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EXISTENCE AND NONEXISTENCE RESULTS FOR A $2n$-TH ORDER p-LAPLACIAN DISCRETE DIRICHLET BOUNDARY VALUE PROBLEM

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Abstract. [1] In this paper $2n$-th order p-Laplacian difference equations are considered. Using the critical point method, we establish various sufficient conditions for the existence and nonexistence of solutions for Dirichlet boundary value problem. Recent results in the literature are generalized and significantly complemented, as well as, some new results are obtained.

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1. Introduction

Throughout the paper the letters **N**, **Z** and **R** denote the sets of all natural, integer and real numbers, respectively. The letter k stands for a positive integer. For any a, $b \in \mathbf{Z}$ $(a < b)$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$ and $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$. Also, the symbol * denotes the transpose of a vector.

Recently, the difference equations have widely occurred as the mathematical models describing real life situations in many fields, such as: probability theory, matrix theory, electrical circuit analysis, combinatorial analysis, queuing theory, number theory, psychology and sociology, etc.

For the general background of difference equations, one can refer the monograph [1, 8]. Since the last decade, there has been much progress on the study of qualitative

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properties of difference equations, which includes results on stability, attractivity, oscillation and other topics (see, [6, 11, 12, 16], and reference therein).

In this paper we consider the following $2n$-th order p-Laplacian difference equation

$$\Delta^n (\gamma_{i-n+1}\varphi_p (\Delta^n u_{i-1})) = (-1)^n f(i, u_{i+1}, u_i, u_{i-1}),\ n \in \mathbf{Z}(1),\ i \in \mathbf{Z}(1,k), \tag{1.1}$$

with boundary value conditions:

$$u_{1-n} = u_{2-n} = \cdots = u_0 = 0,\ u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0, \tag{1.2}$$

where Δ is the forward difference operator: $\Delta u_i = u_{i+1} - u_i$, $\Delta^n u_i = \Delta^{n-1}(\Delta u_i)$, γ_i is nonzero and real-valued for each $i \in \mathbf{Z}(2-n, k+1)$, $\varphi_p(s)$ is the p-Laplacian operator: $\varphi_p(s) = |s|^{p-2}s(1 < p < \infty)$, and $f \in C(\mathbf{R}^4, \mathbf{R})$.

We may think of (1.1) as a discrete analogue of the following $2n$-th order p-Laplacian functional differential equation

$$\frac{d^n}{dt^n}\left[\gamma(t)\varphi_p\left(\frac{d^n u(t)}{dt^n}\right)\right] = (-1)^n f(t, u(t+1), u(t), u(t-1)),\ t \in [a,b], \tag{1.3}$$

with boundary value conditions:

$$u(a) = u'(a) = \cdots = u^{(n-1)}(a) = 0,\ u(b) = u'(b) = \cdots = u^{(n-1)}(b) = 0. \tag{1.4}$$

Note that equations of type (1.3) arise in the study of solitary waves [15], in lattice differential equations and periodic solutions [9], and in the study of functional differential equations.

In recent years, the boundary value problems for differential equations were in the focus of a number of researchers. By using various methods and techniques, such as the Schauder fixed point theory, the topological degree theory, the coincidence degree theory, a series of existence results of nontrivial solutions for differential equations have been obtained (see, [4, 9]). Another important and powerful tool that was used to deal with problems on differential equations is critical point theory (see [7, 13]). However, only since 2003, the critical point theory has been employed to establish sufficient conditions for the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [10] and Shi et al. [14] have found sufficient conditions for the existence of periodic solutions of second-order nonlinear difference equations. We also refer to [16] for the discrete boundary value problems.

Compared to the first- or second-order difference equations, the study of higher-order equations, and in particular, $2n$-th order equations, has received considerably less attention (see, [2, 3, 5] and references therein).

The authors [2] studied the following $2n$-th order difference equation:

$$\sum_{j=0}^{n} \Delta^j \left(\gamma_j(i-j)\Delta^j u(i-j)\right) = 0 \tag{1.5}$$

in the context of the discrete calculus of variations, and Peil and Peterson [12] studied the asymptotic behavior of solutions of (1.5) with $\gamma_j(i) \equiv 0$ for $1 \leq j \leq n-1$. In 1998, Anderson [3] considered (1.5) for $i \in \mathbf{Z}(a)$, and obtained a formulation of generalized zeros and (n, n)-disconjugacy for (1.5). In 2004, Migda [11] studied an m-th order linear difference equation. In 2007, Cai and Yu [5] have obtained some criteria for the existence of periodic solutions of the following $2n$-th order difference equation:

$$\Delta^n \left(\gamma_{i-n}\Delta^n u_{i-n}\right) + f(i, u_i) = 0,\ n \in \mathbf{Z}(3),\ i \in \mathbf{Z}, \tag{1.6}$$

in the case where f grows superlinearly both at 0 and at ∞. In 2007, Chen and Fang [6], using the critical point theory, have obtained a sufficient condition for the existence of periodic and subharmonic solutions of the following p-Laplacian difference equation:

$$\Delta\left(\varphi_p\left(\Delta u_{i-1}\right)\right) + f(i, u_{i+1}, u_i, u_{i-1}) = 0,\ i \in \mathbf{Z}. \tag{1.7}$$

The study of boundary value problems (BVP) to determine the existence of solutions of difference equations has been a very active research area in the last twenty years. For the surveys of recent results in this area, we refer the reader to the monographs [1, 8]. However, to the best of our knowledge, results on solutions to boundary value problems of higher-order nonlinear difference equations are very scarce in the literature. Furthermore, since the equation (1.1) contains both advance and retardation, not surprisingly, there are only few papers dealing with this subject.

Motivated by the above results, we use the critical point theory to give some sufficient conditions for the existence and nonexistence of solutions for the BVP (1.1), (1.2). We study both the superlinear and sublinear cases. The main idea used in this paper is to transfer the existence of solutions of the BVP (1.1), (1.2) into the existence of the critical points of some functional. The proofs are based on the celebrated Mountain Pass Lemma in combination with variational technique. The purpose of this paper is two-folded. On one hand, we further demonstrate the powerfulness of critical point theory in the study of solutions for boundary value problems of difference equations. On the other hand, we complement the existing results. The motivation for the present work stems from the recent papers [7, 9].

For the basic concepts of variational methods, we refer the reader to monographs [8, 13]. Let

$$\bar{\gamma} = \max\{\gamma_i : i \in \mathbf{Z}(2-n, k+1)\}, \quad \underline{\gamma} = \min\{\gamma_i : i \in \mathbf{Z}(2-n, k+1)\}.$$

Our main results are as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*

(γ) $\gamma_i < 0$ *for any* $i \in Z(2-n, k+1)$;

(F_1) *there exists a functional* $F(i, \cdot) \in C^1(Z \times R^2, R)$ *with* $F(0, \cdot) = 0$, *such that*

$$\frac{\partial F(i-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(i, v_1, v_2)}{\partial v_2} = f(i, v_1, v_2, v_3), \ \forall i \in Z(1, k);$$

(F_2) *there exists a constant* $M_0 > 0$, *such that for all* $(i, v_1, v_2) \in Z(1, k) \times R^2$

$$\left|\frac{\partial F(i, v_1, v_2)}{\partial v_1}\right| \le M_0, \ \left|\frac{\partial F(i, v_1, v_2)}{\partial v_2}\right| \le M_0.$$

Then the BVP (1.1), (1.2) possesses at least one solution.

Remark 1.1. Assumption (F_2) implies that there exists a constant $M_1 > 0$, such that

(F_2') $|F(i, v_1, v_2)| \le M_1 + M_0(|v_1| + |v_2|), \ \forall (i, v_1, v_2) \in \mathbf{Z}(1, k) \times \mathbf{R}^2$.

Theorem 1.2. *Suppose that* (F_1) *and the following hypotheses are satisfied:*

(γ') $\gamma_i > 0$ *for any* $i \in Z(2-n, k+1)$;

(F_3) *there exists a functional* $F(i, \cdot) \in C^1(Z \times R^2, R)$, *such that*

$$\lim_{r \to 0} \frac{F(i, v_1, v_2)}{r^p} = 0, \ r = \sqrt{v_1^2 + v_2^2}, \ \forall i \in Z(1, k);$$

(F_4) *there exists a constant* $\beta > p$, *such that for any* $i \in Z(1, k)$

$$0 < \frac{\partial F(i, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(i, v_1, v_2)}{\partial v_2} v_2 < \beta F(i, v_1, v_2), \ \forall (v_1, v_2) \neq 0.$$

Then the BVP (1.1), (1.2) possesses at least two nontrivial solutions.

Remark 1.2. Assumption (F_4) implies that there exist constants $a_1 > 0$ and $a_2 > 0$, such that

(F_4') $F(i, v_1, v_2) > a_1 \left(\sqrt{v_1^2 + v_2^2}\right)^{\beta} - a_2, \ \forall i \in \mathbf{Z}(1, k)$.

Theorem 1.3. *Suppose that* (γ'), (F_1) *and the following assumption are satisfied:*

(F_5) *there exist constants* $R > 0$ *and* $1 < \alpha < 2$, *such that for* $i \in Z(1, k)$ *and* $\sqrt{v_1^2 + v_2^2} \ge R$,

$$0 < \frac{\partial F(i, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(i, v_1, v_2)}{\partial v_2} v_2 \le \frac{\alpha}{2} p F(i, v_1, v_2).$$

Then the BVP (1.1), (1.2) possesses at least one solution.

Remark 1.3. Assumption (F_5) implies that for each $i \in \mathbf{Z}(1,k)$ there exist constants $a_3 > 0$ and $a_4 > 0$, such that

(F_5') $F(i, v_1, v_2) \le a_3 \left(v_1^2 + v_2^2\right)^{\frac{q}{2}p} + a_4,\ \forall (i, v_1, v_2) \in \mathbf{Z}(1,k) \times \mathbf{R}^2.$

Theorem 1.4. *Suppose that* (γ), (F_1) *and the following assumption are satisfied:*

(F_6) $v_2 f(i, v_1, v_2, v_3) > 0$, *for* $v_2 \neq 0$, $\forall i \in Z(1,k)$.

Then the BVP (1.1), (1.2) has no nontrivial solutions.

Remark 1.4. As it was mentioned above, results on the nonexistence of solutions of problem (1.1), (1.2) are very scarce. Hence, Theorem 1.4 complements the existing results.

The rest of the paper is organized as follows. In Section 2 we establish the variational framework for the BVP (1.1), (1.2), and transfer the problem of existence of solutions of BVP (1.1), (1.2) into that of the existence of critical points of the corresponding functional. Some related fundamental results are also recalled. Finally, in Section 3 we prove our main results, by using the critical point method.

2. Variational structure and some lemmas

In order to apply the critical point theory, we first establish the corresponding variational framework for the BVP (1.1) with (1.2), and state some lemmas, which are used in the proofs of our main results. We start with some basic notation.

Let $\mathbf{R}^k$ be the real Euclidean space of dimension k. Define the inner product on $\mathbf{R}^k$ as follows:

$$\langle u, v\rangle = \sum_{j=1}^{k} u_j v_j,\quad u, v \in \mathbf{R}^k, \tag{2.1}$$

which induced the norm $\|\cdot\|$:

$$\|u\| = \left(\sum_{j=1}^{k} u_j^2\right)^{\frac{1}{2}},\quad u \in \mathbf{R}^k. \tag{2.2}$$

On the other hand, for all $u \in \mathbf{R}^k$ and $s > 1$, we define the norm $\|u\|_s$ on $\mathbf{R}^k$ as follows:

$$\|u\|_s = \left(\sum_{j=1}^{k} |u_j|^s\right)^{\frac{1}{s}}. \tag{2.3}$$

Since the norms $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants c_1, c_2 ($c_2 \geq c_1 > 0$), such that

$$c_1\|u\|_2 \leq \|u\|_s \leq c_2\|u\|_2, \ \ u \in \mathbf{R}^k. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For the BVP (1.1), (1.2), consider the functional J defined on $\mathbf{R}^k$ as follows:

$$J(u) = \frac{1}{p}\sum_{i=1-n}^{k} \gamma_{i+1}\,|\Delta^n u_i|^p - \sum_{i=1}^{k} F(i, u_{i+1}, u_i), \ u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k, \tag{2.5}$$

where $u_{1-n} = u_{2-n} = \cdots = u_0 = 0, \ u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0$ and

$$\frac{\partial F(i-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(i, v_1, v_2)}{\partial v_2} = f(i, v_1, v_2, v_3).$$

It is easy to see that $J \in C^1(\mathbf{R}^k, \mathbf{R})$, and for any $u = \{u_i\}_{i=1}^k = (u_1, u_2, \ldots, u_k)^*$, by using $u_{1-n} = u_{2-n} = \cdots = u_0 = 0, \ u_{k+1} = u_{k+2} = \cdots = u_{k+n} = 0$, and

$$\Delta^n u_i = \sum_{j=0}^{n} (-1)^j \binom{n}{j} u_{i+n-j},$$

we can compute the partial derivatives of J by formula:

$$\frac{\partial J}{\partial u_i} = (-1)^n \Delta^n \left(\gamma_{i-n+1}\varphi_p\left(\Delta^n u_{i-1}\right)\right) - f(i, u_{i+1}, u_i, u_{i-1}), \ \forall i \in \mathbf{Z}(1, k).$$

Thus, u is a critical point of J on $\mathbf{R}^k$ if and only if

$$\Delta^n \left(\gamma_{i-n+1}\varphi_p\left(\Delta^n u_{i-1}\right)\right) = (-1)^n f(i, u_{i+1}, u_i, u_{i-1}), \ \ i \in \mathbf{Z}(1, k),$$

and so, we can reduce the existence of solutions of the BVP (1.1), (1.2) to the existence of critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (1.1), (1.2).

Let D be the $(k+n) \times (k+n)$ matrix defined by

$$D = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Clearly, D is positive definite. Let $\lambda_{1-n}, \lambda_{2-n}, \cdots, \lambda_k$ be the eigenvalues of D. Applying matrix theory, we see that $\lambda_j > 0, \ j = 1-n, 2-n, \cdots, k$, and, without loss of generality, we can assume that

$$0 < \lambda_{1-n} \leq \lambda_{2-n} \leq \cdots \leq \lambda_k. \tag{2.6}$$

Let E be a real Banach space. We assume that $J \in C^1(E, \mathbf{R})$, that is, J is a continuously Fréchet-differentiable functional defined on E. The functional J is said to satisfy the Palais-Smale condition, (PS)-condition, for short, if any sequence $\{u^{(l)}\} \subset E$ for which $\{J(u^{(l)})\}$ is bounded and $J'(u^{(l)}) \to 0$ as $l \to \infty$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E of radius ρ centered at 0, and let ∂B_ρ denote its boundary.

Lemma 2.1 *(Mountain Pass Lemma [13]). Let E be a real Banach space and let $J \in C^1(E, R)$ satisfy the (PS)-condition. If $J(0) = 0$ and*
(J_1) there exist constants ρ, $a > 0$ such that $J|_{\partial B_\rho} \geq a$,
(J_2) there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.
Then J possesses a critical value $c \geq a$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} J(g(s)), \tag{2.7}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}. \tag{2.8}$$

Lemma 2.2. *Suppose that the conditions (γ'), (F_1), (F_3) and (F_4) are satisfied. Then the functional J satisfies the (PS)-condition.*

Proof. Let $u^{(l)} \in \mathbf{R}^k$ and $l \in \mathbf{Z}(1)$ be such that $\{J(u^{(l)})\}$ is bounded. Then there exists a positive constant M_2, such that

$$-M_2 \leq J\left(u^{(l)}\right) \leq M_2,\ \forall l \in \mathbf{N}.$$

By (F_4'), we can write

$$\begin{aligned} -M_2 \leq J\left(u^{(l)}\right) &= \frac{1}{p} \sum_{i=1-n}^{k} \gamma_{i+1} \left|\Delta^n u_i^{(l)}\right|^p - \sum_{i=1}^{k} F\left(i, u_{i+1}^{(l)}, u_i^{(l)}\right) \\ &\leq \frac{\bar{\gamma}}{p} c_2^p \left[\sum_{i=1-n}^{k} \left(\Delta^{n-1} u_{i+1}^{(l)} - \Delta^{n-1} u_i^{(l)}\right)^2\right]^{\frac{p}{2}} - a_1 \sum_{i=1}^{k} \left[\sqrt{\left(u_{i+1}^{(l)}\right)^2 + \left(u_i^{(l)}\right)^2}\right]^{\beta} + a_2 k \\ &\leq \frac{\bar{\gamma}}{p} c_2^p \left[\left(x^{(l)}\right)^* D x^{(l)}\right]^{\frac{p}{2}} - a_1 c_1^{\beta} \left\|u^{(l)}\right\|^{\beta} + a_2 k \\ &\leq \frac{\bar{\gamma}}{p} c_2^p \lambda_k^{\frac{p}{2}} \left\|x^{(l)}\right\|^p - a_1 c_1^{\beta} \left\|u^{(l)}\right\|^{\beta} + a_2 k, \end{aligned}$$

where $x^{(l)} = \left(\Delta^{n-1} u_{1-n}^{(l)}, \Delta^{n-1} u_{2-n}^{(l)}, \cdots, \Delta^{n-1} u_k^{(l)}\right)^*$. Taking into account that

$$\left\|x^{(l)}\right\|^p = \left[\sum_{i=1-n}^{k} \left(\Delta^{n-2} u_{i+1}^{(l)} - \Delta^{n-2} u_i^{(l)}\right)^2\right]^{\frac{p}{2}} \leq \left[\lambda_k \sum_{i=1-n}^{k} \left(\Delta^{n-2} u_i^{(l)}\right)^2\right]^{\frac{p}{2}} \leq \lambda_k^{\frac{(n-1)p}{2}} \left\|u^{(l)}\right\|^p,$$

we obtain

$$J\left(u^{(l)}\right) \leq \frac{\bar{\gamma}}{p} c_2^p \lambda_k^{\frac{np}{2}} \left\|u^{(l)}\right\|^p - a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta + a_2 k.$$

That is,

$$a_1 c_1^\beta \left\|u^{(l)}\right\|^\beta - \frac{\bar{\gamma}}{p} c_2^p \lambda_k^{\frac{np}{2}} \left\|u^{(l)}\right\|^p \leq M_2 + a_2 k.$$

Since $\beta > p$, there exists a constant $M_3 > 0$ to satisfy $\|u^{(l)}\| \leq M_3$, $l \in \mathbf{N}$. Therefore, $\{u^{(l)}\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\{u^{(l)}\}$ possesses a convergence subsequence in $\mathbf{R}^k$, implying the (PS)-condition. Lemma 2.2 is proved. □

3. Proofs of the main results

In this Section, we prove our main results by using the critical point theory.

Proof of Theorem 1.1. By (F_2'), for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned}
J(u) &= \frac{1}{p} \sum_{i=1-n}^{k} \gamma_{i+1} |\Delta^n u_i|^p - \sum_{i=1}^{k} F(i, u_{i+1}, u_i) \\
&\leq \frac{\bar{\gamma}}{p} c_1^p \left[\sum_{i=1-n}^{k} \left(\Delta^{n-1} u_{i+1} - \Delta^{n-1} u_i\right)^2\right]^{\frac{p}{2}} + M_0 \sum_{i=1}^{k} (|u_{i+1}| + |u_i|) + M_1 k \\
&\leq \frac{\bar{\gamma}}{p} c_1^p (x^* D x)^{\frac{p}{2}} + 2M_0 \sum_{i=1}^{k} |u_i| + M_1 k \leq \frac{\bar{\gamma}}{p} c_1^p \lambda_{1-n}^{\frac{p}{2}} \|x\|^p + 2M_0 \|u\| + M_1 k,
\end{aligned}$$

where $x = \left(\Delta^{n-1} u_{1-n}, \Delta^{n-1} u_{2-n}, \cdots, \Delta^{n-1} u_k\right)^*$. Since

$$\|x\|^p = \left[\sum_{i=1-n}^{k} \left(\Delta^{n-2} u_{i+1} - \Delta^{n-2} u_i\right)^2\right]^{\frac{p}{2}} \geq \left[\lambda_{1-n} \sum_{i=1-n}^{k} \left(\Delta^{n-2} u_i\right)^2\right]^{\frac{p}{2}} \geq \lambda_{1-n}^{\frac{(n-1)p}{2}} \|u\|^p,$$

we have

$$J(u) \leq \tfrac{\bar{\gamma}}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p + 2M_0 \sqrt{k} \|u\| + M_1 k \to -\infty \text{ as } \|u\| \to +\infty.$$

The above inequality means that $-J(u)$ is coercive. By the continuity of $J(u)$, J attains its maximum at some point, which we denote by $\check{u}$, that is,

$$J(\check{u}) = \max\left\{J(u) | u \in \mathbf{R}^k\right\}.$$

Clearly, $\check{u}$ is a critical point of the functional J, and the result follows. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By (F_3), for any $\epsilon = \frac{\underline{\gamma}}{2p} c_1^p \lambda_{1-n}^{\frac{np}{2}}$, where λ_{1-n} is as in (2.6), there exists $\rho > 0$, such that

$$|F(i, v_1, v_2)| \leq \frac{\underline{\gamma}}{2p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \left(v_1^2 + v_2^2\right)^{\frac{p}{2}}, \forall i \in \mathbf{Z}(1, k),$$

for $\sqrt{v_1^2+v_2^2} \le \sqrt{2}\rho$.

For any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$ and $\|u\| \le \rho$, we have $|u_i| \le \rho,\ i \in \mathbf{Z}(1,k)$.

It follows from the proof of the Theorem 1.1 that for any $u \in \mathbf{R}^k$,

$$J(u) = \frac{1}{p}\sum_{i=1-n}^{k} \gamma_{i+1}\,|\Delta^n u_i|^p - \sum_{i=1}^{k} F(i, u_{i+1}, u_i)$$

$$\ge \frac{\gamma}{p}c_1^p\lambda_{1-n}^{\frac{np}{2}}\|u\|^p - \frac{\gamma}{2p}c_1^p\lambda_{1-n}^{\frac{np}{2}}\sum_{i=1}^{k}\left(u_{i+1}^2+u_i^2\right)^{\frac{p}{2}}$$

$$\ge \frac{\gamma}{p}c_1^p\lambda_{1-n}^{\frac{np}{2}}\|u\|^p - \frac{\gamma}{2p}c_1^p\lambda_{1-n}^{\frac{np}{2}}\|u\|^p = \frac{\gamma}{2p}c_1^p\lambda_{1-n}^{\frac{np}{2}}\|u\|^p.$$

Taking $a = \frac{\gamma}{2p}c_1^p\lambda_{1-n}^{\frac{np}{2}}\rho^p > 0$, we obtain

$$J(u) \ge a > 0,\ \forall u \in \partial B_\rho.$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho} \ge a$, implying that J satisfies the condition (J_1) of Lemma 2.1.

For our setting, clearly $J(0) = 0$. In order to exploit the Mountain Pass Lemma in critical point theory, we need to verify other conditions of this lemma. By Lemma 2.2, J satisfies the (PS)-condition. So; it remains to verify the condition (J_2).

It follows from the proof of Lemma 2.2 that

$$J(u) \le \frac{\bar{\gamma}}{p}c_2^p\lambda_k^{\frac{np}{2}}\|u\|^p - a_1c_1^\beta\|u\|^\beta + a_2k.$$

Since $\beta > p$, we can choose $\bar{u}$ large enough to ensure that $J(\bar{u}) < 0$.

Now we can apply the Mountain Pass Lemma to conclude that the functional J possesses a critical value $c \ge a > 0$, where

$$c = \inf_{h\in\Gamma}\sup_{s\in[0,1]} J(h(s)),$$

and

$$\Gamma = \{h \in C([0,1], \mathbf{R}^k) \mid h(0) = 0,\ h(1) = \bar{u}\}.$$

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated with the critical value c of J, that is, $J(\tilde{u}) = c$. Similar to the proof of Lemma 2.2 ((PS)- condition), we can conclude that $J(u)$ is bounded on $\mathbf{R}^k$. As a consequence, there exists $\hat{u} \in \mathbf{R}^k$ such that

$$J(\hat{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s)).$$

Clearly, $\hat{u} \ne 0$. If $\tilde{u} \ne \hat{u}$, then the conclusion of Theorem 1.2 holds. Otherwise, $\tilde{u} = \hat{u}$, and we have $c = J(\tilde{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s))$. That is,

$$\sup_{u\in\mathbf{R}^k} J(u) = \inf_{h\in\Gamma}\sup_{s\in[0,1]} J(h(s)).$$

Therefore,

$$c_{\max} = \max_{s\in[0,1]} J(h(s)),\ \forall h \in \Gamma.$$

By the continuity of $J(h(s))$ with respect to s, $J(0) = 0$ and $J(\bar{u}) < 0$ imply that there exists $s_0 \in (0,1)$, such that $J(h(s_0)) = c_{\max}$. Choose $h_1,\ h_2 \in \Gamma$ such that $\{h_1(s) \mid s \in (0,1)\} \cap \{h_2(s) \mid s \in (0,1)\}$ is empty, then there exists $s_1,\ s_2 \in (0,1)$ to satisfy $J(h_1(s_1)) = J(h_2(s_2)) = c_{\max}$. Thus, we get two different critical points of J on $\mathbf{R}^k$ denoted by

$$u^1 = h_1(s_1),\ u^2 = h_2(s_2).$$

The above arguments can be applied to conclude that the BVP (1.1), (1.2) possesses at least two nontrivial solutions. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. In view of above arguments, we only need to find at least one critical point of the functional J defined by (2.5).

By (F_5'), for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we can write

$$\begin{aligned} J(u) &= \frac{1}{p}\sum_{i=1-n}^{k} \gamma_{i+1}\,|\Delta^n u_i|^p - \sum_{i=1}^{k} F(i, u_{i+1}, u_i) \\ &\geq \frac{\gamma}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - a_3 \sum_{i=1}^{k}\left(\sqrt{u_{i+1}^2 + u_i^2}\right)^{\frac{\alpha}{2}p} - a_4 k \\ &\geq \frac{\gamma}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - a_3 \left\{\left[\sum_{i=1}^{k}\left(\sqrt{u_{i+1}^2 + u_i^2}\right)^{\frac{\alpha}{2}p}\right]^{\frac{2}{\alpha p}}\right\}^{\frac{\alpha}{2}p} - a_4 k \\ &\geq \frac{\gamma}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - a_3 c_2^{\frac{\alpha}{2}p} \left\{\left[\sum_{i=1}^{k}\left(u_{i+1}^2 + u_i^2\right)\right]^{\frac{1}{2}}\right\}^{\frac{\alpha}{2}p} - a_4 k \\ &\geq \frac{\gamma}{p} c_1^p \lambda_{1-n}^{\frac{np}{2}} \|u\|^p - 2^{\frac{\alpha}{2}p} a_3 c_2^{\frac{\alpha}{2}p} \|u\|^{\frac{\alpha}{2}p} - a_4 k \to +\infty \quad \text{as} \quad \|u\| \to +\infty. \end{aligned}$$

By the continuity of J, the above inequality implies that there exist lower bounds of the values of J. This means that J attains its minimal value at some point, which is just the critical point of J with the finite norm. Theorem 1.3 is proved. □

Proof of Theorem 1.4. Assume the opposite, that the BVP (1.1), (1.2) has a nontrivial solution. Then J has a nonzero critical point u^*. Since

$$\frac{\partial J}{\partial u_i} = (-1)^n \Delta^n \left(\gamma_{i-n+1} \varphi_p \left(\Delta^n u_{i-1}\right)\right) - f(i, u_{i+1}, u_i, u_{i-1}),$$

we get

$$\sum_{i=1}^{k} f(i, u_{i+1}^*, u_i^*, u_{i-1}^*) u_i^* = \sum_{i=1}^{k} \left[(-1)^n \Delta^n \left(\gamma_{i-n+1} \varphi_p \left(\Delta^n u_{i-1}^*\right)\right)\right] u_i^* \tag{3.1}$$

$$= \sum_{i=1-n}^{k} \gamma_{i+1} \left|\Delta^n u_i^*\right|^p \leq 0.$$

On the other hand, it follows from (F_6) that

$$\sum_{i=1}^{k} f(i, u_{i+1}^*, u_i^*, u_{i-1}^*) u_i^* > 0. \tag{3.2}$$

This contradicts (3.1), and the result follows. Theorem 1.4 is proved. □

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