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INTEGRALS OF CHEBYSHEV POLYNOMIALS OF THIRD AND FOURTH KINDS: AN APPLICATION TO SOLUTION OF BOUNDARY VALUE PROBLEMS WITH POLYNOMIAL COEFFICIENTS

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Abstract. Two new formulae expressing explicitly the repeated integrals of Chebyshev polynomials of third and fourth kinds of arbitrary degree in terms of the same polynomials are derived. The method of proof is novel and essentially based on making use of the power series representation of these polynomials and their inversion formulae. Using the Galerkin spectral method, we show that these formulae can be used to solve some high-order boundary value problems with varying coefficients, and propose two Galerkin-type algorithms for solving the integrated forms of some high-order boundary value problems with polynomial coefficients. A numerical example is discussed, which shows that the proposed algorithms are more accurate and efficient compared with the analytical ones.

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1. Introduction

Spectral methods have been extensively used in applied mathematics and scientific computing to obtain numerical solutions of ordinary and partial differential equations (see Boyd [5] and Canuto et al. [6]). These numerical solutions are written as expansions in terms of certain "basis functions which may be expressed in terms of various orthogonal polynomials. Spectral methods have advantage that they take on a global approach, while finite-element methods use a local approach, and as a consequence, spectral methods have "good" error properties and converge exponentially.

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ play an important role in mathematical analysis and its applications (see Abramowitz and Stegun [3], Andrews et al. [4] and Boyd [5]). In particular, the Legendre, the Chebyshev and the ultraspherical polynomials, which are special classes of Jacobi polynomials, have already played an important role in the spectral methods for solving ordinary and partial differential equations.

The Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. It is well-known that there are four kinds of Chebyshev polynomials. They all are special cases of the classical Jacobi polynomials. A large number of books and research articles deal with the first and second kinds of Chebyshev polynomials $T_n(x)$ and $U_n(x)$ and their various applications (see Boyd [5], Doha et al. [14], Julien and Watson [17], and references therein). However, there is only a few number of publications devoted to the Chebyshev polynomials of third and fourth kinds $V_n(x)$ and $W_n(x)$ (see, e.g., Doha et al. [13] and Eslahchi et al. [16]). This motivates our interest to such polynomials.

The study of both high even-order and high odd-order boundary-value problems (BVP's) is of interest. For instance, the third order equations are of mathematical and physical interest, since they lack symmetry and, in addition, contain an important type of operators which appears in many commonly occurring partial differential equations, such as the Kortewege-de Vries equation. Another important example is the sixth-order boundary-value problem, which arise in astrophysics. Due to their great importance in various applications in many fields, high-order boundary-value problems have been extensively discussed by a number of authors (see, e.g., [15, 19, 22, 24, 25, 26]). In a series of papers [1, 2, 9, 10, 11, 13], the authors dealt with such equations by using the Galerkin or Petrov-Galerkin methods. Using compact combinations of various orthogonal polynomials, they have constructed suitable bases functions which satisfy the boundary conditions of the given differential equation.

An alternative approach is to integrate the differential equation q times, where q is the order of the equation. The advantage of this approach is that the underlying equation resulted an algebraic system that contains a finite number of terms. Doha et al. [12] have followed this approach, to solve the integrated forms of third- and fifth-order elliptic differential equations using general parameters of the generalized Jacobi polynomials. Some other papers were concerned with obtaining analytical formulae for the q times repeated integration of some orthogonal polynomials (see, e.g. Doha [7, 8], and Phillips and Karageorghis [21]).

In this paper, we derive two new formulae that express explicitly the repeated integrals of Chebyshev polynomials of third and fourth kinds in terms of the same polynomials. Then using these formulae, we develop two Galerkin-type algorithms, (C3GM) and (C4GM), for solving the integrated forms of some high even-order differential equations with polynomial coefficients.

The paper is organized as follows. In Section 2, some properties of Chebyshev polynomials of third and fourth kinds are given, and some new relations of these polynomials are stated and proved. In Section 3, we derive two new formulae which express explicitly the repeated integrals of Chebyshev polynomials of third and fourth kinds in terms of the same polynomials. In Section 4, we present two Galerkin-type algorithms for solving the integrated forms of some high-order boundary value problems with polynomial coefficients. In Section 5, a numerical example is discussed to demonstrate the accuracy and efficiency of the algorithms proposed in Section 4.

2. Some properties of Chebyshev polynomials of third and fourth kinds

Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of third and fourth kinds are polynomials in x, which can be defined by one of the following two equivalent forms (see Mason and Handscomb [20]):

$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}} = \frac{2^{2n}}{\binom{2n}{n}}\, P_n^{(-\frac{1}{2},\frac{1}{2})}(x),$$

and

$$W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} = \frac{2^{2n}}{\binom{2n}{n}}\, P_n^{(\frac{1}{2},-\frac{1}{2})}(x),$$

where $x = \cos\theta$, and $P_n^{(\alpha,\beta)}(x)$ is the classical Jacobi polynomial of degree n.

It is clear that

$$W_n(x) = (-1)^n\, V_n(-x). \tag{2.1}$$

The polynoamils $V_n(x)$ and $W_n(x)$ are orthogonal on $(-1,1)$ with respect to the weight functions $\sqrt{\frac{1+x}{1-x}}$ and $\sqrt{\frac{1-x}{1+x}}$, respectively, that is, we have

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\, V_n(x)\, V_m(x)\, dx = \int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\, W_n(x)\, W_m(x)\, dx = \begin{cases} 0, & \text{if } m \neq n, \\ \pi, & \text{if } m = n, \end{cases} \tag{2.2}$$

and can be generated by using the following two recurrence relations:

$$V_n(x) = 2x\, V_{n-1}(x) - V_{n-2}(x), \quad n = 2, 3, \ldots, \tag{2.3}$$

with $V_0(x) = 1$, $V_1(x) = 2x - 1$, and

$$W_n(x) = 2x\, W_{n-1}(x) - W_{n-2}(x), \quad n = 2, 3, \ldots,$$

with $W_0(x) = 1$, $W_1(x) = 2x+1$. Below the following special values will be of importance:

$$V_n(1) = (-1)^n W_n(-1) = 1, \tag{2.4}$$

$$W_n(1) = (-1)^n V_n(-1) = 2n+1, \tag{2.5}$$

$$D^q V_n(1) = (-1)^{n+q} D^q W_n(-1) = \prod_{k=0}^{q-1} \frac{(n-k)(n+k+1)}{2k+1},\ q \geq 1, \tag{2.6}$$

$$D^q W_n(1) = (-1)^{n+q} D^q V_n(-1) = (2n+1) \prod_{k=0}^{q-1} \frac{(n-k)(n+k+1)}{2k+3},\ q \geq 1. \tag{2.7}$$

The following two theorems and lemma are needed in the sequel.

Theorem 2.1. *The explicit power form of the polynomial $V_n(x)$, $n \geq 1$ is given by the formula*

$$V_n(x) = \sum_{k=0}^{[\frac{n}{2}]} a_{n,k}\, x^{n-2k} + \sum_{k=0}^{[\frac{n}{2}]} b_{n,k}\, x^{n-2k-1}, \tag{2.8}$$

where

$$a_{n,k} = \frac{(-1)^k (n-k)!\, 2^{n-2k}}{k!\,(n-2k)!}, \quad b_{n,k} = \frac{(-1)^{k+1} (n-k-1)!\, 2^{n-2k-1}}{k!\,(n-2k-1)!}. \tag{2.9}$$

Proof. We proceed by induction on n. Assume that the relation (2.8) holds for $(n-1)$ and $(n-2)$. Then starting with the recurrence relation (2.3) and applying the induction hypothesis twice, we obtain

$$V_n(x) = 2 \sum_{k=0}^{[\frac{n-1}{2}]} a_{n-1,k}\, x^{n-2k} - \sum_{k=0}^{[\frac{n}{2}]-1} a_{n-2,k}\, x^{n-2k-2} + \tag{2.10}$$

$$+2 \sum_{k=0}^{[\frac{n-1}{2}]} b_{n-1,k}\, x^{n-2k-1} - \sum_{k=0}^{[\frac{n}{2}]-1} b_{n-2,k}\, x^{n-2k-3},$$

which can be written in the form

$$V_n(x) = \sum_1 + \sum_2,$$

where

$$\sum_1 = -a_{n-2,\frac{n}{2}-1}\, \delta_n + 2\, a_{n-1,0}\, x^n + \sum_{k=1}^{[\frac{n-1}{2}]} \{2\, a_{n-1,k} - a_{n-2,k-1}\}\, x^{n-2k},$$

$$\sum_2 = 2\, b_{n-1,0}\, x^{n-1} + \sum_{k=1}^{[\frac{n-1}{2}]} \{2\, b_{n-1,k} - b_{n-2,k-1}\}\, x^{n-2k-1},$$

and $\delta_n = \begin{cases} 1, & \text{n even}, \\ 0, & \text{n odd}. \end{cases}$

It is not difficult to show that

$2\,a_{n-1,0} = a_{n,0},\ -a_{n-2,\frac{n}{2}-1} = a_{n,\frac{n}{2}},\ 2\,a_{n-1,k} - a_{n-2,k-1} = a_{n,k},\ 1 \le k \le \left[\frac{n-1}{2}\right],$

$2\,b_{n-1,0} = b_{n,0},\ 2\,b_{n-1,k} - b_{n-2,k-1} = b_{n,k},\ 1 \le k \le \left[\frac{n-1}{2}\right],$

and therefore we can write

$$\sum_1 = \sum_{k=0}^{[\frac{n}{2}]} a_{n,k}\, x^{n-2k},$$

and

$$\sum_2 = \sum_{k=0}^{[\frac{n}{2}]} b_{n,k}\, x^{n-2k-1},$$

where $a_{n,k}$ and $b_{n,k}$ are given in (2.9). This completes the proof of Theorem 2.1. □

The next theorem was proved in Doha et al. [13].

Theorem 2.2. *For all $k, m \in \mathbf{Z}^+$, we have*

$$x^m\, V_k(x) = \frac{1}{2^m} \sum_{s=0}^{m} \binom{m}{s} V_{k+m-2s}(x). \tag{2.11}$$

In particular, the following inversion formula holds

$$x^m = \frac{1}{2^m} \sum_{s=0}^{m} \binom{m}{s} V_{m-2s}(x). \tag{2.12}$$

Lemma 2.1. *For every nonnegative integer r and a natural $n > r$, we have*

$$\sum_{j=0}^{r} \frac{(-1)^j\,(n-j-1)!}{j!\,(n-j+q-r)!\,(r-j)!} = \frac{(-1)^r\, q!\,(n-r-1)!}{r!\,(q-r)!\,(n+q-r)!}. \tag{2.13}$$

Proof. Setting

$$M_{n,q,r} = \sum_{j=0}^{r} \frac{(-1)^j\,(n-j-1)!}{j!\,(n-j+q-r)!\,(r-j)!},$$

and using Zeilberger's algorithm (see, e.g., Koepf [18]), we conclude that $M_{n,q,r}$ satisfies the following difference equation of order one:

$$(r+1)(n-r-1)\, M_{n,q,r+1} + (q-r)(n+q-r)\, M_{n,q,r} = 0, \quad M_{n,q,0} = \frac{(n-1)!}{(n+q)!},$$

which can be solved to obtain

$$M_{n,q,r} = \frac{(-1)^r\, q!\,(n-r-1)!}{r!\,(q-r)!\,(n+q-r)!}.$$

Lemma 2.1 is proved. □

Remark 2.1. *The counterparts of Theorems 2.1 and 2.2 for the polynomials $W_n(x)$ can easily be deduced with the aid of relation* (2.1).

3. Formulas for repeated integrals of Chebyshev polynomials $V_n(x)$ and $W_n(x)$

The objective of this section is to state and prove two theorems, which express explicitly the repeated integrals of Chebyshev polynomials $V_n(x)$ and $W_n(x)$ in terms of the same polynomials.

Given a natural number q, the q-times repeated integral of the third kind Chebyshev polynomial $V_n(x)$ is denoted by

$$I_n^{(q)}(x) = \int^{(q)} V_n(x)\,(dx)^q = \overbrace{\int\int\dots\int}^{q\ times} V_n(x)\,\overbrace{dx\,dx\,...dx}^{q\ times}.$$

Theorem 3.1. *The following formula holds.*

$$I_n^{(q)}(x) = \sum_{r=0}^{q} A_{n,r,q}\,V_{n+q-2r}(x) + \sum_{r=0}^{q} B_{n,r,q}\,V_{n+q-2r-1}(x) + \pi_{q-1}(x), \tag{3.1}$$

where

$$A_{n,r,q} = \frac{(-1)^r\,(n-r)!\,q!}{2^q\,r!\,(q-r)!\,(n+q-r)!},\quad B_{n,r,q} = \frac{(-1)^{r+1}\,(n-r-1)!\,q!}{2^q\,r!\,(q-r-1)!\,(n+q-r)!},$$

and $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

Proof. Integrating the relation (2.8) q-times, and using the equality

$$\int^{(q)} x^i\,(dx)^q = \frac{x^{i+q}}{(i+1)_q} + \pi_{q-1}(x),$$

we get

$$I_n^{(q)}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} e_{n,k,q}\,x^{n-2k-q} + \sum_{k=0}^{\left[\frac{n}{2}\right]} f_{n,k,q}\,x^{n-2k+q-1} + \pi_{q-1}(x),$$

where

$$e_{n,k,q} = \frac{(-1)^k\,2^{n-2k}(n-k)!}{k!\,(n-2k+q)!},\quad f_{n,k,q} = \frac{(-1)^{k+1}\,2^{n-2k-1}(n-k-1)!}{k!\,(n-2k+q-1)!}. \tag{3.2}$$

and $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

Taking into account the relation (2.12), we can write

$$I_n^{(q)}(x) = \sum_1 + \sum_2 + \pi_{q-1}(x),$$

where

$$\sum_1 = \sum_{k=0}^{[\frac{n}{2}]} e_{n,k,q} \sum_{i=0}^{[\frac{n+q-1}{2}]-k} c_{i,n-2k+q}\, V_{n+q-2k-2i}(x) +$$

$$\sum_{k=0}^{[\frac{n}{2}]} f_{n,k,q} \sum_{i=0}^{[\frac{n+q}{2}]-k-1} c_{i,n-2k+q-1}\, V_{n+q-2k-2i-2}(x),$$

$$\sum_2 = \sum_{k=0}^{[\frac{n}{2}]} e_{n,k,q} \sum_{i=0}^{[\frac{n+q-1}{2}]-k} c_{i,n-2k+q}\, V_{n+q-2k-2i-1}(x) +$$

$$\sum_{k=0}^{[\frac{n}{2}]} f_{n,k,q} \sum_{i=0}^{[\frac{n+q-1}{2}]-k} c_{i,n-2k+q-1}\, V_{n-q-2k-2i-3}(x),$$

the coefficients $e_{n,k,q}$ and $f_{n,k,q}$ are given in (3.2) and

$$c_{i,m} = \frac{\binom{m}{i}}{2^m}.$$

Expanding $\sum_1$ and $\sum_2$, and collecting similar terms, after some algebra we get

$$I_n^{(q)}(x) = \sum_{r=0}^{q} A_{n,r,q}\, V_{n+q-2r}(x) + \sum_{r=0}^{q} B_{n,r,q}\, V_{n+q-2r-1}(x) + \pi_{q-1}(x),$$

where

$$A_{n,r,q} = \sum_{j=0}^{r} \left\{ e_{n,j,q}\, c_{r-j,n+q-2j} + f_{n,j,q}\, c_{r-j-1,n+q-2j-1} \right\}, \tag{3.3}$$

$$B_{n,r,q} = \sum_{j=0}^{r} \left\{ e_{n,j,q}\, c_{r-j,n+q-2j} + f_{n,j,q}\, c_{r-j,n+q-2j-1} \right\}. \tag{3.4}$$

Next, it is not difficult to show that

$$e_{n,j,q}\, c_{r-j,n+q-2j} + f_{n,j,q}\, c_{r-j-1,n+q-2j-1} = \frac{(-1)^j\, 2^{-q}\, (n-r)\, (n-j-1)!}{j!\, (n+q-j-r)!\, (r-j)!}, \tag{3.5}$$

and

$$e_{n,j,q}\, c_{r-j,n+q-2j} + f_{n,j,q}\, c_{r-j,n+q-2j-1} = \frac{(-1)^{j+1}\, 2^{-q}\, (q-r)\, (n-j-1)!}{j!\, (n+q-j-r)!\, (r-j)!}. \tag{3.6}$$

Finally, substituting the relations (3.5) and (3.6) into (3.3) and (3.4), and using (2.13), for $A_{n,r,q}$ and $B_{n,r,q}$ we obtain

$$A_{n,r,q} = \frac{(-1)^r\, 2^{-q}\, (n-r)!\, q!}{r!\, (q-r)!\, (n+q-r)!},$$

$$B_{n,r,q} = \frac{(-1)^{r+1}\, 2^{-q}\, (n-r-1)!\, q!}{r!\, (q-r-1)!\, (n+q-r)!},$$

and the result follows. Theorem 3.1 is proved. □

Remark 3.1. Note that the relation (3.1) may be written in the following equivalent form:

$$I_n^{(q)}(x) = \sum_{i=0}^{2q} E_{n,i,q}\, V_{n+q-i}(x) + \pi_{q-1}(x), \quad n \geq q \geq 1, \tag{3.7}$$

where

$$E_{n,i,q} = \frac{q!}{2^q} \begin{cases} \dfrac{(-1)^{\frac{i}{2}}\ (n-\frac{i}{2})!}{(\frac{i}{2})!\ (q-\frac{i}{2})!\ (n+q-\frac{i}{2})!}, & \text{if } i \text{ is even,} \\ \dfrac{(-1)^{\frac{i+1}{2}}\ (n-(\frac{i+1}{2}))!}{(\frac{i-1}{2})!\ (q-(\frac{i+1}{2}))!\ (n+q-(\frac{i-1}{2}))!}, & \text{if } i \text{ is odd,} \end{cases} \tag{3.8}$$

and $\pi_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

Using the arguments of the proof of Theorem 3.1 and formula (2.1), we can obtain a formula that express explicitly the repeated integral of Chebyshev fourth kind polynomial $W_n(x)$ in terms of the same polynomial. The corresponding result is stated in the following theorem.

Theorem 3.2. *Let $J_n^{(q)}(x)$ be the q-times repeated integral of the polynomial $W_n(x)$:*

$$J_n^{(q)}(x) = \int^{(q)} W_n(x)(dx)^q,$$

then

$$J_n^{(q)}(x) = \sum_{i=0}^{2q} S_{n,i,q}\, W_{n+q-i}(x) + \bar{\pi}_{q-1},$$

where

$$S_{n,i,q} = (-1)^i\, E_{n,i,q}, \tag{3.9}$$

and $\bar{\pi}_{q-1}(x)$ is a polynomial of degree at most $(q-1)$.

4. An application to a high-order two point boundary value problem

In this section, we are interested in applying the formulas, obtained in Section 3, to solve the following high-order boundary value problem:

$$(-1)^n\, u^{(2n)}(x) + \gamma\, p(x)\, u(x) = f(x), \quad x \in (-1,1),\ n \geq 1, \tag{4.1}$$

subject to the nonhomogeneous Dirichlet boundary conditions

$$u^{(j)}(\pm 1) = \pm\alpha_j, \quad 0 \leq j \leq n-1, \tag{4.2}$$

where $p(x)$ is a given polynomial and γ is a real constant.

It is worth to note that if we use the transformation:

$$y(x) = u(x) + \sum_{i=0}^{2n-1} \xi_i\, x^i,$$

where $\xi_i,\ 0 \le i \le 2n-1$, are coefficients to be determined such that $y(x)$ satisfies the homogeneous boundary conditions

$$y^{(j)}(\pm 1) = 0, \quad 0 \le j \le n-1, \tag{4.3}$$

then the equation (4.1) takes the form

$$(-1)^n\, y^{(2n)}(x) + \gamma\, p(x)\, y(x) = g(x), \quad x \in (-1,1),\ n \ge 1, \tag{4.4}$$

where

$$g(x) = f(x) + \sum_{i=0}^{2n-1} \eta_i\, x^i,$$

and $\eta_i,\ 0 \le i \le 2n-1$ are some constants that are determined in terms of ξ_i. For details we refer to Doha et al. [13].

In what follows, we take $p(x) = x^\mu,\ \mu \in \mathbb{Z}^{\ge 0}$, and instead of the problem (4.4) subject to (4.3), consider its integrated form:

(4.5)

$$\left.\begin{aligned} &(-1)^n\, y(x) + \gamma \int^{(2n)} x^\mu\, y(x)(dx)^{(2n)} = h(x) + \sum_{i=0}^{2n-1} \alpha_i\, x^i, \quad x \in (-1,1),\\ &y^{(j)}(\pm 1) = 0, \quad 0 \le j \le n-1, \quad h(x) = \int^{(2n)} g(x)(dx)^{(2n)}, \end{aligned}\right\}$$

where α_i are arbitrary constants, and

$$\int^{(q)} y(x)(dx)^q = \overbrace{\int\int \ldots \int}^{q\ times} y(x)\, \overbrace{dx\, dx\, \ldots dx}^{q\ times}.$$

Define the following spaces

$$\begin{aligned} S_N &= \operatorname{span}\{V_0(x), V_1(x), V_2(x), \ldots, V_N(x)\},\\ \bar{S}_N &= \operatorname{span}\{W_0(x), W_1(x), W_2(x), \ldots, W_N(x)\},\\ X_N &= \{v(x) \in S_N\ : D^j\, v(\pm 1) = 0,\ 0 \le j \le n-1\},\\ \bar{X}_N &= \{\bar{v}(x) \in \bar{S}_N\ : D^j\, \bar{v}(\pm 1) = 0,\ 0 \le j \le n-1\}. \end{aligned}$$

Then the Chebyshev third and fourth kinds Galerkin procedures for solving (4.5) consist of finding $y_N^n(x) \in X_N$ and $\bar{y}_N^n(x) \in \bar{X}_N$ to satisfy

$$\begin{aligned}(4.6)\qquad & \left((-1)^n\, y_N^n(x), v(x)\right)_{w_1(x)} + \gamma \left(\int^{(2n)} x^\mu\, y_N^n(x)(dx)^{(2n)}, v(x)\right)_{w_1(x)} \\ & = \left(h(x) + \sum_{i=0}^{2n-1} b_i\, V_i(x), v(x)\right)_{w_1(x)}, \quad 0 \le k \le N-2n, \quad \forall\, v(x) \in X_N,\end{aligned}$$

and

$$\begin{aligned}(4.7)\qquad & \left((-1)^n\, \bar{y}_N^n(x), \bar{v}(x)\right)_{w_2(x)} + \gamma \left(\int^{(2n)} x^\mu\, \bar{y}_N^n(x)(dx)^{(2n)}, \bar{v}(x)\right)_{w_2(x)} \\ & = \left(h(x) + \sum_{i=0}^{2n-1} \bar{b}_i\, W_i(x), \bar{v}(x)\right)_{w_2(x)}, \quad 0 \le k \le N-2n, \quad \forall\, \bar{v}(x) \in \bar{X}_N,\end{aligned}$$

where $w_1(x) = \sqrt{\frac{1+x}{1-x}}$, $w_2(x) = \sqrt{\frac{1-x}{1+x}}$, $(u(x), v(x))_{w_i(x)} = \int_{-1}^{1} w_i(x)\, u(x)\, v(x)\, dx$ is the inner product in the weighted space $L^2_{w_i(x)}(-1,1)$, and b_i, $\bar{b}_i$, $i = 1, 2$ are some constants.

We can construct two kinds of bases functions as compact combinations of the Chebyshev polynomials of third and fourth kinds by setting

$$(4.8)\qquad \phi_{k,n}(x) = V_k(x) + \sum_{m=1}^{2n} d_{m,k}\, V_{k+m}(x), \quad 0 \le k \le N-2n,\ n \ge 1,$$

$$(4.9)\qquad \psi_{k,n}(x) = W_k(x) + \sum_{m=1}^{2n} \bar{d}_{m,k}\, W_{k+m}(x), \quad 0 \le k \le N-2n,\ n \ge 1,$$

where the coefficients $\{d_{m,k}\}$ and $\{\bar{d}_{m,k}\}$ are chosen such that $\phi_{k,n}(x) \in X_{k+2n}$ and $\psi_{k,n}(x) \in \bar{X}_{k+2n}$. In view of relations (2.4)-(2.7), the boundary conditions (4.3) lead to the following linear system to determine the constants $\{d_{m,k}\}$:

$$\begin{cases} 1+\sum_{m=1}^{2n+1} d_{m,k}=0, \\ \sum_{m=1}^{2n+1}(-1)^m\,(2\,k+2\,m+1)\,d_{m,k}=-(2\,k+1), \\ \prod_{s=0}^{q-1}(k-s)\,(k+s+1)+\sum_{m=1}^{2\,n+1} d_{m,k}\prod_{s=0}^{q-1}(k+m-s)(k+m+s+1)=0, \\ (2\,k+1)\,\prod_{s=0}^{q-1}(k-s)\,(k+s+1)+\sum_{m=1}^{2\,n+1}(-1)^m\,d_{m,k}\,(2\,k+2m+1)\times \\ \prod_{s=0}^{q-1}(k+m-s)(k+m+s+1)=0, \\ \qquad 0\le k\le N-2n, \quad \text{and} \quad 1\le q\le n-1. \end{cases}$$

The determinant of the above system is different from zero, hence $\{d_{m,k}\}$ can be uniquely determined to obtain

$$d_{m,k}=\begin{cases} \dfrac{(-1)^{\frac{m}{2}}\binom{n}{\frac{m}{2}}\,(k+1)_{\frac{m}{2}}}{(k+n+2)_{\frac{m}{2}}}, & \text{if } m \text{ is even}, \\ \dfrac{(-1)^{\frac{m+1}{2}}\binom{n}{\frac{m-1}{2}}\,(k+1)_{\frac{m-1}{2}}}{(k+n+2)_{\frac{m-1}{2}}}, & \text{if } m \text{ is odd}, \end{cases} \tag{4.10}$$

and hence the basis functions $\phi_{k,n}(x)$ take the form:

$$\phi_{k,n}(x)=\sum_{m=0}^{n}\frac{(-1)^m\binom{n}{m}\,(k+1)_m}{(k+n+1)_m}\,V_{k+2m}(x)+ \\ \sum_{m=0}^{n-1}\frac{(-1)^m\binom{n}{m}\,(m-n)\,(k+1)_m}{(k+n+1)_{m+1}}\,V_{k+2m+1}(x). \tag{4.11}$$

Similarly, the constants $\bar{d}_{m,k}$ can be uniquely determined to obtain

$$\bar{d}_{m,k}=(-1)^m\,d_{m,k}, \tag{4.12}$$

and hence

$$\psi_{k,n}(x)=\sum_{m=0}^{n}\frac{(-1)^m\binom{n}{m}\,(k+1)_m}{(k+n+1)_m}\,W_{k+2m}(x)+ \\ \sum_{m=0}^{n-1}\frac{(-1)^{m+1}\binom{n}{m}\,(m-n)\,(k+1)_m}{(k+n+1)_{m+1}}\,W_{k+2m+1}(x). \tag{4.13}$$

It is obvious that $\{\phi_{k,n}(x)\}$ and $\{\psi_{k,n}(x)\}$ are linearly independent. Therefore we have

$$X_N = \text{span}\{\phi_{k,n}(x) : 0 \le k \le N-2n\}, \quad \bar{X}_N = \text{span}\{\psi_{k,n}(x) : 0 \le k \le N-2n\}.$$

Thus, the variational relations (4.6) and (4.7) are respectively equivalent to the following:

$$\begin{aligned}&\left((-1)^n\, y_N^n(x), \phi_{k,n}(x)\right)_{w_1(x)} + \left(\int^{(2n)} x^{\mu}\, y_N^n(x)(dx)^{(2n)}, \phi_{k,n}(x)\right)_{w_1(x)} \\ &= \left(h(x) + \sum_{i=0}^{2n-1} b_i\, V_i(x), \phi_{k,n}(x)\right)_{w_1(x)}, \quad 0 \le k \le N-2n,\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}&\left((-1)^n\, \bar{y}_N^n(x), \psi_{k,n}(x)\right)_{w_2(x)} + \left(\int^{(2n)} x^{\mu}\, \bar{y}_N^n(x)(dx)^{(2n)}, \psi_{k,n}(x)\right)_{w_2(x)} \\ &= \left(h(x) + \sum_{i=0}^{2n-1} \bar{b}_i\, W_i(x), \psi_{k,n}(x)\right)_{w_2(x)}, \quad 0 \le k \le N-2n.\end{aligned} \tag{4.15}$$

Noting that the constants $b_i,\ \bar{b}_i,\ 0 \le i \le 2n-1$, should not appear if we take $k \ge 2n$ in (4.14) and (4.15), we can write

$$\begin{aligned}&\left((-1)^n\, y_N^n(x), \phi_{k,n}(x)\right)_{w_1(x)} + \left(\int^{(2n)} x^{\mu}\, y_N^n(x)(dx)^{(2n)}, \phi_{k,n}(x)\right)_{w_1(x)} \\ &= \left(h(x), \phi_{k,n}(x)\right)_{w_1(x)}, \quad 2n \le k \le N,\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}&\left((-1)^n\, \bar{y}_N^n(x), \psi_{k,n}(x)\right)_{w_2(x)} + \left(\int^{(2n)} x^{\mu}\, \bar{y}_N^n(x)(dx)^{(2n)}, \psi_{k,n}(x)\right)_{w_2(x)} \\ &= \left(h(x), \psi_{k,n}(x)\right)_{w_2(x)}, \quad 2n \le k \le N.\end{aligned} \tag{4.17}$$

Denoting

$$h_k^n = (h(x), \phi_{k,n}(x))_{w_1(x)}, \qquad \mathbf{h}^n = (h_{2n}^n, h_{2n+1}^n, \dots, h_N^n)^T,$$

$$\bar{h}_k^n = (h(x), \psi_{k,n}(x))_{w_2(x)}, \qquad \bar{\mathbf{h}}^n = (\bar{h}_{2n}^n, \bar{h}_{2n+1}^n, \dots, \bar{h}_N^n)^T,$$

$$y_N^n(x) = \sum_{m=0}^{N-2n} c_m^n \, \phi_{m,n}(x), \qquad \mathbf{c}^n = (c_0^n, c_1^n, \dots, c_{N-2n}^n)^T,$$

$$\bar{y}_N^n(x) = \sum_{m=0}^{N-2n} \bar{c}_m^n \, \psi_{m,n}(x), \qquad \bar{\mathbf{c}}^n = (\bar{c}_0^n, \bar{c}_1^n, \dots, \bar{c}_{N-2n}^n)^T,$$

$$A_n = (a_{kj}^n)_{2n \le k,j \le N}, \qquad B_n = (b_{kj}^n)_{2n \le k,j \le N},$$

$$R_n = (r_{kj}^n)_{2n \le k,j \le N}, \qquad S_n = (s_{kj}^n)_{2n \le k,j \le N},$$

the equations (4.16) and (4.17) can be written in the following equivalent matrix forms:

$$(A_n + \gamma B_n)\, \mathbf{c}^n = \mathbf{h}^n,$$

and

$$(R_n + \gamma S_n)\, \bar{\mathbf{c}}^n = \bar{\mathbf{h}}^n,$$

where the nonzero elements of the matrices A_n, B_n, R_n and S_n are given explicitly in the following two theorems.

Theorem 4.1. *Let the basis functions $\phi_{k,n}(x)$ be defined as in* (4.11), *and let*

$$a_{kj}^n = (-1)^n \left(\phi_{j-2n,n}(x), \phi_{k,n}(x)\right)_{w_1(x)}$$

and

$$b_{kj}^n = \left(\int^{(2n)} x^\mu \, \phi_{j-2n,n}(x) (dx)^{(2n)}, \phi_{k,n}(x) \right)_{w_1(x)}.$$

Then

$$X_{N+2n} = span\{\phi_{0,n}(x), \phi_{1,n}(x), \dots, \phi_{N,n}(x)\},$$

and the nonzero elements of the matrices A_n and B_n are given by

$$a_{kj}^n = (-1)^n \pi \sum_{m=0}^{2n} d_{m,j-2n}\, d_{j-k+m-2n,k}, \tag{4.18}$$

$$b_{kj}^n = \frac{\pi}{2^\mu} \sum_{m=0}^{2n} \sum_{s=0}^{\mu} \sum_{i=0}^{4n} \binom{\mu}{s} d_{m,j-2n}\, d_{\mu+j+m-i-k-2s,k}\, E_{\mu+j+m-2n-2s,i,2n}, \tag{4.19}$$

where $d_{m,k}$ and $E_{n,i,q}$ are as in (4.10) *and* (3.8), *respectively.*

Theorem 4.2. *Let the basis functions $\psi_{k,n}(x)$ be defined as in (4.13), and let*

$$r^n_{kj} = (-1)^n \left(\psi_{j-2n,n}(x), \psi_{k,n}(x)\right)_{w_2(x)},$$

and

$$s^n_{kj} = \left(\int^{(2n)} x^{\mu}\, \psi_{j-2n,n}(x)(dx)^{(2n)}, \psi_{k,n}(x)\right)_{w_2(x)}.$$

Then

$$\bar{X}_{N+2n} = span\{\psi_{0,n}(x), \psi_{1,n}(x), \dots, \psi_{N,n}(x)\},$$

and the nonzero elements of the matrices R_n and S_n are given by

$$r^n_{kj} = (-1)^n \pi \sum_{m=0}^{2n} \bar{d}_{m,j-2n}\, \bar{d}_{j-k+m-2n,k},$$

$$s^n_{kj} = \frac{\pi}{2^{\mu}} \sum_{m=0}^{2n} \sum_{s=0}^{\mu} \sum_{i=0}^{4n} \binom{\mu}{s} \bar{d}_{m,j-2n}\, \bar{d}_{\mu+j+m-i-k-2s,k}\, S_{\mu+j+m-2n-2s,i,2n},$$

where $\bar{d}_{m,k}$ and $S_{k,n,q}$ are as in (4.12) and (3.9), respectively.

The proofs of Theorems 4.1 and 4.2 are similar, so it suffices to prove only Theorem 4.1.

Proof of Theorem 4.1. The basis functions $\phi_{k,n}(x)$ we choose such that $\phi_{k,n}(x) \in X_{N+2n}$ for $k = 0, 1, \dots, N$. On the other hand, it is clear that $\{\phi_{k,n}(x)\}_{0\leq k\leq N}$ are linearly independent and the dimension of X_{N+2n} is equal to $N+1$. Hence, we have

$$X_{N+2n} = \text{span}\{\phi_{0,n}(x), \phi_{1,n}(x), \dots, \phi_{N,n}(x)\}.$$

To obtain the nonzero elements (a^n_{kj}) for $2n \leq k, j \leq N$, we use formula (4.8) to get

$$a^n_{kj} = (-1)^n \sum_{m=0}^{2n} \sum_{i=0}^{2n} d_{m,j-2n}\, d_{i,k} \left(V_{j-2n+m}(x), V_{k+i}(x)\right)_{w_1(x)},$$

which in turn, with the aid of the orthogonality relation (2.2), yields

$$a^n_{kj} = (-1)^n \pi \sum_{m=0}^{2n} d_{m,j-2n}\, d_{j-2n+m-k,k}, \quad j = k+s, s \geq 0.$$

Thus, the formula (4.18) is proved. To prove (4.19), observe that since

$$b^n_{kj} = \left(\int^{(2n)} x^{\mu}\, \phi_{j-2n,n}(x)(dx)^{(2n)}, \phi_{k,n}(x)\right)_{w_1(x)},$$

we can use the formulas (2.11) and (4.8) to obtain

$$x^{\mu}\,\phi_{j-2n}(x)=\frac{1}{2^{\mu}}\sum_{m=0}^{2n}\sum_{s=0}^{\mu}\binom{\mu}{s}d_{m,j-2n}\;V_{\mu+j+m-2n-2s}(x),$$

and therefore relation (3.7) gives

$$\int^{(2n)} x^{\mu}\,\phi_{j-2n,n}(x)(dx)^{(2n)}$$
$$=\frac{1}{2^{\mu}}\sum_{m=0}^{2n}\sum_{s=0}^{\mu}\sum_{i=0}^{4n}\binom{\mu}{s}\;d_{m,j-2n}\,E_{\mu+j+m-2n-2s,i,2n}\,V_{\mu+j+m-i-2s}(x).$$

Finally, using the orthogonality relation (2.2), we obtain

$$b_{kj}^{n}=\frac{\pi}{2^{\mu}}\sum_{m=0}^{2n}\sum_{s=0}^{\mu}\sum_{i=0}^{4n}\binom{\mu}{s}\;d_{m,j-2n}\,d_{\mu+j+m-i-k-2s,k}\,E_{\mu+j+m-2n-2s,i,2n}.$$

This completes the proof of Theorem 4.1. □

5. Numerical results

In this section we give a numerical example to show the accuracy and the efficiency of the proposed algorithms.

Example 1. *Consider the following linear sixth-order boundary value problem (see, Siddiqi and Akram [23]):*

(5.1)

$$y^{(6)}(x)+(5x+1)\,y(x)=(185\,x-25\,x^2+10\,x^4)\cos(x)+(270-36\,x^2)\,\sin(x),\quad x\in[-1,1],$$

subject to the boundary conditions:

$$\begin{aligned} &y(-1)=4\cos(1), && y(1)=-2\cos(1),\\ &y^{(1)}(-1)=\cos(1)+4\sin(1), && y^{(1)}(1)=\cos(1)+2\sin(1),\\ &y^{(2)}(-1)=-16\cos(1)+2\sin(1), && y^{(2)}(1)=14\cos(1)-2\sin(1).\end{aligned}$$

The analytical solution of this problem is given by

$$y(x)=(2x^3-5x+1)\cos(x).$$

Table 1 below contains the maximum pointwise error E of $|u-u_N|$ using our algorithms C3GM and C4GM for various values of N, while Table 2 contains the

TABLE 1. Maximum pointwise error for Example 1, $N = 14, 16, 18, 20, 22$.

N	C3GM	C4GM
14	2.16553×10^{-9}	2.16799×10^{-9}
16	7.20557×10^{-12}	7.21269×10^{-12}
18	1.65128×10^{-14}	1.70084×10^{-14}
20	1.3765×10^{-15}	1.38995×10^{-15}

TABLE 2. Comparison between best error for Example 1 by different methods

Best error	C3GM	C4GM	Siddiqi and Akram [23]
E	1.3765×10^{-15}	1.38995×10^{-15}	8.68×10^{-7}

best errors obtained by our methods (C3GM and C4GM) and by the septic spline method developed in [23].

Comparing the errors given in Table 2, we conclude that our two methods, C3GM and C4GM, are more accurate than the method developed in [23].

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