac{1}{\theta}E_\alpha[\xi] + \frac{x}{\theta}, \qquad \frac{\partial l_\alpha(x)}{\partial c} = -\Big\{E_\alpha[h_{0,1}(\xi,\alpha)] - h_{0,1}(x,\alpha)\Big\},$$

$$\frac{\partial l_\alpha(x)}{\partial b} = \rho\Big\{E_\alpha[H(\xi,\alpha)] - H(x,\alpha)\Big\}, \quad \frac{\partial l_\alpha(x)}{\partial\rho} = E_\alpha[\Lambda(\xi,\alpha)] - \Lambda(x,\alpha).$$

For the second order partial derivatives we have

$$l_\alpha^{11}(x) = \frac{\partial^2 l_\alpha(x)}{\partial\theta^2} = \frac{1}{\theta^2}\Big\{(E_\alpha[\xi]-x) - Var_\alpha[\xi]\Big\}, \quad l_\alpha^{12}(x) = \frac{\partial^2 l_\alpha(x)}{\partial\theta\partial c} = -\frac{1}{\theta}Cov_\alpha[\xi, h_{0,1}(\xi,\alpha)],$$

$$l_\alpha^{13}(x) = \frac{\partial^2 l_\alpha(x)}{\partial\theta\partial b} = \frac{\rho}{\theta}Cov_\alpha[\xi, H(\xi,\alpha)], \quad l_\alpha^{14}(x) = \frac{\partial^2 l_\alpha(x)}{\partial\theta\partial\rho} = \frac{1}{\theta}Cov_\alpha[\xi, \Lambda(\xi,\alpha)],$$

$$l_\alpha^{22}(x) = \frac{\partial^2 l_\alpha(x)}{\partial c^2} = \Big(E_\alpha[h_{0,2}(\xi,\alpha)] - h_{0,2}(x)\Big) - Var_\alpha[h_{0,1}(\xi,\alpha)],$$

$$l_\alpha^{23}(x) = \frac{\partial^2 l_\alpha(x)}{\partial c\partial b} = \rho\Big\{(E_\alpha[h_{1-\rho,2}(\xi)] - h_{1-\rho,2}(x)) + Cov_\alpha[h_{0,1}(\xi,\alpha), H(\xi,\alpha)]\Big\},$$

$$l_\alpha^{24}(x) = \frac{\partial^2 l_\alpha(x)}{\partial c\,\partial\rho} = \{E_\alpha[l_{\rho,2,1}(\xi,\alpha)] - l_{\rho,2,1}(x,\alpha)\} + Cov_\alpha[h_{0,1}(\xi,\alpha), \Lambda(\xi,\alpha)],$$

$$l_\alpha^{33}(x) = \tfrac{\partial^2 l_\alpha(x)}{\partial b^2} = \rho^2 Var_\alpha[H(\xi,\alpha)] - \rho\{E_\alpha[\tfrac{1}{\xi+b}]^2 - \tfrac{1}{x+b}\} - \rho(c-1)\{E_\alpha[h_{2,1}(\xi,\alpha)]$$

$$-h_{2,1}(x,\alpha)\} - \rho^2(c-1)\{E_\alpha[h_{2-\rho,2}(\xi,\alpha)] - h_{2-\rho,2}(x,\alpha)\},$$

$$l_\alpha^{34}(x) = \tfrac{\partial^2 l_\alpha(x)}{\partial b \partial \rho} = \{E_\alpha[H(\xi,\alpha)] - H(x,\alpha)\} + \rho(c-1)\{E_\alpha[l_{\rho-1,2,1}(\xi,\alpha)]$$

$$-l_{\rho-1,2,1}(x,\alpha)\} - \rho\, Cov_\alpha[H(\xi,\alpha),\Lambda(\xi,\alpha)],$$

$$l_\alpha^{44}(x) = \tfrac{\partial^2 l_\alpha(x)}{\partial \rho^2} = -Var_\alpha[\Lambda(\xi,\alpha)] - (c-1)\{E_\alpha[l_{\rho,2,2}(\xi,\alpha)] - l_{\rho,2,2}(x,\alpha)\}.$$

Lemma 3.1. *For all $x \subset \mathbb{N}$, $\alpha \subset \Omega$, $j,k \subset \mathbb{N}$ and $\gamma \subset \mathbb{R}$ the following inequalities hold:*

(i) $h_{\gamma,j}(x,\alpha) < \frac{x}{b_0^{R_0 j+\gamma}}$, *if* $\rho j + \gamma > 0$;

(ii) $l_{\gamma,j,k}(x,\alpha) < \frac{x}{b_0^{R_0 j-\gamma-k}}$, *if* $\rho j - \gamma - k > 0$;

(iii) $H(x,\alpha) < \frac{(C_0-1)\, x}{b_0^{R_0+1}} + \frac{1}{x+b_0}$;

(iv) $\Lambda(x,\alpha) < \frac{(C_0-1)\, x}{b_0^{R_0-1}} + x + B_0$.

Proof. We have

$$h_{\gamma,k}(x,\alpha) = \sum_{m=0}^{x-1} \frac{1}{(m+b)^\gamma} \cdot \frac{1}{[(m+b)^\rho + c - 1]^j} < \sum_{m=0}^{x-1} \frac{1}{(m+b)^{j\rho+\gamma}} < \frac{x}{b_0^{jR_0+\gamma}},$$

implying the inequality (i).

To prove (ii), observe that

$$l_{\gamma,j,k}(x,\alpha) = \sum_{m=0}^{x-1} (m+b)^\gamma \cdot \frac{[\ln(m+b)]^k}{[(m+b)^\rho + c - 1]^j} < \sum_{m=0}^{x-1} \frac{1}{(m+b)^{j\rho-\gamma-k}} < \frac{x}{b_0^{R_0 j-\gamma-k}},$$

and the result follows.

To prove the inequality (iii), we use the inequality (i), to obtain

$$H(x,\alpha) = (c-1)h_{1,1}(x,\alpha) + (x+b)^{-1} < (C_0-1)\cdot \frac{x}{b_0^{R_0+1}} + (x+b_0)^{-1}.$$

Finally, using the inequality (ii), we can write

$$\Lambda(x,\alpha) = (c-1)l_{0,1,1}(x,\alpha) + \ln(x+b) < (C_0-1)\frac{x}{b_0^{R_0-1}} + x + B_0,$$

implying the inequality (iv). Lemma 3.1 is proved.

4. The Main Results

Let, as above, ξ be a discrete random variable with probability distribution $\mathbb{P}_\alpha$ given by (1.1), and let $X^n = (X_1, ..., X_n)$ be a random sample from the distribution $\mathbb{P}_\alpha$. We first prove a lemma, which will be used in the proof of our main result.

Lemma 4.1. *For all $\alpha \in \Omega$ there exist positive numbers A_{ij} and B_{ij} $(i, j = 1, 2, 3, 4)$ such that the second order partial derivatives $l_\alpha^{ij}(x)$ satisfy the following inequalities:*

$$| l_\alpha^{ij}(x) | \leq A_{ij} + B_{ij} \cdot x, \quad i, j = 1, 2, 3, 4$$

for all $x \in \mathbb{N}$.

Proof. The following inequalities from [5] we use repeatedly:

$$E_\alpha[\xi^k] \leq \zeta_k(0) \exp\{(C_0 - 1)\zeta(0)\} \equiv S_k(0), \quad k = 1, 2,$$

where

$$\zeta_k(0) := \zeta_k(\rho_0, b_0) = \sum_{n=0}^{\infty} \frac{n^k}{(n + b_0)^{\rho_0}} < \infty,$$

$$\zeta(0) := \zeta_0(\rho_0, b_0) = \sum_{n=0}^{\infty} \frac{1}{(n + b_0)^{\rho_0}} < \frac{1}{b_0^{\rho_0}} + \sum_{n=1}^{\infty} \frac{1}{n^{\rho_0}} < \infty,$$

and $\mathbb{Z}(\rho_0) = \sum_{n=1}^{\infty} \frac{1}{n^{\rho_0}}$ is the Riemann's Zeta-Function.

Now, we estimate the partial derivatives $l_\alpha^{ij}(x)$ for $i, j = 1, ..., 4$. First, for $l_\alpha^{11}(x)$ we have

$$| l_\alpha^{11}(x) | \leq \frac{1}{\theta_0^2}\Big(E_\alpha[\xi] + E_\alpha[\xi]^2 + x\Big) \leq \frac{1}{\theta_0^2}\Big[S_1(0) + S_2(0)\Big] + \frac{1}{\theta_0^2} \cdot x \equiv A_{11} + B_{11} \cdot x.$$

To estimate $l_\alpha^{12}(x)$, observe first that

$$| l_\alpha^{12}(x) | \leq \frac{1}{\theta_0} E_\alpha[\xi \cdot h_{0,1}(\xi, \alpha)] + \frac{1}{\theta_0} E_\alpha[\xi] \cdot E_\alpha[h_{0,1}(\xi, \alpha)].$$

Taking into account that by Lemma 3.1 (i)

$$h_{0,1}(x, \alpha) < \frac{x}{b_0^{R_0}}, \tag{4.1}$$

we obtain

$$| l_\alpha^{12}(x) | < \frac{1}{\theta_0 \, b_0^{R_0}} \cdot E_\alpha[\xi^2] + \frac{1}{\theta_0 \, b_0^{R_0}} \cdot (E_\alpha[\xi])^2 \leq \frac{2}{\theta_0 \, b_0^{R_0}} S_2(0) \equiv A_{12}.$$

To estimate $l_\alpha^{13}(x)$, we use the inequality (see Lemma 3.1 (iii))

$$H(x, \alpha) < \frac{C_0 - 1}{b_0^{R_0+1}} \cdot x + \frac{1}{x + b_0}, \tag{4.2}$$

to obtain

$$| l_\alpha^{13}(x) | \le \tfrac{R_0}{\theta_0} \cdot E_\alpha[\xi \cdot H(\xi,\alpha)] + \tfrac{R_0}{\theta_0} \cdot E_\alpha[\xi] \cdot E_\alpha[H(\xi,\alpha)]$$

$$< \tfrac{R_0}{\theta_0} + \tfrac{R_0}{\theta_0} \cdot E_\alpha[\xi] + \tfrac{2R_0(C_0-1)}{\theta_0\ b_0^{R_0+1}} \cdot E_\alpha[\xi^2] \le \tfrac{R_0}{\theta_0}\Big(1 + S_1(0) + \tfrac{2(C_0-1)}{b_0^{R_0+1}} S_2(0)\Big) \equiv A_{13}.$$

Next, using now the inequality (see Lemma 3.1 (iv))

$$\Lambda(x,\alpha) < \frac{C_0-1}{b_0^{R_0-1}} \cdot x + x + B_0, \tag{4.3}$$

we get

$$| l_\alpha^{14}(x) | \le \tfrac{1}{\theta_0}\Big\{E_\alpha[\xi \cdot \Lambda(\xi,\alpha)] + E_\alpha[\xi] \cdot E_\alpha[\Lambda(\xi,\alpha)]\Big\}$$

$$< \tfrac{2}{\theta_0}\Big\{(\tfrac{C_0-1}{b_0^{R_0-1}} + 1)E_\alpha[\xi^2] + B_0 \cdot E_\alpha[\xi]\Big\}$$

$$\le \tfrac{2}{\theta_0}\Big\{B_0 \cdot S_1(0) + (\tfrac{C_0-1}{b_0^{R_0-1}} + 1)S_2(0)\Big\} \equiv A_{14}.$$

To estimate $l_\alpha^{22}(x)$, we use the inequality (see Lemma 3.1 (i))

$$h_{0,2}(x,\alpha) < \frac{x}{b_0^{2R_0}} \tag{4.4}$$

and the inequality (4.1) to obtain

$$| l_\alpha^{22}(x) | \le E_\alpha[h_{0,2}(\xi,\alpha)] + E_\alpha[h_{0,1}(\xi,\alpha)]^2 + h_{0,2}(x,\alpha)$$

$$< \tfrac{1}{b_0^{2R_0}}\Big(E_\alpha[\xi] + E_\alpha[\xi^2] + x\Big)$$

$$\le \tfrac{1}{b_0^{2R_0}}\Big(S_1(0) + S_2(0)\Big) + \tfrac{1}{b_0^{2R_0}} x \equiv A_{22} + B_{22}\ x.$$

Now we estimate $l_\alpha^{23}(x)$. We have

$$| l_\alpha^{23}(x) | \le R_0 \cdot \Big\{E_\alpha[h_{0,1}(\xi,\alpha) \cdot H(\xi,\alpha)] + E_\alpha[h_{0,1}(\xi,\alpha)] \cdot E_\alpha[H(\xi,\alpha)]$$

$$+ E_\alpha[h_{1-\rho,2}(\xi,\alpha)] + h_{1-\rho,2}(x,\alpha)\Big\}.$$

Hence, taking into account that by Lemma 3.1 (i)

$$h_{1-\rho,2}(x,\alpha) < \frac{x}{b_0^{1+R_0}}, \tag{4.5}$$

from (4.1), (4.2) and (4.5), we obtain

$$| l_\alpha^{23}(x) | < R_0 \cdot \Big\{E_\alpha\Big[\tfrac{\xi}{b_0^{R_0}} \cdot (\tfrac{C_0-1}{b_0^{R_0+1}}\xi + \tfrac{1}{\xi+b_0})\Big] + E_\alpha[\tfrac{\xi}{b_0^{R_0}}] \cdot E_\alpha\Big[\tfrac{C_0-1}{b_0^{R_0+1}}\xi + \tfrac{1}{\xi+b_0}\Big]$$

$$+ \tfrac{1}{b_0^{1+R_0}} \cdot E_\alpha[\xi] + \tfrac{x}{b_0^{1+R_0}}\Big\}$$

$$\le \tfrac{R_0}{b_0^{R_0}} \cdot \Big\{1 + (1 + \tfrac{1}{b_0}) \cdot S_1(0) + \tfrac{2(C_0-1)}{b_0^{R_0+1}} \cdot S_2(0)\Big\} + \tfrac{R_0}{b_0^{R_0+1}}\ x \equiv A_{23} + B_{23}\ x.$$

For $l_\alpha^{24}(x)$ we have

$$| l_\alpha^{24}(x) | \le E_\alpha[h_{0,1}(\xi,\alpha)\cdot\Lambda(\xi,\alpha)] + E_\alpha[h_{0,1}(\xi,\alpha)]\cdot E_\alpha[\Lambda(\xi,\alpha)] + E_\alpha[l_{\rho,2,1}(\xi,\alpha)] + l_{\rho,2,1}(x).$$

Hence, using (4.1), (4.3) and the following inequality (see Lemma 3.1 (i))

$$l_{\rho,2,1}(x,\alpha) < \frac{x}{b_0^{R_0-1}},$$

we get

$$| l_\alpha^{24}(x) | < \tfrac{1}{b_0^{R_0}}\cdot E_\alpha\Big[\xi\cdot(\tfrac{C_0-1}{b_0^{R_0-1}}\xi + \xi + B_0)\Big] + \tfrac{1}{b_0^{R_0}}\cdot E_\alpha[\xi]\cdot E_\alpha\Big[(1+\tfrac{C_0-1}{b_0^{R_0-1}})\xi + B_0\Big]$$

$$+\tfrac{1}{b_0^{R_0-1}}\cdot E_\alpha[\xi] + \tfrac{1}{b_0^{R_0-1}}\, x$$

$$\le \tfrac{1}{b_0^{R_0}}\cdot\Big\{(2B_0+b_0)\cdot S_1(0) + 2(1+\tfrac{C_0-1}{b_0^{R_0-1}})\cdot S_2(0)\Big\} + \tfrac{1}{b_0^{R_0-1}}\, x \equiv A_{24} + B_{24}\, x.$$

To estimate $l_\alpha^{33}(x)$, we use (4.2) and the inequalities (see Lemma 3.1 (i))

$$h_{2-\rho,2}(x,\alpha) < \frac{x}{b_0^{R_0+2}}, \qquad h_{2,1}(x,\alpha) < \frac{x}{b_0^{R_0+2}},$$

to obtain

$$| l_\alpha^{33}(x) | < R_0^2\cdot E_\alpha[H(\xi,\alpha)]^2 + 2R_0 + R_0^2(C_0-1)\Big(\tfrac{1}{b_0^{R_0+2}}\cdot E_\alpha[\xi] + \tfrac{x}{b_0^{R_0+2}}\Big)$$

$$+R_0(C_0-1)\Big(\tfrac{1}{b_0^{R_0+2}}E_\alpha[\xi] + \tfrac{x}{b_0^{R_0+2}}\Big) \le 2R_0(1+R_0)$$

$$+\tfrac{R_0(C_0-1)}{b_0^{R_0+2}}\cdot\Big\{(R_0+1)S_1(0) + \tfrac{2R_0(C_0-1)}{b_0^{R_0}}S_2(0)\Big\} + \tfrac{R_0(R_0+1)(C_0-1)}{b_0^{R_0+2}}\cdot x \equiv A_{33} + B_{33}\, x.$$

For $l_\alpha^{34}(x)$ we have

$$| l_\alpha^{34}(x) | \le R_0\Big\{E_\alpha[H(\xi,\alpha)\cdot\Lambda(\xi,\alpha)] + E_\alpha[H(\xi,\alpha)]\cdot E_\alpha[\Lambda(\xi,\alpha)]\Big\} + E_\alpha[H(\xi,\alpha)] + H(x,\alpha)$$

$$+R_0(C_0-1)\Big\{E_\alpha[l_{\rho-1,2,1}(\xi,\alpha)] + l_{\rho-1,2,1}(x,\alpha)\Big\}.$$

Hence, from (4.2), (4.3) and the inequality (see Lemma 3.1 (i))

$$l_{\rho-1,2,1}(x,\alpha) < \frac{x}{b_0^{R_0}},$$

we obtain

$$| l_\alpha^{34}(x) | < \frac{2R_0(C_0-1)}{b_0^{R_0}}\Big(\frac{C_0-1}{b_0^{R_0}}+\frac{1}{b_0}\Big)\cdot E_\alpha[\xi^2]+\Big(\frac{2B_0(C_0-1)R_0}{b_0^{1+R_0}}+\frac{R_0(C_0-1)}{b_0^{R_0-1}}+R_0+\frac{C_0-1}{b_0^{1+R_0}}$$

$$+\frac{R_0(C_0-1)}{b_0^{R_0}}\Big)E_\alpha[\xi]+R_0\Big(\frac{C_0-1}{b_0^{R_0-1}}+1+B_0\Big)+R_0B_0+2+\frac{C_0-1}{b_0^{R_0}}\Big(\frac{1}{b_0}+R_0\Big)\cdot x$$

$$\leq \frac{2R_0(C_0-1)}{b_0^{R_0}}\Big(\frac{C_0-1}{b_0^{R_0}}+\frac{1}{b_0}\Big)\cdot S_2(0)+R_0\Big\{\frac{(C_0-1)}{b_0^{R_0}}\Big[\frac{2B_0}{b_0}+b_0+1+\frac{1}{b_0^{R_0}}\Big]+1\Big\}\cdot S_1(0)$$

$$+R_0\Big(\frac{C_0-1}{b_0^{R_0-1}}+1+2B_0\Big)+2+\frac{C_0-1}{b_0^{R_0}}\Big(\frac{1}{b_0}+R_0\Big)\cdot x \equiv A_{34}+B_{34}\,x.$$

Finally, for $l_\alpha^{44}(x)$ we have

$$| l_\alpha^{44}(x) | \leq E_\alpha[\Lambda(\xi,\alpha)]^2+(C_0-1)\Big\{E_\alpha[l_{\rho,2,2}(\xi,\alpha)]+l_{\rho,2,2}(x,\alpha)\Big\}.$$

So, using (4.3) and the inequality (see Lemma 3.1 (i))

$$l_{\rho,2,2}(x,\alpha) < \frac{x}{b_0^{R_0-2}},$$

we obtain

$$| l_\alpha^{44}(x) | \leq 2\Big[\frac{(C_0-1)}{b_0^{2(R_0-1)}}\cdot E_\alpha[\xi^2]+E_\alpha[\xi+B_0]^2\Big]+\frac{(C_0-1)}{b_0^{R_0-2}}\cdot E_\alpha[\xi]+\frac{(C_0-1)}{b_0^{R_0-2}}\cdot x$$

$$=2\Big(\frac{(C_0-1)}{b_0^{2(R_0-1)}}+1\Big)\cdot S_2(0)+\Big(4B_0+\frac{(C_0-1)}{b_0^{R_0-2}}\Big)\cdot S_1(0)+2B_0^2+\frac{(C_0-1)}{b_0^{R_0-2}}\cdot x \equiv A_{44}+B_{44}\,x.$$

Thus Lemma 4.1 is proved.

The main result of this paper is the following statement.

Theorem 4.1. *The distribution $\mathbb{P}_\alpha$ given by (1.1) satisfies the RR–conditions.*

Proof. Observe first that the Conditions 1-3 are obviously satisfied. All the expressions on the right-hand side of the representations for the functions $l_\alpha^{ij}(x)$, introduced in Section 3, are finite and are continuous functions in $\alpha \in \Omega$ (see [5]). So, we have to verify only the Conditions 4 and 5.

Proof of Condition 4: We look for a function $M(x)$ to satisfy $| l_\alpha^{ij}(x) | \leq M(x)$ for all $\alpha \in \Omega$ and $x \in Supp\,\mathbb{P}_\alpha$. To this end, denote

$$L_0=\max\{A_{ij}\},\ 1\leq i\leq 4,\ i\leq j\leq 4,$$

$$N_0=\max\{B_{11},B_{ij}\},\ 2\leq i\leq 4,\ \ i\leq j\leq 4,\quad M(x)=L_0+N_0\,x.$$

Then using Lemma 4.1 we obtain

$$| l_\alpha^{ij}(x) | \leq M(x)\ \ \textit{for all}\ \alpha\in\Omega\ \textit{and}\ i,j=1,...,4.$$

Besides, we have $E_\alpha[M(\xi)] \leq L_0 + N_0 \cdot S_1(0)$ for all $\alpha \in \Omega$. Now we verify the condition

$$\lim_{N \longrightarrow \infty} \sup_{\alpha \in \Omega} E_\alpha \left[M(\xi) \cdot \mathbb{I}_{M(\xi) \geq N} \right] = 0. \tag{4.6}$$

Observe first that for arbitrary $N > 0$

$$E_\alpha \left[M(\xi) \cdot \mathbb{I}_{M(\xi) \geq N} \right] = L_0 + N_0 \cdot E_\alpha[\xi \cdot \mathbb{I}_{\xi \geq \tilde{N}}],$$

where $\tilde{N} = \frac{N - L_0}{N_0}$. On the other hand, we have

$$E_\alpha \left[\xi \cdot \mathbb{I}_{\xi \geq \tilde{N}} \right] = [g(\alpha)]^{-1} \sum_{n \geq \tilde{N}} \frac{\theta^n}{(n+b)^\rho} \prod_{m=0}^{n-1} \left(1 + \frac{c-1}{(m+b)^\rho} \right) \leq \sum_{n \geq \tilde{N}} \frac{1}{(n+b_0)^{R_0}} \leq \sum_{n \geq \tilde{N}} \frac{1}{n^{R_0}},$$

which implies (4.6), and thus completes the proof of Condition 4.

Proof of Condition 5: Using the representations for functions $l_\alpha^{ij}(x)$, given in the Section 3, we can write

$$I_{11}(\alpha) = \tfrac{1}{\theta^2} Var_\alpha(\xi), \quad I_{12}(\alpha) = I_{21}(\alpha) = \tfrac{1}{\theta} Cov_\alpha[\xi, h_{0,1}(\xi, \alpha)],$$

$$I_{13}(\alpha) = I_{31}(\alpha) = -\tfrac{\rho}{\theta} Cov_\alpha[\xi, H(\xi, \alpha)], \quad I_{14}(\alpha) = I_{41}(\alpha) = -\tfrac{1}{\theta} Cov_\alpha[\xi, \Lambda(\xi, \alpha)],$$

$$I_{22}(\alpha) = Var_\alpha[h_{0,1}(\xi, \alpha)], \quad I_{23}(\alpha) = I_{32}(\alpha) = -\rho \; Cov_\alpha[h_{0,1}(\xi, \alpha), H(\xi, \alpha)],$$

$$I_{24}(\alpha) = I_{42}(\alpha) = -Cov_\alpha[h_{0,1}(\xi, \alpha), \Lambda(\xi, \alpha)], \quad I_{44}(\alpha) = Var_\alpha[\Lambda(\xi, \alpha)]$$

$$I_{33}(\alpha) = -\rho^2 \; Var_\alpha[H(\xi, \alpha)], \quad I_{34}(\alpha) = I_{43}(\alpha) = \rho \; Cov_\alpha[H(\xi, \alpha), \Lambda(\xi, \alpha)].$$

Using the results from [5], we conclude that for sufficiently large n the matrix $I(\alpha)$ is positive definite, and so $\mid I(\alpha) \mid > 0$ for all $\alpha \in \Omega$. The continuity of $I(\alpha)$ on the set Ω is obvious. Thus, the Condition 5 is fulfilled. Theorem 4.1 is proved.

As an immediate consequence of Theorems 2.1 and 4.1 we obtain the following result.

Corollary 4.1. *Let $X^n = (X_1, ..., X_n)$ be a random sample from the generalized Pareto-type distribution $\mathbb{P}_\alpha$ given by (1.1) with unknown vector parameter $\alpha = (\theta, c, b, \rho)$, and let $\widehat{\alpha}_n = \widehat{\alpha}(X^n)$ be the MLE of α. Then for sufficiently large sample size n the statistic $\widehat{\alpha}_n$ is asymptotically unbiased, weak consistence, asymptotically normal and asymptotically efficient point estimator for α, and for any $k \geq 1$ the k^{th} moment of $\widehat{\alpha}_n$ converges to the k^{th} moment of the limiting normal distribution as $n \to \infty$.*

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