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# A CHARACTERIZATION OF TIGHT WAVELET FRAMES ON LOCAL FIELDS OF POSITIVE CHARACTERISTIC

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Abstract. The objective of this paper is to establish a complete characterization of tight wavelet frames on local fields of positive characteristic by means of two basic equations in the Fourier domain.

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## 1. Introduction

Tight wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, the tight wavelet frames become much easier to construct than the orthonormal wavelets. Tight wavelet frames provide representations of signals and images in applications, where redundancy of the representation is preferred and the perfect reconstruction property of the associated filter bank algorithm, as in the case of orthonormal wavelets, is kept.

In recent years there has been a considerable interest in the problem of constructing wavelet bases on locally compact Abelian groups. For example, Dahlke [4] introduced multiresolution analysis and wavelets on locally compact Abelian groups, Lang [12], by following the procedure of Daubechies [5], has constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathfrak{C}$ via scaling filters, and these wavelets turn out to be certain lacunary Walsh series on the real line. Later on, Farkov [6] extended the results of Lang [12] on the wavelet analysis on the Cantor dyadic group $\mathfrak{C}$ to the locally compact Abelian group $G_p$, which is defined for an integer $p \geq 2$ and coincides with $\mathfrak{C}$ when $p = 2$. Concerning the construction of wavelets on the half-line $\mathbb{R}^+$, Farkov [7] has given the general construction of all compactly supported orthogonal $p$-wavelets in $L^2(\mathbb{R}^+)$ and obtained necessary and sufficient conditions for scaling filters with $p^n$ many terms $(p, n \geq 2)$ to generate a

$p$-MRA analysis in $L^2(\mathbb{R}^+)$. These studies were continued by Shah and Debnath in [15-17], where they have given some new algorithms for constructing the wavelet and Gabor frames on the positive half-line $\mathbb{R}^+$. More results in this direction can also be found in [8, 9] and in the references therein.

A field $\mathbb{K}$ equipped with a topology is called a *local field* if both the additive and multiplicative groups of $\mathbb{K}$ are locally compact Abelian groups. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). The local fields of characteristic zero include the $p$-adic field $\mathbb{Q}_p$. Examples of local fields of positive characteristic are the Cantor dyadic group and the Vilenkin $p$-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. Local fields have attracted the attention of several mathematicians, and have found innumerable applications not only in the number theory, but also in the representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as a part of the standard repertoire of contemporary mathematics. For more details we referr to [14, 19].

Recently, R. L. Benedetto and J. J. Benedetto [3] developed a wavelet theory for local fields and related groups. Jiang *et al.*[11] pointed out a method for constructing orthogonal wavelets on a local field $\mathbb{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$. Subsequently, the tight wavelet frames on the local fields were constructed by Li and Jiang in [13]. They have obtained a necessary condition and sufficient conditions for tight wavelet frame on local fields in the frequency domain. Behera and Jahan [1] have constructed wavelet packets and wavelet frame packets on a local field $\mathbb{K}$ of positive characteristic, and show how to construct an orthonormal basis from a Riesz basis. Further, Behera and Jahan [2] have given a characterization of scaling functions associated with given multiresolution analysis of positive characteristic on a local field $\mathbb{K}$. Recently, Shah and Debnath [18], by following the procedure of Daubechies [5], have constructed tight wavelet frames on a local field $\mathbb{K}$ via extension principles.

Finally, E. Hermandes and Weiss (see [10]) have given a general characterization of all tight wavelet frames in $L^2(\mathbb{R})$ by means of the Fourier transform. As for the corresponding counterpart for a local field $\mathbb{K}$, such a result is not yet reported. So in this paper, we give a complete characterization of tight wavelet frames on local fields of positive characteristic, using the Fourier transform with different machinery as that of used in [10].

The paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and also some results which are required in the subsequent section. A characterization of tight wavelet frames on local fields of positive characteristic is given in Section 3.

## 2. Preliminaries on local fields

Let $\mathbb{K}$ be both a field and a topological space. Then $\mathbb{K}$ is called a *local field* if both $\mathbb{K}^+$ and $\mathbb{K}^*$ are locally compact Abelian groups, where $\mathbb{K}^+$ and $\mathbb{K}^*$ denote the additive and multiplicative groups of $\mathbb{K}$, respectively. If $\mathbb{K}$ is any field and is endowed with the discrete topology, then $\mathbb{K}$ is a local field. Further, if $\mathbb{K}$ is connected, then $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{K}$ is not connected, then it is totally disconnected. Hence by a local field, we mean a field $\mathbb{K}$ which is locally compact, non-discrete and totally disconnected. The $p$-adic fields are examples of local fields. For more details we refer the monographs [14, 19]. In the rest of the paper, we use the symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let $\mathbb{K}$ be a fixed local field. Then there is an integer $q = \mathfrak{p}^r$, where $\mathfrak{p}$ is a fixed prime element of $\mathbb{K}$ and $r$ is a positive integer, and a norm $|\cdot|$ on $\mathbb{K}$ such that for all $x \in \mathbb{K}$ we have $|x| \geq 0$ and for each $x \in \mathbb{K} \setminus \{0\}$ we can write $|x| = q^k$ for some integer $k$. This norm is non-Archimedean, that is, $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$ and $|x+y| = \max\{|x|, |y|\}$ whenever $|x| \neq |y|$. Let $dx$ be the Haar measure on the locally compact topological group $(\mathbb{K}, +)$. This measure is normalized so that $\int_{\mathfrak{D}} dx = 1$, where $\mathfrak{D} = \{x \in \mathbb{K} : |x| \leq 1\}$ is the *ring of integers* in $\mathbb{K}$. Define $\mathfrak{B} = \{x \in \mathbb{K} : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in $\mathbb{K}$. The prime ideal in $\mathbb{K}$ is the unique maximal ideal in $\mathfrak{D}$, and hence as result $\mathfrak{B}$ is both principal and prime. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in \mathbb{K} : |x| = 1\}$. Then, it is easy to verify that $\mathfrak{D}^*$ is a group of units in $\mathbb{K}^*$ and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. Moreover, $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \{x \in \mathbb{K} : |x| < q^{-k}\}$ are compact subgroups of $\mathbb{K}^+$, and are known as the *fractional ideals* of $\mathbb{K}^+$ (see [14]). Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in \mathbb{K}$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let $\chi$ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{B}^{-1}$. Therefore, $\chi$ is constant on cosets of $\mathfrak{D}$, implying that if

$y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx)$ for $x \in \mathbb{K}$. Suppose that $\chi_u$ is any character on $\mathbb{K}^+$, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in $\mathbb{K}^+$, then, as it was proved in [19], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n\in\mathbb{N}_0}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q) = \Gamma$, where $GF(q)$ is a $c$-dimensional vector space over the field $GF(p)$ (see [19]). We choose a set $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, ..., \epsilon_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, ..., \epsilon_{c-1}\} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q, \ \ n = a_0 + a_1 p + ... + a_{c-1}p^{c-1}, \ 0 \le a_k < p \ \text{ and } \ k = 0, 1, ..., c-1,$$

we define

$$u(n) = (a_0 + a_1\epsilon_1 + ... + a_{c-1}\epsilon_{c-1})\mathfrak{p}^{-1}. \tag{2.1}$$

Also, for $n = b_0 + b_1 q + ... + b_s q^s, n \ge 0, 0 \le b_k < q$, we set

$$u(n) = u(b_0) + \mathfrak{p}^{-1}u(b_1) + ... + \mathfrak{p}^{-s}u(b_s).$$

Then, it is easy to verify that for $\ell \in \mathbb{N}_0$

$$\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\} = \{u(k) + u(\ell) : k \in \mathbb{N}_0\},$$

and $u(n) = 0$ iff $n = 0$ (see [19]). Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \ge 0$. Also, by $\Omega$ we denote the test function space on $\mathbb{K}$, that is, each function $f$ in $\Omega$ is a finite linear combination of functions of the form $\mathbf{1}_k(x-h)$, $h \in \mathbb{K}$, $k \in \mathbb{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathfrak{B}^k$. Then, it is clear that $\Omega$ is dense in $L^p(\mathbb{K})$, $1 \le p < \infty$, and each function in $\Omega$ is of compact support and so is its Fourier transform. The Fourier transform of a function $f \in L^1(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\,\overline{\chi_\xi(x)}dx.$$

Note that

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\,\overline{\chi_\xi(x)}dx = \int_{\mathbb{K}} f(x)\,\chi(-\xi x)dx.$$

The properties of the Fourier transform on the local field $\mathbb{K}$ are quite similar to those of the Fourier analysis on the real line (see [14, 19]). In particular, if $f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$, then $\hat{f} \in L^2(\mathbb{K})$ and $\|\hat{f}\|_2 = \|f\|_2$. For a given $\psi \in L^2(\mathbb{K})$, define the wavelet system

$$X(\Psi) = \{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0\}, \tag{2.2}$$

where $\psi_{j,k} = q^{j/2}\psi(\mathrm{p}^j. - u(k))$. The wavelet system (2.2) is called a *wavelet frame*, if there exist positive numbers $0 < A \le B < \infty$ such that for all $f \in L^2(\mathbb{K})$

$$A\|f\|^2 \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \psi_{j,k}\rangle|^2 \le B\|f\|^2. \tag{2.3}$$

The largest constant $A$ and the smallest constant $B$ satisfying (2.3) are called the *lower and upper wavelet frame bounds*, respectively. A wavelet frame is a *tight wavelet frame* if $A$ and $B$ are chosen so that $A = B$, and the wavelet frame is called a *Parseval's wavelet frame* if $A = B = 1$, that is, for all $f \in L^2(\mathbb{K})$

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \psi_{j,k}\rangle|^2 = \|f\|^2, \tag{2.4}$$

and in this case, every function $f \in L^2(\mathbb{K})$ can be written as

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} \langle f, \psi_{j,k}\rangle \psi_{j,k}(x).$$

Since $\Omega$ is dense in $L^2(\mathbb{K})$ and is closed under the Fourier transform, the set

$$\Omega^0 = \left\{ f \in \Omega : \operatorname{supp}\hat{f} \subset \mathbb{K}\backslash\{0\} \right\}$$

is also dense in $L^2(\mathbb{K})$. Therefore, it is enough to verify that the system $X(\Psi)$ given by (2.2) is a frame and tight frame for $L^2(\mathbb{K})$ if (2.3) and (2.4) hold for all $f \in \Omega^0$.

In order to prove the main result to be presented in next section, we need the following lemma whose proof can be found in [13].

**Lemma 2.1.** *Let $f \in \Omega^0$ and $\psi \in L^2(\mathbb{K})$. If ess sup$\{\sum_{j\in\mathbb{Z}} |\hat{\psi}(\mathrm{p}^j\xi)|^2 : \xi \in \mathfrak{B}^{-1}\backslash\mathfrak{D}\} < \infty$, then*

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |\langle f, \psi_{j,k}\rangle|^2 = \int_{\mathbb{K}} \left|\hat{f}(\xi)\right|^2 \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(\mathrm{p}^j\xi)\right|^2 d\xi + R_\psi(f), \tag{2.5}$$

*where*

$$R_\psi(f) = \sum_{j\in\mathbb{Z}} \int_{\mathbb{K}} \overline{\hat{f}(\xi)}\hat{\psi}(\mathrm{p}^j\xi) \left[\sum_{l\in\mathbb{N}} \hat{f}\left(\xi + \mathrm{p}^{-j}u(l)\right) \overline{\hat{\psi}(\mathrm{p}^j\xi + u(l))}\right] d\xi$$

$$= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \int_{\mathbb{K}} \overline{\hat{f}(\xi)}\hat{\psi}(\mathrm{p}^j\xi)\, \hat{f}\left(\xi + p^{-j}u(l)\right) \overline{\hat{\psi}(\mathrm{p}^j\xi + u(l))}\, d\xi. \tag{2.6}$$

*Furthermore, the iterated series in (2.6) is absolutely convergent.*

**Remark 2.1.** The left hand side of (2.5) converges for all $f \in \Omega^0$ if and only if $\sum_{j\in\mathbb{Z}} |\hat{\psi}(\mathrm{p}^j\xi)|^2$ is locally integrable in $\mathbb{K}\backslash \cup_{j\in\mathbb{Z}} E_j^c$, where $E_j$ is the set of regular points of $\left|\psi(\mathrm{p}^j\xi)\right|^2$, which means that for each $x \in E_j$, we have

$$q^n \int_{\xi - x\in\mathfrak{B}^n} |\hat{\psi}(\mathrm{p}^j\xi)|^2 d\xi \to |\hat{\psi}(\mathrm{p}^j\xi)|^2 \text{ as } n \to \infty.$$

## 3. The main result

In this section, we establish our main result concerning the characterization of the wavelet system $X(\Psi)$ given by (2.2) to be a tight frame for $L^2(\mathbb{K})$.

**Theorem 3.1.** *The wavelet system $X(\Psi)$ given by (2.2) is a tight wavelet frame for $L^2(\mathbb{K})$ if and only if $\psi$ satisfies*

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(\mathfrak{p}^j\xi\right)\right|^2=1 \quad \text{for a.e. } \xi\in\mathcal{B}^{-1}\backslash\mathcal{D} \tag{3.1}$$

*and*

$$\sum_{j\in\mathbb{N}_0}\hat{\psi}\left(\mathfrak{p}^{-j}\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^{-j}(\xi+u(m))\right)}=0 \quad \text{for a.e. } \xi\in\mathcal{B}^{-1}\backslash\mathcal{D},\ \ m\in q\mathbb{N}_0+\tilde{Q}, \tag{3.2}$$

*where $q\mathbb{N}_0=\{qk:\ k=0,1,2,...\}$ and $\tilde{Q}=\{1,2,...,q-1\}$.*

**Proof.** Let

$$t_\psi(u(m),\xi)=\sum_{k\in\mathbb{N}_0}\hat{\psi}\left(\mathfrak{p}^{-k}\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^{-k}(\xi+u(m))\right)}.$$

Assume $f\in\Omega^0$, then for each $l\in\mathbb{N}$, there exists $k\in\mathbb{N}_0$ and a unique $m\in q\mathbb{N}_0+\tilde{Q}$ such that $l=q^km$. Thus, by virtue of (2.1) we have that $\{u(l)\}_{l\in\mathbb{N}}=\{\mathfrak{p}^{-k}u(m)\}_{(k,m)\in\mathbb{N}_0\times\{q\mathbb{N}_0+\tilde{Q}\}}$. Since the series in (2.4) is absolutely convergent, we can estimate $R_\psi(f)$, defined by (2.6), as follows:

$$R_\psi(f)=\sum_{j\in\mathbb{Z}}\int_{\mathbb{K}}\overline{\hat{f}(\xi)}\hat{\psi}\left(\mathfrak{p}^j\xi\right)\left\{\sum_{l\in\mathbb{N}}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(l)\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi+u(l)\right)}\right\}d\xi$$

$$=\sum_{j\in\mathbb{Z}}\int_{\mathbb{K}}\overline{\hat{f}(\xi)}\hat{\psi}\left(\mathfrak{p}^j\xi\right)\left\{\sum_{k\in\mathbb{N}_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\hat{f}\left(\xi+\mathfrak{p}^{-j-k}u(m)\right)\overline{\hat{\psi}\left(\mathfrak{p}^j\xi+\mathfrak{p}^{-k}u(m)\right)}\right\}d\xi$$

$$=\int_{\mathbb{K}}\overline{\hat{f}(\xi)}\left\{\sum_{k\in\mathbb{N}_0}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\sum_{j\in\mathbb{Z}}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(m)\right)\hat{\psi}\left(\mathfrak{p}^{j-k}\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^{j-k}\xi+\mathfrak{p}^{-k}u(m)\right)}\right\}d\xi$$

$$=\int_{\mathbb{K}}\overline{\hat{f}(\xi)}\left\{\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(m)\right)\sum_{k\in\mathbb{N}_0}\hat{\psi}\left(\mathfrak{p}^{j-k}\xi\right)\overline{\hat{\psi}\left(\mathfrak{p}^{-k}\left(\mathfrak{p}^j\xi+u(m)\right)\right)}\right\}d\xi$$

$$=\int_{\mathbb{K}}\overline{\hat{f}(\xi)}\left\{\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(m)\right)t_\psi\left(u(m),\mathfrak{p}^j\xi\right)\right\}d\xi.$$

Let us collect the results we have obtained: if $\psi\in L^2(\mathbb{K})$ and $f\in\Omega^0$, then

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|\langle f,\psi_{j,k}\rangle|^2=\int_{\mathbb{K}}|\hat{f}(\xi)|^2\sum_{j\in\mathbb{Z}}|\hat{\psi}\left(\mathfrak{p}^j\xi\right)|^2d\xi$$

$$+\int_{\mathbb{K}} \overline{\hat{f}(\xi)} \sum_{j\in\mathbb{Z}} \sum_{m\in q\mathbb{N}_0+\tilde{Q}} \hat{f}\left(\xi+\mathrm{p}^{-j}u(m)\right) t_{\psi}(u(m),\mathrm{p}^j\xi)d\xi. \tag{3.3}$$

The last integrand is integrable, and so is the first when $\sum_{j\in\mathbb{Z}} |\hat{\psi}(\mathrm{p}^j\xi)|^2$ is locally integrable in $\mathbb{K}\setminus \cup_{j\in\mathbb{Z}} E_j^c$. Further, equation (3.2) implies that

$$t_{\psi}(u(m),\xi)=0 \text{ for all } m\in q\mathbb{N}_0+\tilde{Q}.$$

Combining all together with (3.1) and (3.2) we get

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f,\psi_{j,k}\rangle|^2 = \|f\|_2^2, \quad \forall\, f\in\Omega^0.$$

Since $\Omega^0$ is dense in $L^2(\mathbb{K})$, we conclude that the wavelet system $X(\Psi)$ given by (2.2) is a tight frame for $L^2(\mathbb{K})$. Conversely, suppose that the system $X(\Psi)$ given by (2.2) is a tight wavelet frame for $L^2(\mathbb{K})$, then we need to show that both equations (3.1) and (3.2) are satisfied. Since $\{\psi_{j,k}(x) : j\in\mathbb{Z}, k\in\mathbb{N}_0\}$ is a tight wavelet frame for $L^2(\mathbb{K})$, then for all $f\in\Omega^0$ we have

$$\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f,\psi_{j,k}\rangle|^2 = \|f\|_2^2. \tag{3.4}$$

By remark 2.1, $\sum_{j\in\mathbb{Z}} |\hat{\psi}(\mathrm{p}^j\xi)|^2$ is locally integrable in $\mathbb{K}\setminus \cup_{j\in\mathbb{Z}} E_j^c$. Therefore, for each $\xi_0\in\mathbb{K}\setminus \cup_{j\in\mathbb{Z}} E_j^c$, we consider

$$\hat{f}_1(\xi)=q^{\frac{M}{2}}\Phi_M(\xi-\xi_0),$$

where $f=f_1$ and $\Phi_M(\xi-\xi_0)$ is the characteristic function of $\xi_0+\mathcal{B}^M$. Then, it follows that $\overline{\hat{f}(\xi)}\hat{f}(\xi+\mathrm{p}^{-j}u(l))\equiv 0$ for $l\in\mathbb{N}$, since $\xi$ and $\xi+\mathrm{p}^{-j}u(l)$ can not be in $\xi_0+\mathcal{B}^M$ simultaneously, and hence $\|f_1\|_2^2=1$. Furthermore, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f,\psi_{j,k}\rangle|^2 &= \|f_1\|_2^2=\|\hat{f}_1\|_2^2=1\\ &= \int_{\xi_0+\mathcal{B}^M} \sum_{j\in\mathbb{Z}} q^M \left|\hat{\psi}(\mathrm{p}^j\xi)\right|^2 d\xi + R_{\psi}(f_1).\end{aligned}$$

By letting $M\to\infty$, we obtain

$$1=\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(\mathrm{p}^j\xi_0)\right|^2 + \lim_{M\to\infty} R_{\psi}(f_1). \tag{3.5}$$

Now, we estimate $R_{\psi}(f_1)$ as follows:

$$R_{\psi}(f_1)=\sum_{j\in\mathbb{Z}} \int_{\mathbb{K}} \overline{\hat{f}_1(\xi)}\hat{\psi}(\mathrm{p}^j\xi) \left\{\sum_{l\in\mathbb{N}} \hat{f}_1\left(\xi+\mathrm{p}^{-j}u(l)\right) \overline{\hat{\psi}(\mathrm{p}^j\xi+u(l))}\right\} d\xi$$

$$|R_\psi(f_1)| \le \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_{\mathbb{K}}\left|\hat{f}_1(\xi)\hat{\psi}\left(\mathfrak{p}^j\xi\right)\hat{f}_1\left(\xi+\mathfrak{p}^{-j}u(l)\right)\hat{\psi}\left(\mathfrak{p}^j\xi+u(l)\right)\right|d\xi$$
$$=\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}q^j\int_{\mathbb{K}}\left|\hat{f}_1(\mathfrak{p}^{-j}\xi)\hat{f}_1\left(\mathfrak{p}^{-j}(\xi+u(l))\right)\hat{\psi}(\xi)\hat{\psi}(\xi+u(l))\right|d\xi.$$

Note that

$$\left|\hat{\psi}(\xi)\hat{\psi}(\xi+u(l))\right| \le \frac{1}{2}\left(\left|\hat{\psi}(\xi)\right|^2+\left|\hat{\psi}(\xi+u(l))\right|^2\right).$$

Therefore, we have

$$|R_\psi(f_1)| \le \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}q^j\int_{\mathbb{K}}|\hat{f}_1(\mathfrak{p}^{-j}\xi)\hat{f}_1\left(\mathfrak{p}^{-j}(\xi+u(l))\right)||\hat{\psi}(\xi)|^2d\xi. \tag{3.6}$$

Since $u(l)\neq 0$ ($l\in\mathbb{N}$) and $f_1\in\Omega^0$, there exists a constant $J>0$ such that

$$\hat{f}_1(\mathfrak{p}^{-j}t)\hat{f}_1\left(\mathfrak{p}^{-j}t+\mathfrak{p}^{-j}u(l)\right)=0, \quad \forall\, |j|>J.$$

On the other hand, for each $|j|\le J$, there exists a constant $L$ such that

$$\hat{f}_1\left(\mathfrak{p}^{-j}t+\mathfrak{p}^{-j}u(l)\right)=0, \quad \forall\, l>L.$$

This means that only a finite number of terms of the series on the right hand side of (3.6) are non-zero. Consequently, there exits a constant $C$ such that

$$|R_\psi(f_1)| \le C\|\hat{f}_1\|_\infty^2\|\hat{\psi}\|_2^2 = Cq^m\|\hat{\psi}\|_2^2,$$

implying

$$\lim_{M\to\infty}|R_\psi(f_1)|=0.$$

Hence equation (3.5) becomes

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(p^j\xi_0\right)\right|^2=1.$$

Finally, we must show that if (3.4) holds for all $f\in\Omega^0$, then equation (3.2) is true. From equalities (3.3), (3.4) and just established equality (3.1), for all $f\in\Omega^0$ we have

$$\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\int_{\mathbb{K}}\overline{\hat{f}(\xi)}\hat{f}\left(\xi+\mathfrak{p}^{-j}u(m)\right)t_\psi(u(m),\mathfrak{p}^j\xi)d\xi=0.$$

Also, by polarization, for all $f,g\in\Omega^0$ we have

$$\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\tilde{Q}}\int_{\mathbb{K}}\overline{\hat{f}(\xi)}\hat{g}\left(\xi+\mathfrak{p}^{-j}u(m)\right)t_\psi(u(m),\mathfrak{p}^j\xi)\,d\xi=0. \tag{3.7}$$

Let us fix $m_0\in q\mathbb{N}_0+\tilde{Q}$ and $\xi_0\in\mathbb{K}\setminus\cup_{j\in\mathbb{Z}}E_j^c$ such that neither $\xi_0\neq 0$ nor $\xi_0+u(m_0)\neq 0$. Setting $f=f_1$ and $g=g_1$ such that

$$\hat{f}_1(\xi)=q^{\frac{M}{2}}\Phi_M(\xi-\xi_0) \quad\text{and}\quad \hat{g}_1(\xi)=\hat{f}_1(\xi-u(m_0)),$$

we obtain $\hat{f}_1(\xi)\hat{g}_1(\xi+u(m_0)) = q^M \Phi_M(\xi-\xi_0)$. Now, equality (3.7) can be written as

$$0 = q^M \int_{\xi_0+\mathfrak{B}^M} t_\psi(u(m_0),\xi)d\xi + J_1, \tag{3.8}$$

where

$$J_1 = \sum_{\substack{j\in\mathbb{Z} \\ (j,m)\neq(0,m_0)}} \sum_{m\in q\mathbb{N}_0+\tilde{Q}} \int_{\mathbb{K}} \overline{\hat{f}_1(\xi)}\hat{g}_1\left(\xi+\mathfrak{p}^{-j}u(m)\right) t_\psi(u(m),\mathfrak{p}^j\xi)d\xi.$$

Since the first summand in (3.8) tends to $t_\psi(u(m_0),\xi_0)$ as $M\to\infty$, we have to prove that

$$\lim_{M\to\infty} J_1 = 0.$$

Since $u(m)\neq 0$, $(m\in\mathbb{N})$ and $f_1, g_1 \in \Omega^0$, there exists a constant $J_0>0$ such that

$$\overline{\hat{f}_1(\xi)}\,\hat{g}_1\left(\xi+\mathfrak{p}^{-j}u(m)\right) = 0 \;\; \forall\, j > J_0.$$

Therefore, we have

$$J_1 = \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\tilde{Q}} \int_{\mathbb{K}} \overline{\hat{f}_1(\xi)}\hat{g}_1\left(\xi+\mathfrak{p}^{-j}u(m)\right) t_\psi(u(m),\mathfrak{p}^j\xi)d\xi$$

$$|J_1| \le \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\tilde{Q}} q^j \int_{\mathbb{K}} \left|\overline{\hat{f}_1(\mathfrak{p}^{-j}\xi)}\hat{g}_1\left(\mathfrak{p}^{-j}(\xi+u(m))\right)\right| |t_\psi(u(m),\xi)|\, d\xi.$$

Since

$$2\,|t_\psi(u(m),\xi)| \le \sum_{k\in\mathbb{N}_0} |\hat{\psi}\left(\mathfrak{p}^{-k}\xi\right)|^2 + \sum_{k\in\mathbb{N}_0} |\hat{\psi}\left(\mathfrak{p}^{-k}(\xi+u(m))\right)|^2,$$

we have

$$|J_1| \le J_1^{(1)} + J_1^{(2)},$$

where

$$J_1^{(1)} = \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\tilde{Q}} q^j \int_{\mathbb{K}} \left|\hat{f}_1(\mathfrak{p}^{-j}\xi)\right| \left|\hat{g}_1\left(\mathfrak{p}^{-j}(\xi+u(m))\right)\right| [\tau(\xi)]^2 d\xi$$

with

$$\int_{\mathbb{K}} [\tau(\xi)]^2 d\xi = \frac{1}{2}\sum_{k\in\mathbb{N}_0} \int_{\mathbb{K}} \left|\hat{\psi}\left(\mathfrak{p}^{-k}\xi\right)\right|^2 d\xi = \|\hat{\psi}\|_2^2 < \infty,$$

and

$$J_1^{(2)} = \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\tilde{Q}} q^j \int_{\mathbb{K}} \left|\hat{f}_1(\mathfrak{p}^{-j}\xi)\right| \left|\hat{g}_1\left(\mathfrak{p}^{-j}(\xi+u(m))\right)\right| [\tau(\xi+u(m))]^2 d\xi$$

$$= \sum_{j\le J_0} \sum_{m\in q\mathbb{N}_0+\tilde{Q}} q^j \int_{\mathbb{K}} \left|\hat{f}_1(\mathfrak{p}^{-j}(\eta-u(m)))\right| \left|\hat{g}_1\left(\mathfrak{p}^{-j}\eta\right)\right| [\tau(\eta)]^2 d\xi.$$

Thus $J_1^{(2)}$ has the same form as $J_1^{(1)}$ with the roles of $\hat{f}_1$ and $\hat{g}_1$ interchanged. Next, since $\hat{f}_1(\xi) = q^{\frac{M}{2}}\Phi_M(\xi - \xi_0)$, we can write

$$J_1^{(1)} = \sum_{j\le J_0}\sum_{m\in q\mathbb{N}_0+\bar{Q}} q^j q^{\frac{M}{2}} \int_{\mathfrak{p}^{-j}\xi_0+\mathfrak{B}^{-j+M}} \left|\hat{g}_1\left(\mathfrak{p}^{-j}(\xi+u(m))\right)\right| [\tau(\xi)]^2 d\xi.$$

Now, if $\hat{g}_1\left(\mathfrak{p}^{-j}(\xi+u(m))\right) \ne 0$, then we must have $\mathfrak{p}^{-j}\xi+\mathfrak{p}^{-j}u(m) \in \xi_0+\mathfrak{B}^M+u(m_0)$ and $|\mathfrak{p}^{-j}u(m)| \le q^{-M}$, and hence $|u(m)| \le q^{-M-j}$. Thus, we have

(3.9)

$$\begin{aligned} J_1^{(1)} &= \sum_{j\le J_0} q^j q^{\frac{M}{2}} \int_{\mathfrak{p}^{-j}\xi_0+\mathfrak{B}^{-j+M}} [\tau(\xi)]^2 \sum_{m\in q\mathbb{N}_0+\bar{Q}} \left|\hat{g}_1\left(\mathfrak{p}^{-j}(\xi+u(m))\right)\right| d\xi \\ &\le \sum_{j\le J_0} q^j q^{\frac{M}{2}} \int_{\mathfrak{p}^{-j}\xi_0+\mathfrak{B}^{-j+M}} [\tau(\xi)]^2 q^{-M-j} q^{\frac{M}{2}} d\xi = \sum_{j\le J_0} \int_{\mathfrak{p}^{-j}\xi_0+\mathfrak{B}^{-j+M}} [\tau(\xi)]^2 d\xi\,. \end{aligned}$$

For given $\xi_0 \ne 0$, we choose $q^{J_0} < |\xi_0| = q^{-M}$ to obtain

$$\mathfrak{p}^{-j}\xi_0 + \mathfrak{B}^{-j+M} \subset \mathfrak{B}^{-J_0+M} \quad \forall\, j \le J_0, \tag{3.10}$$

as $|\mathfrak{p}^{-j}\xi_0| = q^j q^{-M} \le q^{J_0} q^{-M}$ and $\mathfrak{B}^{-j+M} \subset \mathfrak{B}^{-J_0+M}$. On the other hand, for any $j_1 < j_2 \le J_0$, we claim that

$$\{\mathfrak{p}^{-j_1}\xi_0 + \mathfrak{B}^{-j_1+M}\} \cap \{\mathfrak{p}^{-j_2}\xi_0 + \mathfrak{B}^{-j_2+M}\} = \emptyset. \tag{3.11}$$

Indeed, for any $x \in \mathfrak{p}^{-j_1}\xi_0+\mathfrak{B}^{-j_1+M}$ and $y \in \mathfrak{p}^{-j_2}\xi_0+\mathfrak{B}^{-j_2+M}$, write $x = \mathfrak{p}^{-j_1}\xi_0+x_1$ and $y = \mathfrak{p}^{-j_2}\xi_0 + y_1$, then we have $|x-y| = \max\{|\mathfrak{p}^{-j_1}\xi_0 - \mathfrak{p}^{-j_2}\xi_0|, |x_1-y_1|\} = q^{j_2-M} \ne 0$, implying that (3.11) holds. Combining (3.9) - (3.11), we obtain

$$J_1^{(1)} \le \int_{\mathfrak{B}^{-J_0+M}} [\tau(\xi)]^2 d\xi \to 0 \text{ as } M \to \infty.$$

This completes the proof of Theorem 3.1.

**Example 3.1.** Consider the functions

$$\psi_1(x) = \begin{cases} 1 & \text{if } x \in \mathfrak{D}, \\ 0 & \text{if } x \notin \mathfrak{D}, \end{cases} \quad \text{and} \quad \psi_2(x) = \begin{cases} q^{-1} & \text{if } x \in \mathfrak{B}^{-1}, \\ 0 & \text{if } x \notin \mathfrak{B}^{-1}, \end{cases}$$

and define $\psi(x) = \psi_1(x) - \psi_2(x)$. Since $\hat{\psi}_1(\xi) = \psi_1(\xi)$ and

$$\hat{\psi}_2(\xi) = \begin{cases} 1 & x \in \mathfrak{B}, \\ 0 & x \notin \mathfrak{B}, \end{cases}$$

we have

$$\hat{\psi}(\xi) = \begin{cases} 1 & x \in \mathfrak{B}^{-1}\backslash\mathfrak{D}, \\ 0 & \text{otherwise.} \end{cases}$$

Next, for $\xi \neq 0$, we see that $\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(\mathrm{p}^j\xi\right)\right|^2 = 1$, and since $\mathrm{p}^{-j}\xi$ and $\mathrm{p}^{-j}(\xi + u(m))$ can not be in $\mathfrak{B}^{-1}\backslash\mathfrak{D}$ simultaneously, we conclude that

$$\sum_{j=0}^{\infty} \hat{\psi}\left(\mathrm{p}^{-j}\xi\right) \overline{\hat{\psi}\left(\mathrm{p}^{-j}(\xi + u(m))\right)} = 0.$$

## Список литературы


[1] B. Behera and Q. Jahan, "Wavelet packets and wavelet frame packets on local fields of positive characteristic", J. Math. Anal. Appl., **395**, 1 – 14 (2012).

[2] B. Behera and Q. Jahan, "Multiresolution analysis on local fields and characterization of scaling functions", Adv. Pure Appl. Math., **3**, 181 – 202 (2012).

[3] J. J. Benedetto and R. L. Benedetto, "A wavelet theory for local fields and related groups", J. Geomet. Anal., 14, 423 – 456 (2004).

[4] S. Dahlke, "Multiresolution analysis and wavelets on locally compact abelian groups", Wavelets, Images, and Surface Fitting, A. K. Peters, 141 – 156 (1994).

[5] I. Daubechies, Ten Lectures on Wavelets, CBMS-NSF Regional Conference in Applied Mathematics, SIAM, Philadelphia (1992).

[6] Yu. A. Farkov, "Orthogonal wavelets with compact support on locally compact Abelian groups", Izvest. Math., **69**(3), 623 – 650 (2005).

[7] Yu. A. Farkov, "On wavelets related to Walsh series", J. Approx. Theory. **161**, 259 – 279 (2009).

[8] Yu. A. Farkov, "Wavelets and frames based on Walsh-Dirichlet type kernels", Commun. Math. Appl., **1**, 27 – 46 (2010).

[9] Yu. A. Farkov, "Examples of frames on the Cantor dyadic group", J. Math. Sci. **187**, 22 – 34 (2012).

[10] E. Herñyandez and G. Weiss, A First Course on Wavelets, CRC Press (1996).

[11] H. K. Jiang, D. F. Li and N. Jin, "Multiresolution analysis on local fields", J. Math. Anal. Appl., **294**, 523 – 532 (2004).

[12] W. C. Lang, "Orthogonal wavelets on the Cantor dyadic group", SIAM J. Math. Anal., **27**, 305 – 312 (1996).

[13] D. F. Li and H. K. Jiang, "The necessary condition and sufficient conditions for wavelet frame on local fields", J. Math. Anal. Appl., **345**, 500 – 510 (2008).

[14] D. Ramakrishnan and R. J. Valenza, Fourier Analysis on Number Fields, Graduate Texts in Mathematics 186, Springer-Verlag, New York (1999).

[15] F. A. Shah, "Gabor frames on a half-line", J. Contemp. Math. Anal. **47**(5), 251 – 260 (2012).

[16] F. A. Shah, "Tight wavelet frames generated by the Walsh polynomials", Int. J. Wavelets, Multiresolut. Inf. Process., **11**(6), 15 pages, (2013).

[17] F. A. Shah and L. Debnath, "Dyadic wavelet frames on a half-line using the Walsh-Fourier transform", Integ. Trans. Special Funct. **22**(7), 477 – 486 (2011).

[18] F. A. Shah and L. Debnath, "Tight wavelet frames on local fields", Analysis. **33**, 293 – 307, (2013).

[19] M. H. Taibleson, Fourier Analysis on Local Fields, Princeton University Press, Princeton, NJ (1975).


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