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ALMOST EVERYWHERE STRONG SUMMABILITY OF DOUBLE WALSH-FOURIER SERIES

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Abstract. [1]In this paper we study a question of almost everywhere strong convergence of the quadratic partial sums of two-dimensional Walsh-Fourier series. Specifically, we prove that the asymptotic relation $\frac{1}{n}\sum_{m=0}^{n-1}|S_{mm}f-f|^p\to 0$ as $n\to\infty$ holds a.e. for every function of two variables belonging to $L\log L$ and for $0<p\leq 2$. Then using a theorem by Getsadze [6] we infer that the space $L\log L$ can not be enlarged by preserving this strong summability property.
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1. Introduction

Let $\mathbb{P}$ denote the set of positive integers and let $\mathbb{N}:=\mathbb{P}\cup\{0\}$. Denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is, $\mathbb{Z}_2=\{0,1\}$, where the group operation is modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is defined such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of G are of the form $x=(x_0,x_1,...,x_k,...)$ with $x_k\in\{0,1\}$, $k\in\mathbb{N}$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology, respectively. The compact Abelian group G is called

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Walsh group. A base for the neighborhoods of G can be given in the following way:

$$I_0(x) = G, \ \ I_n(x) = I_n(x_0, ..., x_{n-1})$$
$$= \{y \in G : y = (x_0, ..., x_{n-1}, y_n, y_{n+1}, ...)\}, \quad x \in G, n \in \mathbb{N}.$$

These sets are called dyadic intervals. Let $0 := (0 : i \in \mathbb{N}) \in G$ denote the null element of G, $I_n := I_n(0)$, and $e_n := (0, ..., 0, x_n = 1, 0, ...) \in G$, $n \in \mathbb{N}$.

For $k \in \mathbb{N}$ and $x \in G$, by $r_k(x)$ we denote the k-th Rademacher function:

$$r_k(x) = (-1)^{x_k}, \quad x \in G, \ k \in \mathbb{N}.$$

If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ $(i \in \mathbb{N})$, that is, n is expressed in the number system of base 2. For $n > 0$ denote $|n| = \max\{j \in \mathbb{N}\colon n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

Throughout the paper the notation $a \lesssim b$ will stand for $a \leq c \cdot b$, where c is an absolute constant.

The Walsh-Paley system is defined to be the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k} \quad (x \in G, n \in \mathbb{P}),$$

and $w_0 := 1$. The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [13] and [33])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } \ x \in I_n, \\ 0, & \text{if } \ x \in \overline{I}_n. \end{cases} \tag{1.1}$$

We consider the double system $\{w_n(x) \times w_m(y) : n, m \in \mathbb{N}\}$ on $G \times G$.

The rectangular partial sums of the two-dimensional Walsh-Fourier series are defined as follows:

$$S_{M,N}(x, y, f) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i, j)\, w_i(x)\, w_j(y),$$

where the number

$$\widehat{f}(i, j) = \int_{G \times G} f(x, y)\, w_i(x)\, w_j(y)\, d\mu(x, y)$$

is called the (i, j)-th Walsh-Fourier coefficient of the function f.

Denote

$$S_n^{(1)}(x,y,f) := \sum_{l=0}^{n-1} \widehat{f}(l,y)\, w_l(x), \quad S_m^{(2)}(x,y,f) := \sum_{r=0}^{m-1} \widehat{f}(x,r)\, w_r(y),$$

where

$$\widehat{f}(l,y) = \int_G f(x,y)\, w_l(x)\, d\mu(x), \quad \widehat{f}(x,r) = \int_G f(x,y)\, w_r(y)\, d\mu(y).$$

The norm (or pre-norm) of the space $L_p(G \times G)$ is defined by

$$\|f\|_p := \left(\int_{G\times G} |f(x,y)|^p\, d\mu(x,y) \right)^{1/p} \quad (0 < p < +\infty).$$

We denote by $L \log L(G \times G)$ the class of measurable functions f satisfying

$$\int_{G\times G} |f| \log^+ |f| < \infty,$$

where $\log^+ u = \mathbb{I}_{(1,\infty)}(u) \log u$, and $\mathbb{I}_E$ is the characteristic function of the set E.

Denote by $S_n^T(x,f)$ the partial sums of the trigonometric Fourier series of f, and define the $(C,1)$ means by

$$\sigma_n^T(x,f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k^T(x,f).$$

Fejér [1] proved that $\sigma_n^T(x,f)$ converges to $f(x)$ uniformly for any 2π-periodic continuous function f. Lebesgue [17] established almost everywhere convergence of $(C,1)$ means for $f \in L_1(\mathbb{T}), \mathbb{T} := [-\pi, \pi)$. The strong summability problem, that is, the convergence of the strong means

$$(1.2) \qquad \frac{1}{n+1} \sum_{k=0}^{n} \left| S_k^T(x,f) - f(x) \right|^p, \quad x \in \mathbb{T}, \quad p > 0,$$

was first considered by Hardy and Littlewood in [14]. They showed that for any $f \in L_r(\mathbb{T})$ $(1 < r < \infty)$ the strong means tend to 0 a.e. as $n \to \infty$. The Fourier series of $f \in L_1(\mathbb{T})$ is said to be (H,p)-summable at $x \in T$, if the strong means, defined by (1.2), converge to 0 as $n \to \infty$. The (H,p)-summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [22] for $p = 2$, and later by Zygmund [42] for the general case $1 \le p < \infty$. In [24], Oskolkov proved the following result: if $f \in L_1(\mathbb{T})$ and Φ is a continuous positive convex function on $[0, +\infty)$ with $\Phi(0) = 0$ and

$$(1.3) \qquad \ln \Phi(t) = O(t / \ln \ln t) \quad (t \to \infty),$$

then for almost all x

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\Phi\left(\left|S_k^T(x,f)-f(x)\right|\right)=0. \tag{1.4}$$

It was noted in [24] that Totik announced a conjecture that (1.4) holds almost everywhere for any $f\in L_1(\mathbb{T})$, provided that

$$\ln\Phi(t)=O(t)\quad(t\to\infty). \tag{1.5}$$

The next result was proved by Rodin in [25].

Theorem A. *Let $f\in L_1(\mathbb{T})$. Then for any $A>0$*

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left(\exp\left(A\left|S_k^T(x,f)-f(x)\right|\right)-1\right)=0\quad\text{for a.e.}\quad x\in\mathbb{T}.$$

In [15] Karagulyan proved the following theorem.

Theorem B. *Let $\Phi:[0,\infty)\to[0,\infty)$ be a continuous increasing function satisfying the conditions: $\Phi(0)=0$ and*

$$\limsup_{t\to+\infty}\frac{\log\Phi(t)}{t}=\infty.$$

Then there exists a function $f\in L_1(\mathbb{T})$ for which the relation

$$\limsup_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\Phi\left(\left|S_k^T(x,f)\right|\right)=\infty$$

holds everywhere on $\mathbb{T}$.

For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [23] has proved that if $f\in L\log L\left(\mathbb{T}^2\right)$, $\mathbb{T}:=[-\pi,\pi)^2$, then

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left(S_{kk}^T(x,y,f)-f(x,y)\right)=0$$

for a. e. $(x,y)\in\mathbb{T}^2$. In [40] Zhizhiashvili improved this result by showing that the class $L\log L\left(\mathbb{T}^2\right)$ can be replaced by $L_1\left(\mathbb{T}^2\right)$.

From a result of Konyagin [16] it follows that for every $\varepsilon>0$ there exists a function $f\in L\log^{1-\varepsilon}\left(\mathbb{T}^2\right)$ such that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left|S_{kk}^T(x,y,f)-f(x,y)\right|\neq0\quad\text{for a. e. }(x,y)\in\mathbb{T}^2. \tag{1.6}$$

The aforementioned results show that in the one-dimensional case, the $(C,1)$ summability and the $(C,1)$ strong summability have the same maximal convergence spaces. Namely, in both cases we have the space L_1. But, the situation changes as we step further to the two-dimensional case. In other words, the spaces of functions with almost everywhere summable Marcinkiewicz and strong Marcinkiewicz means are different.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems. For instance, concerning the Walsh system we refer to Schipp [29, 30, 31], Fridli [2, 3], Leindler [17]-[21], Totik [34] - [36], Fridli and Schipp [3], Rodin [26], Weisz [38, 39], Gabisonia [4].

The problem of summability of cubic partial sums of multiple Fourier series have been studied by Gogoladze [10] - [12], Wang [37], Zhag [41], Glukhov [7], Goginava [8], Gát, Goginava, Tkebuchava [5], Goginava, Gogoladze [9].

For Walsh system Rodin [27] (see also Schipp [28]) proved the following result.

Theorem C. *Let $f \in L_1(G)$. Then for any $A > 0$*

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left(\exp\left(A\left|S_k\left(x,f\right)-f\left(x\right)\right|\right)-1\right)=0$$

for a. e. $x \in G$.

In [28], Schipp introduced the operator

$$V_nf\left(x\right):=\left(\frac{1}{2^n}\int_G\left(\sum_{j=0}^{n-1}2^{j-1}\mathbb{I}_{I_j}\left(t\right)S_{2^n}f\left(x+t+e_j\right)\right)^2d\mu\left(t\right)\right)^{1/2},$$

and proved the following theorem.

Theorem D. ([28]) *Let $f \in L_1(G)$, and let $Vf := \sup_n V_nf$. Then*

$$\mu\left\{|Vf|>\lambda\right\}\lesssim\frac{\|f\|_1}{\lambda}.$$

The exponential uniform strong approximation of the Marcinkiewicz means of two-dimensional Walsh-Fourier series has been studied in [9]. Recall that a function ψ is said to belong the class Ψ if it increases on $[0,+\infty)$ and satisfies the condition:

$$\lim_{u\to 0}\psi\left(u\right)=\psi\left(0\right)=0.$$

Theorem E. ([9]) a) *Let $\varphi \in \Psi$ be such that*

$$\overline{\lim_{u\to\infty}}\frac{\varphi\left(u\right)}{\sqrt{u}}<\infty.$$

Then for any function $f \in C(G \times G)$

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{l=1}^{n}\left(e^{\varphi(|S_{ll}(f)-f|)}-1\right)\right\|_C=0.$$

b) For any function $\varphi \in \Psi$ *satisfying the condition*

$$\overline{\lim_{u\to\infty}}\frac{\varphi(u)}{\sqrt{u}}=\infty$$

there exists a function $F \in C(G \times G)$ *such that*

$$\overline{\lim_{m\to\infty}}\frac{1}{m}\sum_{l=1}^{m}\left(e^{\varphi(|S_{ll}(0,0,F)-F(0,0)|)}-1\right)=+\infty.$$

For the two-dimensional Walsh-Fourier series Weisz [39] proved that if $f \in L_1(G \times G)$, then

$$\frac{1}{n}\sum_{j=0}^{n-1}\left(S_{j,j}(x,y;f)-f(x,y)\right)\to 0$$

for a. e. $(x,y) \in G \times G$.

In this paper we consider the strong means

$$H_n^p f := \left(\frac{1}{2^n}\sum_{m=0}^{2^n-1}|S_{mm}f|^p\right)^{1/p}$$

and the maximal strong operator

$$H_*^p f := \sup_{n\in\mathbb{N}} H_n^p f,$$

and study the a. e. convergence of strong Marcinkiewicz means of the two-dimensional Walsh-Fourier series. The following theorem is the main result of the present paper.

Theorem 1.1. *Let* $f \in L\log L(G \times G)$ *and* $0 < p \leq 2$. *Then*

$$\mu\{H_*^p f > \lambda\} \lesssim \frac{1}{\lambda}\left(1+\iint_{G\times G}|f|\log^+|f|\right).$$

The weak type $(L\log^+ L, 1)$ inequality and the usual density arguments of Marcinkiewicz and Zygmund imply the next result.

Theorem 1.2. *Let* $f \in L\log L(G \times G)$ *and* $0 < p \leq 2$. *Then*

$$\left(\frac{1}{n}\sum_{m=0}^{n-1}|S_{mm}(x,y,f)-f(x,y)|^p\right)^{1/p}\to 0 \quad \text{for a.e.} \quad (x,y)\in G\times G \quad \text{as} \quad n\to\infty.$$

It is worthwhile to note that from a theorem by Getsadze [6] it follows that the class $L\log L$ in the last theorem is necessary in the context of strong summability in question. In other words, it is impossible to give a larger convergence space (of the form $L\log L\phi(L)$ with $\phi(\infty)=0$) than the space $L\log L$. This emphasizes a sharp contrast between the one- and two-dimensional strong summability properties.

We also note that in the case of trigonometric system Sjölin [32] proved that for every $p>1$ and $f\in L_p(\mathbb{T}^2)$ the almost everywhere convergence $S_{nn}f\to f$ $(n\to\infty)$ holds. Since for Walsh system this issue is still an open problem, from this point of view Theorem 1.2 becomes more interesting.

2. Proof of Theorem 1.1

Let $f\in L_1(G\times G)$. The dyadic maximal function is given by

$$Mf(x,y):=\sup_{n\in\mathbb{N}}2^{2n}\int\limits_{I_n\times I_n}|f(x+s,y+t)|\,d\mu(s,t).$$

For an integrable function f of two variables, we need to introduce the following hybrid maximal functions:

$$M_1f(x,y):=\sup_{n\in\mathbb{N}}2^{n}\int\limits_{I_n}|f(x+s,y)|\,d\mu(s),$$

$$M_2f(x,y):=\sup_{n\in\mathbb{N}}2^{n}\int\limits_{I_n}|f(x,y+t)|\,d\mu(t),$$

$$V_1(x,y,f)=\sup_{n\in\mathbb{N}}\left(\frac{1}{2^n}\int\limits_G\left(\sum_{j=0}^{n-1}2^{j-1}\mathbb{I}_{I_j}(t)S_{2^n}^{(1)}f(x+t+e_j,y)\right)^2d\mu(t)\right)^{1/2}\tag{2.1}$$

$$V_2(x,y,f)=\sup_{n\in\mathbb{N}}\left(\frac{1}{2^n}\int\limits_G\left(\sum_{j=0}^{n-1}2^{j-1}\mathbb{I}_{I_j}(t)\,S_{2^n}^{(2)}f(x,y+t+e_j)\right)^2d\mu(t)\right)^{1/2}\tag{2.2}$$

It is well known that for $f\in L\log^+L$ the following estimate holds

$$\begin{aligned}&\lambda\mu\{(x,y)\in G\times G:Mf(x,y)>\lambda\}\\&\lesssim 1+\iint\limits_{G\times G}|f(x,y)|\log^+|f(x,y)|\,d\mu(x,y),\end{aligned}\tag{2.3}$$

and for $s = 1, 2$

$$\iint_{G\times G} |M_s f(x,y)|\, d\mu(x,y) \lesssim 1 + \iint_{G\times G} |f(x,y)| \log^+ |f(x,y)|\, d\mu(x,y). \tag{2.4}$$

Setting

$$\Omega = \{(x,y) \in G\times G : V_1 f(x,y) > \lambda\},$$

we can use Fubini's Theorem and Theorem D to write

$$\mu(\Omega) = \iint_{G\times G} \mathbb{I}_\Omega(x,y)\, d\mu(x,y) = \int_G \left(\int_G \mathbb{I}_\Omega(x,y)\, d\mu(x) \right) d\mu(y) \tag{2.5}$$

$$\lesssim \frac{1}{\lambda} \int_G \left(\int_G |f(x,y)|\, d\mu(x) \right) d\mu(y) \lesssim \frac{\|f\|_1}{\lambda}.$$

Similarly, we can show that

$$\mu\{(x,y) \in G\times G : V_2 f(x,y) > \lambda\} \lesssim \frac{\|f\|_1}{\lambda}. \tag{2.6}$$

For Dirichlet kernel Schipp proved the following representation (see [28, p. 622]):

$$\begin{aligned} D_m(x) &= \sum_{k=0}^{n-1} \mathbb{I}_{I_k\setminus I_{k+1}}(x) \sum_{j=0}^{k} \varepsilon_{kj} 2^{j-1} w_m(x+e_j) \\ &\quad -\frac{1}{2} w_m(x) + (m+1/2)\, \mathbb{I}_{I_n}(x), \end{aligned} \tag{2.7}$$

where $m < 2^n$ and

$$\varepsilon_{kj} = \begin{cases} -1, & \text{if } j = 0,1,\dots,k-1, \\ +1, & \text{if } j = k. \end{cases}$$

Proof of Theorem 1.1. First, we prove that the following estimation holds

$$\begin{aligned} &\left(\frac{1}{2^n} \sum_{m=0}^{2^n-1} |S_{mm}(x,y,f)|^2 \right)^{1/2} \\ &\lesssim V_2(x,y,M_1 f) + V_1(x,y,M_2 f) + Mf(x,y) + V_2(x,y,A) + V_1(x,y,A) + \|f\|_1, \end{aligned} \tag{2.8}$$

where A is an integrable on $G\times G$ function of two variables, which will be defined below.

It is easy to show that

$$\left(\sum_{m=0}^{2^n-1}|S_{mm}(x,y,f)|^2\right)^{1/2} = \left(\sum_{m=0}^{2^n-1}|S_{mm}(x,y,S_{2^n,2^n}f)|^2\right)^{1/2}$$

$$= \left(\sum_{m=0}^{2^n-1}\left|\iint\limits_{G\times G} S_{2^n,2^n}(x+s,y+t,f)\,D_m(s)\,D_m(t)\,d\mu(s,t)\right|^2\right)^{1/2}$$

$$\leq \sup_{\{\alpha_{mn}(x,y)\}}\left|\iint\limits_{G\times G} S_{2^n,2^n}(x+s,y+t,f)\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,D_m(s)\,D_m(t)\,d\mu(s,t)\right|.$$

The last inequality is obtained by taking the supremum over all $\{\alpha_{mn}(x,y)\}$ for which

$$\left(\sum_{m=0}^{2^n-1}|\alpha_{mn}(x,y)|^2\right)^{1/2} \leq 1.$$

In view of (2.7) we can write

$$\iint\limits_{G\times G} S_{2^n,2^n}(x+s,y+t,f)\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,D_m(s)\,D_m(t)\,d\mu(s,t) \tag{2.9}$$

$$=\iint\limits_{G\times G} S_{2^n,2^n}(x+s,y+t,f)\sum_{k_1=0}^{n-1}\sum_{k_2=0}^{n-1}\sum_{j_1=0}^{k_1}\sum_{j_2=0}^{k_2}\mathbb{I}_{I_{k_1}\setminus I_{k_1+1}}(s)$$

$$\times\mathbb{I}_{I_{k_2}\setminus I_{k_2+1}}(t)\,\varepsilon_{k_1j_1}\varepsilon_{k_2j_2}2^{j_1+j_2-2}$$

$$\times\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(s+t+e_{j_1}+e_{j_2})\,d\mu(s,t)$$

$$-\frac{1}{2}\iint\limits_{G\times G} S_{2^n,2^n}(x+s,y+t,f)\sum_{k_1=0}^{n-1}\sum_{j_1=0}^{k_1}\mathbb{I}_{I_{k_1}\setminus I_{k_1+1}}(s)$$

$$\times\varepsilon_{k_1j_1}2^{j_1-1}\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(s+t+e_{j_1})\,d\mu(s,t)$$

$$+\iint\limits_{G\times G} S_{2^n,2^n}(x+s,y+t,f)\sum_{k_1=0}^{n-1}\sum_{j_1=0}^{k_1}\mathbb{I}_{I_{k_1}\setminus I_{k_1+1}}(s)$$

$$\times\varepsilon_{k_1j_1}2^{j_1-1}\mathbb{I}_{I_n}(t)\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(s+e_{j_1})\,(m+1/2)\,d\mu(s,t)$$

$$-\frac{1}{2}\iint\limits_{G\times G} S_{2^n,2^n}f(x+s,y+t)\sum_{k_2=0}^{n-1}\sum_{j_2=0}^{k_2}\mathbb{I}_{I_{k_2}\setminus I_{k_2+1}}(t)$$

$$\times\varepsilon_{k_2j_2}2^{j_2-1}\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(s+t+e_{j_2})\,d\mu(s,t)$$

$$+\frac{1}{4}\iint_{G\times G}S_{2^n,2^n}(x+s,y+t,f)\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(s+t)\,d\mu(s,t)$$

$$-\frac{1}{2}\iint_{G\times G}S_{2^n,2^n}(x+s,y+t,f)\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(s)\left(m+\frac{1}{2}\right)\mathbb{I}_{I_n}(t)\,d\mu(s,t)$$

$$+\iint_{G\times G}S_{2^n,2^n}(x+s,y+t,f)\sum_{k_2=0}^{n-1}\sum_{j_2=0}^{k_2}\mathbb{I}_{I_{k_2}\setminus I_{k_2+1}}(t)$$

$$\times\varepsilon_{k_2j_2}2^{j_2-1}\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t+e_{j_2})\left(m+\frac{1}{2}\right)\mathbb{I}_{I_n}(s)\,d\mu(s,t)$$

$$-\frac{1}{2}\iint_{G\times G}S_{2^n,2^n}(x+s,y+t,f)\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t)\left(m+\frac{1}{2}\right)\mathbb{I}_{I_n}(s)\,d\mu(s,t)$$

$$+\iint_{G\times G}S_{2^n,2^n}(x+s,y+t,f)\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\left(m+\frac{1}{2}\right)^2\mathbb{I}_{I_n}(s)\,\mathbb{I}_{I_n}(t)\,d\mu(s,t)=\sum_{k=1}^{9}J_k.$$

It is easy to show that

$$|J_9|\lesssim\left(\sum_{m=0}^{2^n-1}|\alpha_{mn}(x,y)|^2\right)^{1/2}\times 2^{(5/2)n}\iint_{I_n\times I_n}|f(x+s,y+t)|\,d\mu(s,t)\lesssim 2^{n/2}Mf(x,y),\tag{2.10}$$

$$|J_5|\lesssim 2^{n/2}\left(\sum_{m=0}^{2^n-1}|\alpha_{mn}(x,y)|^2\right)^{1/2}\|f\|_1\lesssim 2^{n/2}\|f\|_1.\tag{2.11}$$

$$\begin{aligned}|J_8|&\lesssim\iint_{I_n\times G}S_{2^n,2^n}(x+s,y+t,|f|)\\&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t)\,(m+1/2)\right|d\mu(s,t)\\&=\iint_{I_n\times G}\left(2^n\int_{I_n}|f(x+s,y+t+v)|\,d\mu(v)\right)\\&\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t)\,(m+1/2)\right|d\mu(s,t)\end{aligned}\tag{2.12}$$

$$= \int_G \left(\int_{I_n} \left(2^n \int_{I_n} |f(x+s, y+t+v)| \, d\mu(s) \right) d\mu(v) \right)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t) \, (m+1/2) \right| d\mu(t) \lesssim \int_G \left(\int_{I_n} M_1 f(x, y+t+v) \, d\mu(v) \right)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t) \, (m+1/2) \right| d\mu(t) \lesssim 2^{-n} \int_G S_{2^n}^{(2)}(x, y+t, M_1 f)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t) \, (m+1/2) \right| d\mu(t)$$

$$\lesssim 2^{-n} \left(\int_G \left(S_{2^n}^{(2)}(x, y+t, M_1 f) \right)^2 d\mu(t) \right)^{1/2}$$

$$\times \left(\int_G \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t) \, (m+1/2) \right|^2 d\mu(t) \right)^{1/2}$$

$$\lesssim 2^{n/2} \left(\sum_{m=0}^{2^n-1} |\alpha_{mn}(x,y)|^2 \right)^{1/2} V_2(x, y, M_1 f) \lesssim 2^{n/2} V_2(x, y, M_1 f).$$

Similarly, we can prove that

$$|J_6| \lesssim 2^{n/2} V_1(x, y, M_2 f). \tag{2.13}$$

Now, we estimate J_7. Since

$$\int_{I_n} S_{2^n,2^n}(x+s, y+t, |f|) \, d\mu(s) = 2^{-n} S_{2^n,2^n}(x, y+t, |f|),$$

we can write

$$|J_7| \lesssim \sum_{j_2=0}^{n-1} \sum_{k_2=j_2}^{n-1} 2^{j_2-1} \iint_{I_n \times \left(I_{k_2} \setminus I_{k_2+1}\right)} S_{2^n,2^n}(x+s, y+t, |f|) \tag{2.14}$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t+e_{j_2}) \, (m+1/2) \right| d\mu(s,t)$$

$$\lesssim \sum_{j_2=0}^{n-1} 2^{j_2-1} \iint_{I_n \times I_{j_2}} S_{2^n,2^n}(x+s, y+t, |f|)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t+e_{j_2}) \, (m+1/2) \right| d\mu(s,t)$$

$$\lesssim \sum_{j_2=0}^{n-1} 2^{j_2-1} \int_{I_{j_2}} \left(\int_{I_n} S_{2^n,2^n} (x+s, y+t, |f|) \, d\mu(s) \right)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t+e_{j_2}) (m+1/2) \right| d\mu(t)$$

$$\lesssim 2^{-n} \sum_{j_2=0}^{n-1} 2^{j_2-1} \int_{I_{j_2}} S_{2^n,2^n} (x, y+t, |f|)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t+e_{j_2}) (m+1/2) \right| d\mu(t)$$

$$\lesssim 2^{-n} \sum_{j_2=0}^{n-1} 2^{j_2-1} \int_{I_{j_2}} S_{2^n,2^n} (x, y+t+e_{j_2}, |f|)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t) (m+1/2) \right| d\mu(t)$$

$$\lesssim 2^{-n} \int_G \sum_{j_2=0}^{n-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}}(t) \, S_{2^n,2^n} (x, y+t+e_{j_2}, |f|)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) \, w_m(t) (m+1/2) \right| d\mu(t)$$

$$\lesssim \left(\int_G \left(\sum_{j_2=0}^{n-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}}(t) \, S_{2^n,2^n} (x, y+t+e_{j_2}, |f|) \right)^2 d\mu(t) \right)^{1/2}.$$

Taking into account that

$$S_{2^n,2^n}(x, y+t+e_{j_2}, |f|) = 2^n \int_{I_n} \left(2^n \int_{I_n} |f(x+u, y+t+e_{j_2}+v)| \, d\mu(u) \right) d\mu(v)$$

$$\lesssim 2^n \int_{I_n} M_1 f(x, y+t+e_{j_2}+v) \, d\mu(v) = S_{2^n}^{(2)}(x, y+t+e_{j_2}, M_1 f),$$

we can use (2.14) to obtain

$$|J_7| \lesssim \left(\int_G \left(\sum_{j_2=0}^{n-1} 2^{j_2-1} \mathbb{I}_{I_{j_2}}(t) \, S_{2^n}^{(2)} (x, y+t+e_{j_2}, M_1 f) \right)^2 d\mu(t) \right)^{1/2} \tag{2.15}$$

$$\lesssim 2^{n/2} V_2(x, y, M_1 f).$$

Similarly, we can prove that

$$|J_3| \lesssim 2^{n/2} V_1(x, y, M_2 f). \tag{2.16}$$

For J_1 we have

$$\begin{aligned} J_1 \lesssim & \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} 2^{j_1+j_2-2} \\ & \times \iint_{(I_{k_1} \setminus I_{k_1+1}) \times (I_{k_2} \setminus I_{k_2+1})} S_{2^n,2^n}(x+s, y+t, |f|) \\ & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) w_m(s+t+e_{j_1}+e_{j_2}) \right| d\mu(s,t) \\ \lesssim & \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} 2^{j_1+j_2-2} \iint_{I_{j_1} \times I_{j_2}} S_{2^n,2^n}(x+s, y+t, |f|) \\ & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) w_m(s+t+e_{j_1}+e_{j_2}) \right| d\mu(s,t) \\ = & \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_1+j_2-2} \iint_{I_{j_1} \times I_{j_2}} S_{2^n,2^n}(x+s, y+t, |f|) \\ & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) w_m(s+t+e_{j_1}+e_{j_2}) \right| d\mu(s,t) \\ & + \sum_{j_1=0}^{n-1} \sum_{j_2=j_1+1}^{n-1} 2^{j_1+j_2-2} \iint_{I_{j_1} \times I_{j_2}} S_{2^n,2^n}(x+s, y+t, |f|) \\ & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) w_m(s+t+e_{j_1}+e_{j_2}) \right| d\mu(s,t) = J_{11} + J_{12}. \end{aligned} \tag{2.17}$$

It is easy to show that $s+t+e_{j_2} = (0, ..., 0, t_{j_2}+1, t_{j_2+1}, ..., t_{j_1-1}, t_{j_1}+s_{j_1}, ...) \in I_{j_2}$ for $s \in I_{j_1}, t \in I_{j_2}$ and $j_2 \leq j_1$. Hence, we can write

$$\begin{aligned} J_{11} \lesssim & \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_1+j_2-2} \iint_{I_{j_1} \times I_{j_2}} S_{2^n,2^n}(x+s, y+t+s+e_{j_2}, |f|) \\ & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) w_m(t+e_{j_1}) \right| d\mu(s,t) \\ \lesssim 2^{2n} & \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} 2^{j_1+j_2-2} \iint_{I_{j_1} \times I_{j_2}} \left(\iint_{I_n \times I_n} |f(x+s+u, y+t+s+e_{j_2}+v)| \, d\mu(u,v) \right) \end{aligned} \tag{2.18}$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t+e_{j_1})\right|d\mu(s,t)\lesssim 2^{2n}\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_2-2}$$

$$\times\int_{I_{j_2}}\left(\iint_{I_n\times I_n}\left(2^{j_1}\int_{I_{j_1}}|f(x+s+u,y+t+s+e_{j_2}+v)|\,d\mu(s)\right)\right)d\mu(u,v)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t+e_{j_1})\right|d\mu(t)\lesssim 2^{2n}\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_2-2}$$

$$\times\int_{I_{j_2}}\left(\iint_{I_n\times I_n}\left(2^{j_1}\int_{I_{j_1}}|f(x+s,y+t+s+e_{j_2}+u+v)|\,d\mu(s)\right)\right)d(u,v)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t+e_{j_1})\right|d\mu(t)\lesssim \sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_2-2}$$

$$\times\int_{I_{j_2}}\left(2^n\int_{I_n}\left(2^{j_1}\int_{I_{j_1}}|f(x+s,y+t+s+e_{j_2}+v)|\,d\mu(s)\right)\right)d(v)$$

$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t+e_{j_1})\right|d\mu(t).$$

We set

$$A_{j_1}(x,y)=2^{j_1}\int_{I_{j_1}}|f(x+s,y+s)|\,d\mu(s).$$

and observe that

$$A_{j_1}(x,y+x)=2^{j_1}\int_{I_{j_1}}|f(x+s,y+x+s)|\,d\mu(s)=2^{j_1}\int_{I_{j_1}}|F_2(x+s,y)|\,d\mu(s),$$

where $F_2(x,y)=f(x,y+x)$. It follows from the condition of the theorem that $F_2\in L\log L(G\times G)$. On the other hand, we have

$$\sup_j A_j(x,x+y)\lesssim M_1F_2(x,y).$$

Let $A(x,y)=\sup_j A_j(x,y)$. It is clear that

$$\iint_{G\times G}A(x,y)\,d\mu(x,y)=\iint_{G\times G}A(x,y+x)\,d\mu(x,y)\tag{2.19}$$

$$\lesssim\iint_{G\times G}M_1F_2(x,y)\,d\mu(x,y)$$

$$\lesssim 1+\iint\limits_{G\times G}|F_2(x,y)|\log^+|F_2(x,y)|\,d\mu(x,y)$$
$$\lesssim 1+\iint\limits_{G\times G}|f(x,y)|\log^+|f(x,y)|\,d\mu(x,y)\,.$$

Then, from (2.18) we have

$$|J_{11}|\lesssim\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_2-2}\int\limits_{I_{j_2}}\left(2^n\int\limits_{I_n}A(x,y+t+v+e_{j_2})\right)d(v) \tag{2.20}$$
$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t+e_{j_1})\right|d\mu(t)\lesssim\sum_{j_1=0}^{n-1}\sum_{j_2=0}^{j_1}2^{j_2-2}\int\limits_{I_{j_2}}S_{2^n}^{(2)}(x,y+t+e_{j_2},A)$$
$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t+e_{j_1})\right|d\mu(t)\lesssim\sum_{j_1=0}^{n-1}\int\limits_{G}\sum_{j_2=0}^{j_1}2^{j_2-2}\mathbb{I}_{I_{j_2}}(t)\,S_{2^n}^{(2)}(x,y+t+e_{j_2},A)$$
$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t+e_{j_1})\right|d\mu(t)$$
$$\lesssim\sum_{j_1=0}^{n-1}\left(\int\limits_{G}\left(\sum_{j_2=0}^{j_1}2^{j_2-2}\mathbb{I}_{I_{j_2}}(t)\,S_{2^n}^{(2)}(x,y+t+e_{j_2},A)\right)^2d\mu(t)\right)^{1/2}$$
$$\lesssim\sum_{j_1=0}^{n-1}2^{j_1/2}V_2(x,y,A)\lesssim 2^{n/2}V_2(x,y,A)\,,$$

where $A\in L_1(G\times G)$. Similarly, we can prove that

$$J_{12}\lesssim 2^{n/2}V_1(x,y,A)\,. \tag{2.21}$$

Combining (2.17), (2.20) and (2.21) we conclude that

$$|J_1|\lesssim 2^{n/2}V_1(x,y,A)+2^{n/2}V_2(x,y,A)\,. \tag{2.22}$$

Similarly, we can write

$$|J_2|\leq\frac{1}{2}\sum_{j_1=0}^{n-1}2^{j_1-1}\iint\limits_{I_{j_1}\times G}S_{2^n,2^n}(x+s,y+t,|f|) \tag{2.23}$$
$$\times\left|\sum_{m=0}^{2^n-1}\alpha_{mn}(x,y)\,w_m(t+s+e_{j_1})\right|d\mu(s,t)$$
$$=\frac{1}{2}\sum_{j_1=0}^{n-1}2^{j_1-1}\iint\limits_{I_{j_1}\times G}S_{2^n,2^n}(x+s,y+t,|f|)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) w_m(e_0) w_m(t+s+e_{j_1}+e_0) \right| d\mu(s,t)$$

$$\leq \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{n-1} 2^{j_1+j_2-2} \iint_{I_{j_1} \times I_{j_2}} S_{2^n,2^n}(x+s,y+t,|f|)$$

$$\times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x,y) w_m(e_0) w_m(t+s+e_{j_1}+e_{j_2}) \right| d\mu(s,t)$$

$$\lesssim 2^{n/2} V_1(x,y,A) + 2^{n/2} V_1(x,y,A),$$

$$|J_4| \lesssim 2^{n/2} V_1(x,y,A) + 2^{n/2} V_1(x,y,A).] \tag{2.24}$$

Combining (2.9), (2.10)-(2.16), and (2.22)-(2.24) we obtain the estimate (2.8). Taking into account the inequality $H_*^p f \leq H_*^2 f$ $(0 < p \leq 2)$ and the estimate

$$\mu\{Mf > \lambda\} \lesssim \frac{\|f\|_1}{\lambda},$$

from (2.3) – (2.6), (2.8), (2.19) and Theorem D we conclude that

$$\mu\{H_*^p f > \lambda\} \lesssim \frac{1}{\lambda}(\|M_1 f\|_1 + \|M_2 f\|_1 + \|A\|_1 + \|f\|_1) \lesssim \frac{1}{\lambda}\left(1 + \iint_{G\times G} |f| \log^+ |f|\right),$$

and the result follows. Theorem 1.1 is proved. □

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