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ON THE CONVERGENCE AND SUMMABILITY OF DOUBLE WALSH-FOURIER SERIES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

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Abstract. [1]The convergence of partial sums and Cesáro means of negative order of double Walsh-Fourier series of functions of bounded generalized variation is investigated.

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1. Classes of functions of bounded generalized variation

In 1881 C. Jordan [17] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation etc. (see, e.g., [2, 18, 27, 29]). In two dimensional case the class of functions of bounded variation (BV) was introduced by G. Hardy [16].

Let f be a real and measurable function of two variables on the unit square. Given intervals $\Delta = (a,b)$, $J = (c,d)$ and points x, y from $I := [0,1)$ we denote

$$f(\Delta, y) = f(b,y) - f(a,y), \qquad f(x, J) = f(x,d) - f(x,c)$$

and

$$f(\Delta, J) = f(a,c) - f(a,d) - f(b,c) + f(b,d).$$

Let $E = \{\Delta_i\}$ be a collection of nonoverlapping intervals from I ordered in arbitrary way and let Ω be the set of all such collections E. Denote by Ω_n the set of all collections of n nonoverlapping intervals $I_k \subset I$.

For the sequences of positive numbers

$$\Lambda^1 = \{\lambda_n^1\}_{n=1}^{\infty}, \qquad \Lambda^2 = \{\lambda_n^2\}_{n=1}^{\infty}$$

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and $I^2 = [0,1)^2$ we denote

$$\Lambda^1 V_1(f; I^2) = \sup_y \sup_{E\in\Omega} \sum_i \frac{|f(\Delta_i, y)|}{\lambda_i^1} \quad (E = \{\Delta_i\}),$$

$$\Lambda^2 V_2(f; I^2) = \sup_x \sup_{F\in\Omega} \sum_j \frac{|f(x, J_j)|}{\lambda_j^2} \qquad (F = \{J_j\}),$$

$$(\Lambda^1\Lambda^2)\, V_{1,2}(f; I^2) = \sup_{F,E\in\Omega} \sum_i \sum_j \frac{|f(\Delta_i, J_j)|}{\lambda_i^1 \lambda_j^2}.$$

Definition 1.1. *We say that a function f has bounded (Λ^1, Λ^2)-variation on I^2 and write $f \in (\Lambda^1, \Lambda^2)\, BV\, (I^2)$, if*

$$(\Lambda^1, \Lambda^2)\, V(f; I^2) := \Lambda^1 V_1(f; I^2) + \Lambda^2 V_2(f; I^2) + (\Lambda^1\Lambda^2)\, V_{1,2}(f; I^2) < \infty.$$

If $\Lambda^1 = \Lambda^2 = \Lambda$, then we say that f has bounded Λ-variation on I^2 and use the notation $\Lambda BV(I^2)$.

We say that a function f has bounded partial Λ-variation on I^2 and write $f \in P\Lambda BV\, (I^2)$, if

$$P\Lambda BV(f; I^2) := \Lambda V_1(f; I^2) + \Lambda V_2(f; I^2) < \infty.$$

If $\Lambda = \{\lambda_n\}$ with $\lambda_n \equiv 1$, or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \ldots$, the classes ΛBV and $P\Lambda BV$ coincide, respectively, with the Hardy class BV and with the class PBV functions of bounded partial variation introduced by Goginava [6]. Hence it is reasonable to assume that $\lambda_n \to \infty$ and since the intervals in $E = \{\Delta_i\}$ are ordered arbitrarily, we can assume, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus, we assume that

$$(1.1) \qquad 1 < \lambda_1 \le \lambda_2 \le \ldots, \qquad \lim_{n\to\infty} \lambda_n = \infty, \qquad \sum_{n=1}^{\infty} (1/\lambda_n) = +\infty.$$

In the case where $\lambda_n = n$, $n = 1, 2 \ldots$ we will use the term *harmonic variation* instead of Λ-variation and will write H instead of Λ, that is, HBV, $PHBV$, $HV(f)$, etc.

The notion of Λ-variation was introduced by Waterman [27] in one dimensional case, and by Sahakian [23] in two dimensional case; the notion of bounded partial Λ-variation ($P\Lambda BV$) was introduced by Goginava and Sahakian [12].

Dyachenko and Waterman [5] introduced another class of functions of generalized bounded variation. Denoting by Γ the set of finite collections of nonoverlapping rectangles $A_k := [\alpha_k, \beta_k] \times [\gamma_k, \delta_k] \subset I^2$, we define

$$\Lambda^* V_{1,2}(f) := \sup_{\{A_k\} \in \Gamma} \sum_k \frac{|f(A_k)|}{\lambda_k}.$$

Definition 1.2 (Dyachenko and Waterman, [5]). *Let f be a real function on I^2. We say that $f \in \Lambda^* BV$, if*

$$\Lambda^* V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda^* V_{1,2}(f) < \infty.$$

In [13], the authors have introduced new classes of functions of generalized bounded variation and investigated the convergence of Fourier series of functions from that classes.

For the sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we define

$$\Lambda^{\#} V_1(f) := \sup_{\{y_i\} \subset I} \sup_{\{I_i\} \in \Omega} \sum_i \frac{|f(I_i, y_i)|}{\lambda_i},$$

$$\Lambda^{\#} V_2(f) := \sup_{\{x_j\} \subset I} \sup_{\{J_j\} \in \Omega} \sum_j \frac{|f(x_j, J_j|}{\lambda_j}.$$

Definition 1.3. *We say that a function f belongs to the class $\Lambda^{\#} BV$, if*

$$\Lambda^{\#} V(f) = \Lambda^{\#} V_1(f) + \Lambda^{\#} V_2(f) < \infty.$$

The notion of continuity of functions in Λ-variation plays an important role in the study of convergence of Fourier series of functions of bounded Λ-variation.

Definition 1.4. *We say that a function f is continuous in (Λ^1, Λ^2)-variation on I^2 and write $f \in C(\Lambda^1, \Lambda^2) V$, if*

$$\lim_{n \to \infty} \Lambda_n^1 V_1(f) = \lim_{n \to \infty} \Lambda_n^2 V_2(f) = 0$$

and

$$\lim_{n \to \infty} (\Lambda_n^1, \Lambda^2) V_{1,2}(f) = \lim_{n \to \infty} (\Lambda^1, \Lambda_n^2) V_{1,2}(f) = 0,$$

where $\Lambda_n^i := \{\lambda_k^i\}_{k=n}^{\infty} = \{\lambda_{k+n}^i\}_{k=0}^{\infty}$, $i = 1, 2$.

Definition 1.5. *A function f is continuous in $\Lambda^{\#}$-variation on I^2 and write $f \in C\Lambda^{\#} V$, if*

$$\lim_{n \to \infty} \Lambda_n^{\#} V(f) = 0,$$

where $\Lambda_n = \{\lambda_k\}_{k=n}^{\infty}$.

Definition 1.6. *We say that a function f is continuous in Λ^*-variation on I^2 and write $f \in C\Lambda^*V$, if*

$$\lim_{n\to\infty} \Lambda_n V_1(f) = \lim_{n\to\infty} \Lambda_n V_2(f) = 0; \quad \lim_{n\to\infty} \Lambda_n^* V_{1,2}(f) = 0.$$

Now, we define

$$v_1^{\#}(n,f) := \sup_{\{y_i\}_{i=1}^n} \sup_{\{I_i\}\in\Omega_n} \sum_{i=1}^{n} |f(I_i, y_i)|, \quad n = 1,2,\ldots,$$

$$v_2^{\#}(m,f) := \sup_{\{x_j\}_{j=1}^m} \sup_{\{J_k\}\in\Omega_m} \sum_{j=1}^{m} |f(x_j, J_j)|, \quad m = 1,2,\ldots.$$

The following theorems hold.

Theorem 1.1 (Goginava, Sahakian [13]). $\left\{\frac{n}{\log n}\right\}^{\#} BV \subset HBV$.

Theorem 1.2 (Goginava, Sahakian [13]). *Suppose*

$$\sum_{n=1}^{\infty} \frac{v_s^{\#}(f;n)\log(n+1)}{n^2} < \infty, \quad s = 1,2.$$

Then $f \in \left\{\frac{n}{\log(n+1)}\right\}^{\#} BV$.

Theorem 1.3 (Goginava [10]). *Let $\alpha, \beta \in (0,1)$, $\alpha+\beta<1$ and*

$$\sum_{j=1}^{\infty} \frac{v_s^{\#}(f;2^j)}{2^{j(1-(\alpha+\beta))}} < \infty, \quad s = 1,2.$$

Then $f \in C\left\{n^{1-(\alpha+\beta)}\right\}^{\#} V$.

Theorem 1.4 (Goginava [10]). *Let $\alpha, \beta \in (0,1)$ and $\alpha+\beta<1$. Then*

$$C\left\{i^{1-(\alpha+\beta)}\right\}^{\#} V \subset C\left\{i^{1-\alpha}\right\}\left\{j^{1-\beta}\right\} V.$$

The next theorem shows, that for some sequences Λ the classes $\Lambda^{\#}V$ and $C\Lambda^{\#}V$ coincide.

Theorem 1.5. *Let the sequence $\Lambda = \{\lambda_n\}$ be as in (1.1) and*

$$\liminf_{n\to\infty} \frac{\lambda_{2n}}{\lambda_n} = q > 1. \tag{1.2}$$

Then $\Lambda^{\#}V = C\Lambda^{\#}V$.

Proof. Assume the opposite, that there exists a function $f \in \Lambda^{\#}V$ for which $\liminf_{n\to\infty} \Lambda_n^{\#}V(f) > 0$ (see Definition 1.5). Without loss of generality, we can assume that $\liminf_{n\to\infty} \Lambda_n^{\#}V_1(f) =$

$\delta > 0$ and that $\delta = 1$. Then, taking into account that the sequence $\{\Lambda_n^{\#} V_1(f)\}$ is decreasing, we have

$$\lim_{n\to\infty} \Lambda_n^{\#} V_1(f) = 1. \tag{1.3}$$

Let a natural k and the numbers $\varepsilon > 0$, $q_0 \in (1, q)$ be fixed.

According to (1.2) and (1.3) there exist a natural $N' > k$ such that

$$\frac{\lambda_{2n}}{\lambda_n} > q_0, \quad \Lambda_n^{\#} V_1(f) > 1 - \varepsilon \quad \text{for} \quad n \geq N'. \tag{1.4}$$

Then for a natural $N > 2N'$ there are a set of points $\{y_i\}_{i=1}^{2i_0}$ and a set of nonoverlapping intervals $\{\delta_i\}_{i=1}^{2i_0} \in \Omega$ such that

$$I := \sum_{i=1}^{2i_0} \frac{|f(\delta_i, y_i)|}{\lambda_{N+i}} \geq 1 - \varepsilon. \tag{1.5}$$

Adding, if necessary, new terms in (1.5) we can assume that

$$\bigcup_{i=1}^{2i_0} \delta_i = (0, 1).$$

Denote

$$I_1 := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i-1}, y_{2i-1})|}{\lambda_{N+2i-1}}, \qquad I_2 := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i}, y_{2i})|}{\lambda_{N+2i}}. \tag{1.6}$$

Since $N > 2N'$ implies that $N + 2i - 1 \geq 2(N' + i)$, from (1.4) and (1.6) we have

$$I_1' := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i-1}, y_{2i-1})|}{\lambda_{N'+i}} = \sum_{i=1}^{i_0} \frac{|f(\delta_{2i-1}, y_{2i-1})|}{\lambda_{N+2i-1}} \cdot \frac{\lambda_{N+2i-1}}{\lambda_{N'+i}} > q_0 I_1 \tag{1.7}$$

and

$$I_2' := \sum_{i=1}^{i_0} \frac{|f(\delta_{2i}, y_{2i})|}{\lambda_{N'+i}} = \sum_{i=1}^{i_0} \frac{|f(\delta_{2i}, y_{2i})|}{\lambda_{N+2i}} \cdot \frac{\lambda_{N+2i}}{\lambda_{N'+i}} > q_0 I_2. \tag{1.8}$$

Consequently, by (1.5) we get

$$I' := I_1' + I_2' \geq q_0(I_1 + I_2) = q_0 I \geq q_0(1 - \varepsilon). \tag{1.9}$$

Now, we take a natural M to satisfy

$$M > N + 2(i_0 + 1) \quad \text{and} \quad \frac{2(2i_0 + 1)}{\lambda_M} \sup_{x\in[0,1]} |f(x)| < \varepsilon, \tag{1.10}$$

and hence using (1.4), we can find a set of points $\{z_j\}_{j=1}^{j_0}$ and a set of nonoverlapping intervals $\{\Delta_j\}_{j=1}^{j_0} \in \Omega$ such that

$$\sum_{j=1}^{j_0} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \geq 1 - \varepsilon. \tag{1.11}$$

Denote by Q the set of indices $j = 1, 2, \cdots, j_0$ for which the corresponding interval Δ_j does not contain an endpoint of the intervals $\delta_i, i = 1, 2, \ldots, 2i_0$, that is, Δ_j lies in one of the intervals $\delta_i, i = 1, 2, \ldots, 2i_0$. Then the number of indices in $[1, j_0] \setminus Q$ does not exceed $2i_0 + 1$, and by (1.10) we get

$$\sum_{j \in [1, j_0] \setminus Q} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \leq \varepsilon.$$

Consequently, by (1.11) we have

$$J := \sum_{j \in Q} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}} \geq 1 - 2\varepsilon. \tag{1.12}$$

Denoting

$$Q_1 = \left\{ j \in Q : \Delta_j \subset \bigcup_{i=1}^{i_0} \delta_{2i-1} \right\}, \qquad Q_2 = \left\{ j \in Q : \Delta_j \subset \bigcup_{i=1}^{i_0} \delta_{2i} \right\}$$

and

$$J_1 := \sum_{j \in Q_1} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}}, \qquad J_2 := \sum_{j \in Q_2} \frac{|f(\Delta_j, z_j)|}{\lambda_{M+j}},$$

from (1.9) and (1.12) we obtain

$$(I_1' + J_2) + (I_2' + J_1) = I' + J \geq q_0(1 - \varepsilon) + 1 - 2\varepsilon \geq q_0 + 1 - 3\varepsilon.$$

Theretofore

$$I_1' + J_2 \geq \frac{q_0 + 1 - 3\varepsilon}{2} \quad \text{or} \quad (I_2' + J_1) \geq \frac{q_0 + 1 - 3\varepsilon}{2},$$

which means that

$$\Lambda_{N'}^{\#} V_1(f) \geq \frac{q_0 + 1 - 3\varepsilon}{2},$$

and hence

$$\Lambda_k^{\#} V_1(f) \geq \frac{q_0 + 1}{2},$$

since ε is any positive number and $N' > k$. Taking into account that k is an arbitrary natural number, the last inequality implies

$$\lim_{n \to \infty} \Lambda_n^{\#} V_1(f) \geq \frac{q_0 + 1}{2} > 1,$$

which contradicts the assumption (1.3), and the result follows. Theorem 1.5 is proved.

It is easy to see that for any $\gamma > 0$ the sequence $\lambda_n = n^\gamma$, $n = 1, 2, \ldots$ satisfies the condition (1.2) with $q = 2^\gamma$. Hence Theorem 1.5 implies the following result.

Corollary 1.1. *If* $0 < \gamma \leq 1$, *then* $\{n^\gamma\}^{\#} V = C\{n^\gamma\}^{\#} V$.

This, combined with Theorem 1.4 implies the next result.

Corollary 1.2. *Let $\alpha, \beta \in (0,1)$ and $\alpha + \beta < 1$. Then*

$$\left\{i^{1-(\alpha+\beta)}\right\}^{\#} V \subset C\left\{i^{1-\alpha}\right\}\left\{j^{1-\beta}\right\} V.$$

2. Walsh functions

Let $\mathbb{P}$ be the set of positive integers, and $\mathbb{N}=\mathbb{P}\cup\{0\}$. We denote the set of all integers by $\mathbb{Z}$ and the set of dyadic rational numbers in the unit interval $I=[0,1)$ by $\mathbb{Q}$. Each element of $\mathbb{Q}$ is of the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N},\ 0 \le p \le 2^n$. By a dyadic interval in I we mean an interval of the form $I_N^l := [l2^{-N}, (l+1)\,2^{-N})$ for some $l \in \mathbb{N}, 0 \le l < 2^N$. Given $N \in \mathbb{N}$ and $x \in I$, we denote by $I_N(x)$ the dyadic interval of length 2^{-N} that contains x. Finally, we set $I_N := [0, 2^{-N})$ and $\overline{I}_N := I \backslash I_N$.

Let $r_0(x)$ be the following function

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0,1/2) \\ -1, & \text{if } x \in [1/2,1) \end{cases}, \qquad r_0(x+1) = r_0(x), \quad x \in \mathbb{R}.$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad x \in I, \qquad n = 1, 2, \ldots.$$

The Walsh functions $w_0, w_1, \ldots$ are defined as follows. Denote $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s \ge 0$, then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x), \quad n = 1, 2, \ldots$$

Recall that (see [15, 25])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 2^{-n}) \\ 0, & \text{if } x \in [2^{-n}, 1) \end{cases} \tag{2.1}$$

and

$$D_{2^n+m}(x) = D_{2^n}(x) + w_{2^n}(x)\, D_m(x), \quad 0 \le m < 2^n, \quad n = 0, 1, \ldots \tag{2.2}$$

It is well known that (see [25])

$$D_n(t) = w_n(t) \sum_{j=0}^{\infty} n_j w_{2^j}(t)\, D_{2^j}(t), \quad \text{if} \quad n = \sum_{j=0}^{\infty} n_j 2^j \tag{2.3}$$

and

$$|D_{q_n}(x)| \ge \frac{1}{4x}, \qquad 2^{-2n-1} \le x < 1, \tag{2.4}$$

where

(2.5)
$$q_n := 2^{2n-2} + 2^{2n-4} + \cdots + 2^2 + 2^0.$$

Given $x \in I$, the expansion

(2.6)
$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)},$$

where each $x_k = 0$ or 1, is called a dyadic expansion of x. If $x \in I \backslash \mathbf{Q}$, then (2.6) is uniquely determined. For $x \in \mathbf{Q}$ we choose the dyadic expansion with $\lim_{k\to\infty} x_k = 0$. The dyadic sum of $x, y \in I$ in terms of the dyadic expansion of x and y is defined by

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

We say that $f(x, y)$ is continuous at (x, y) if

(2.7)
$$\lim_{h,\delta\to 0} f(x \dotplus h, y \dotplus \delta) = f(x, y).$$

We consider the double system $\{w_n(x) \times w_m(y) : n, m \in \mathbf{N}\}$ on the unit square $I^2 = [0,1) \times [0,1)$.

If $f \in L^1(I^2)$, then

$$\hat{f}(n, m) = \int_{I^2} f(x, y) \, w_n(x) w_m(y) dx dy$$

is the (n, m)-th Walsh-Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}(x, y; f) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m, n) \, w_m(x) w_n(y).$$

The Cesàro $(C; \alpha, \beta)$-means of double Walsh-Fourier series are defined as follows

$$\sigma_{n,m}^{\alpha,\beta}(x, y; f) = \frac{1}{A_{n-1}^{\alpha} A_{m-1}^{\beta}} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{\alpha-1} A_{m-j}^{\beta-1} S_{i,j}(x, y; f),$$

where

$$A_0^{\alpha} = 1, \quad A_n^{\alpha} = \frac{(\alpha+1)\cdots(\alpha+n)}{n!}, \quad \alpha \neq -1, -2, \ldots.$$

It is well-known that (see [30])

(2.8)
$$A_n^{\alpha} = \sum_{k=0}^{n} A_{n-k}^{\alpha-1},$$

(2.9)
$$A_n^{\alpha} \sim n^{\alpha}$$

and

$$\sigma_{n,m}^{\alpha,\beta}(x,y;f)=\int\limits_{I^2} f(s,t)\,K_n^{\alpha}(x\dotplus s)\,K_m^{\beta}(y\dotplus t)\,dsdt, \tag{2.10}$$

where

$$K_n^{\alpha}(x):=\frac{1}{A_{n-1}^{\alpha}}\sum_{k=1}^{n}A_{n-k}^{\alpha-1}D_k(x). \tag{2.11}$$

3. Convergence of two-dimensional Walsh-Fourier series

The well known Dirichlet-Jordan theorem (see [30]) states that the Fourier series of a function $f(x)$, $x\in T$ of bounded variation converges at every point x to the value $[f(x+0)+f(x-0)]/2$.

Hardy [16] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if a function $f(x,y)$ has bounded variation in the sense of Hardy ($f\in BV$), then $S[f]$ converges at any point (x,y) to the value $\frac{1}{4}\sum f(x\pm 0,y\pm 0)$. Here and below we consider the convergence of **only rectangular partial sums** of double Fourier series.

Convergence of d-dimensional trigonometric Fourier series of functions of bounded Λ-variation was investigated in details by Sahakian [23], Dyachenko [3, 4, 5], Bakhvalov [1], Sablin [22], Goginava, Sahakian [12, 13], and others.

For the d-dimensional Walsh-Fourier series the convergence of partial sums of functions of bounded harmonic variation and other bounded generalized variations were studied by Moricz [19, 20], Onnewer, Waterman [21], and Goginava [7].

In the two-dimensional case the following result is known.

Theorem 3.1 (Sargsyan [24]). *If $f\in HBV(I^2)$, then the double Walsh-Fourier series of f converges to $f(x,y)$ at any point $(x,y)\in I^2$ where f is continuous.*

The authors have investigated convergence of multiple Walsh-Fourier series of functions of partial Λ-bounded variation. In particular, the following result was proved in [14].

Theorem 3.2 (Goginava, Sahakian [14]). *The following assertions hold:*

a) *If $f\in P\{\frac{n}{\log^{1+\varepsilon}n}\}BV(I^2)$ for some $\varepsilon>0$, then the double Walsh-Fourier series of f converges to $f(x,y)$ at any point $(x,y)\in I^2$ where f is continuous.*

b) *There exists a continuous function $f\in P\{\frac{n}{\log n}\}BV(I^2)$ such that the quadratic partial sums of its Walsh-Fourier series diverge at some point.*

The next theorem contains a similar result for functions of bounded $\Lambda^{\#}$-variation.

Theorem 3.3. *The following assertions hold:*

a) *If $f \in \left\{\frac{n}{\log n}\right\}^{\#} BV$, then the double Walsh-Fourier series of f converges to $f(x,y)$ at any point (x,y) where f is continuous.*

b) *For an arbitrary sequence $\alpha_n \to \infty$ there exists a continuous function $f \in \left\{\frac{n\alpha_n}{\log(n+1)}\right\}^{\#} BV$ such that the quadratic partial sums of its Walsh-Fourier series diverge unboundedly at $(0,0)$.*

Proof. The assertion a) immediately follows from Theorems 1.1 and 3.1.

To prove the assertion b), observe first that for any sequence $\Lambda = \{\lambda_n\}$ satisfying (1.1) the class $C\left(I^2\right) \bigcap \Lambda^{\#}BV$ is a Banach space with the norm

$$\|f\|_{\Lambda^{\#}BV} = \|f\|_C + \Lambda^{\#}BV\,(f),$$

and $S_{N,N}(0,0,f)$, $n = 1, 2, \ldots$, is a sequence of bounded linear functionals on that space. Denote

$$\varphi_{N,j}(x) = \begin{cases} 2^{2N+1}x - 2j, & \text{if } x \in \left[j2^{-2N}, (2j+1)\,2^{-2N-1}\right], \\ -\left(2^{2N+1}x - 2j - 2\right), & \text{if } x \in \left[(2j+1)\,2^{-2N-1}, (j+1)\,2^{-2N}\right], \\ 0, & \text{if } x \in I \setminus \left[j2^{-2N}, (j+1)\,2^{2N}\right], \end{cases}$$

$$\varphi_N(x) = \sum_{j=1}^{2^{2N}-1} \varphi_{N,j}(x), \qquad x \in I, \tag{3.1}$$

$$g_N(x,y) = \varphi_N(x)\,\varphi_N(y)\,\mathrm{sgn}D_{q_N}(x)\,\mathrm{sgn}D_{q_N}(y), \qquad x, y \in I,$$

where q_N is defined in (2.5).

Suppose $\Lambda = \left\{\lambda_n = \frac{n\alpha_n}{\log(n+1)}\right\}_{n=1}^{\infty}$, where $\alpha_n \to \infty$. It is easy to show that for $s = 1, 2$

$$\Lambda^{\#}V_s(g_N) \le c \sum_{i=1}^{2^{2N}-1} \frac{\log(i+1)}{i\alpha_i} = o\left(N^2\right) \text{ as } N \to \infty.$$

Therefore $\|g_N\|_{\Lambda^{\#}BV} = o\left(N^2\right) = \eta_N N^2$, where $\eta_N \to 0$ as $N \to \infty$. Hence, denoting $G_N = \frac{g_N}{\eta_N N^2}$, we conclude that $G_N \in \Lambda^{\#}BV$ and

$$\sup_N \|G_N\|_{\Lambda^{\#}BV} < \infty. \tag{3.2}$$

By construction of the function G_N we have

$$S_{q_N,q_N}(0,0;G_N) = \iint_{I^2} G_N(x,y)\, D_{q_N}(x)\, D_{q_N}(y)\, dxdy$$

$$(3.3) \qquad = \frac{1}{N^2\eta_N}\iint_{I^2} \varphi_N(x)\,\varphi_N(y)\,|D_{q_N}(x)|\,|D_{q_N}(y)|\, dxdy$$

$$= \frac{1}{N^2\eta_N}\left(\int_I \varphi_N(x)\,|D_{q_N}(x)|\, dx\right)^2$$

Next, using (2.4), we can write

$$\int_I \varphi_N(x)\,|D_{m_N}(x)|\, dx = \sum_{j=1}^{2^{2N}-1}\int_{j2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j}(x)\,|D_{m_N}(x)|\, dx =$$

$$= \sum_{j=1}^{2^{2N}-1}\left|D_{m_N}\left(\frac{j}{2^{2N}}\right)\right|\int_{j2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j}(x)\, dx \ge \frac{1}{2^{2N+1}}\sum_{j=1}^{2^{2N}-1}\frac{2^{2N}}{4j} \ge cN.$$

Consequently, from (3.3) we obtain

$$(3.4) \qquad |S_{q_N,q_N}(0,0;G_N)| \ge \frac{c}{\eta_N} \to \infty \quad \text{as} \quad N\to\infty.$$

According to the Banach-Steinhaus Theorem, (3.2) and (3.4) imply that there exists a continuous function $f \in \left\{\frac{n\alpha_n}{\log(n+1)}\right\}^{\#} BV$ such that

$$\sup_N |S_{N,N}(0,0;f)| = +\infty.$$

Theorem 3.3 is proved. □

As an immediate consequence of Theorems 1.2 and 3.3 we have the following result.

Theorem 3.4. *Let the function $f(x,y)$, $(x,y)\in I^2$, satisfy the condition*

$$\sum_{n=1}^{\infty}\frac{v_s^{\#}(f,n)\log(n+1)}{n^2} < \infty, \quad s=1,2.$$

Then the double Walsh-Fourier series of f converges to $f(x,y)$ at any point $(x,y)\in I^2$ where f is continuous.

4. Cesáro means of negative order for two-dimensional Walsh-Fourier series

The problem of summability of Cesáro means of negative order for one dimensional Walsh-Fourier series was studied in the papers [8], [26]. In the two-dimensional case the summability of Walsh-Fourier series by Cesáro method of negative order for functions of partial bounded variation was investigated by the first author in [9], [11]. In particular, the following results were obtained.

Theorem 4.1 (Goginava [9]). *Let $f \in C_w(I^2) \cap PBV$ and $\alpha, \beta > 0$, $\alpha+\beta < 1$. Then the double Walsh-Fourier series of the function f is uniformly $(C; -\alpha, -\beta)$ summable in the sense of Pringsheim.*

Theorem 4.2 (Goginava [9]). *Let $\alpha, \beta > 0$, $\alpha+\beta \geq 1$. Then there exists a continuous function $f_0 \in PBV$ such that the Cesàro $(C; -\alpha, -\beta)$ means $\sigma_{n,n}^{-\alpha,-\beta}(0,0; f_0)$ of the double Walsh-Fourier series of f_0 diverge.*

Theorem 4.3 (Goginava [11]). *Let $f \in C\left(\{i^{1-\alpha}\}, \{i^{1-\beta}\}\right) V(I^2)$ with $\alpha, \beta \in (0,1)$. Then the $(C, -\alpha, -\beta)$-means of double Walsh-Fourier series converge to $f(x,y)$, if f is continuous at (x,y).*

Theorem 4.4 (Goginava [11]). *Let $\alpha, \beta \in (0,1)$, $\alpha+\beta < 1$. The following assertions hold:*

a) *If $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log^{1+\varepsilon}(n+1)}\right\} BV(I^2)$ for some $\varepsilon > 0$, then the double Walsh-Fourier series of the function f is $(C; -\alpha, -\beta)$ summable to $f(x,y)$, if f is continuous at (x,y).*

b) *There exists a continuous function $f \in P\left\{\frac{n^{1-(\alpha+\beta)}}{\log(n+1)}\right\} BV(I^2)$ such that the means $\sigma_{2^n,2^n}^{-\alpha,-\beta}(0,0; f)$ diverge.*

In this paper we prove the following result.

Theorem 4.5. *The following assertions hold:*

a) *Let $\alpha, \beta \in (0,1)$, $\alpha+\beta < 1$ and $f \in \left\{n^{1-(\alpha+\beta)}\right\}^{\#} BV$. Then the means $\sigma_{n,m}^{-\alpha,-\beta}(x,y; f)$ converge to $f(x,y)$, if f is continuous at (x,y).*

b) *Let $\Lambda := \left\{n^{1-(\alpha+\beta)}\xi_n\right\}$, where $\xi_n \uparrow \infty$ as $n \to \infty$. Then there exists a function $f \in C(I^2) \cap C\Lambda^{\#}V$ for which the $(C; -\alpha, -\beta)$-means of double Walsh-Fourier series diverge unboundedly at $(0,0)$.*

Proof. The assertion a) immediately follows from Corollary 1.2 and Theorem 4.3. To prove part b) of the theorem, observe first that

$$\left\{n^{1-(\alpha+\beta)}\sqrt{\xi_n}\right\}^{\#} BV \subset C\left\{n^{1-(\alpha+\beta)}\xi_n\right\}^{\#} V,$$

and since $\xi_n \uparrow \infty$ is arbitrary, it is enough to show that there exists a continuous function $f \in \Lambda^{\#}BV$ for which $(C; -\alpha, -\beta)$-means of double Walsh-Fourier series diverge unboundedly at $(0,0)$.

Denote

$$h_N(x,y) := \varphi_N(x)\varphi_N(y)\,\mathrm{sgn}K_{2^{2N}}^{-\alpha}(x)\,\mathrm{sgn}K_{2^{2N}}^{-\beta}(y),$$

where φ_N is defined in (3.1), and the kernel K_n^α is defined in (2.11). It is easy to show that for $s = 1, 2$ and $N \to \infty$ we have

$$\left\{n^{1-(\alpha+\beta)}\xi_n\right\}^{\#} V_s(h_N) \le c(\alpha,\beta) \sum_{i=1}^{2^{2N}-1} \frac{1}{i^{1-(\alpha+\beta)}\xi_i} = o\left(2^{2N(\alpha+\beta)}\right).$$

Hence

$$\|h_N\|_{\Lambda^{\#}BV} = o\left(2^{2N(\alpha+\beta)}\right) = \eta_N 2^{2N(\alpha+\beta)},$$

where $\eta_N = o(1)$ as $N \to \infty$. Consequently, denoting

$$H_N(x,y) = \frac{h_N(x,y)}{\eta_N 2^{2N(\alpha+\beta)}},$$

we conclude that $H_N \in C(I^2) \cap \Lambda^{\#}BV$ and

$$\sup_N \|H_N\|_{\Lambda^{\#}BV} < \infty. \tag{4.1}$$

By construction of the function H_N, we have

$$\sigma_{2^{2N},2^{2N}}^{-\alpha,-\beta}(0,0;H_N) = \iint_{I^2} H_N(x,y)\,K_{2^{2N}}^{-\alpha}(x)\,K_{2^{2N}}^{-\beta}(y)\,dxdy$$

$$= \frac{1}{\eta_N 2^{2N(\alpha+\beta)}} \iint_{I^2} h_N(x,y)\,K_{2^{2N}}^{-\alpha}(x)\,K_{2^{2N}}^{-\beta}(y)\,dxdy \tag{4.2}$$

$$= \frac{1}{\eta_N 2^{2N(\alpha+\beta)}} \int_I \varphi_N(x)\left|K_{2^{2N}}^{-\alpha}(x)\right| dx \int_I \varphi_N(y)\left|K_{2^{2N}}^{-\beta}(y)\right| dy.$$

Now, using the following estimate from [26]:

$$\int_{2^{m-N-1}}^{2^{m-N}} \left|K_{2^N}^{-\alpha}(x)\right| dx \ge c(\alpha)\,2^{m\alpha}, \quad N\in\mathbb{N}, \quad m = 1,\dots,N, \quad 0<\alpha<1,$$

we can write

$$\int_I \varphi_N(x)\left|K_{2^{2N}}^{-\alpha}(x)\right| dx = \sum_{j=1}^{2^{2N}-1} \int_{j2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j}(x)\left|K_{2^{2N}}^{-\alpha}(x)\right| dx \tag{4.3}$$

$$= \sum_{j=1}^{2^{2N}-1} \left|K_{2^{2N}}^{-\alpha}\left(\frac{j}{2^{2N}}\right)\right| \int_{j2^{-2N}}^{(j+1)2^{-2N}} \varphi_{N,j}(x)\,dx = \frac{1}{2}\sum_{j=1}^{2^{2N}-1} \left|K_{2^{2N}}^{-\alpha}\left(\frac{j}{2^{2N}}\right)\right| \int_{j2^{-2N}}^{(j+1)2^{-2N}} dx$$

$$= \frac{1}{2}\sum_{j=1}^{2^{2N}-1} \int_{j2^{-2N}}^{(j+1)2^{-2N}} \left|K_{2^{2N}}^{-\alpha}(x)\right| dx = \frac{1}{2}\sum_{m=0}^{2N-1}\sum_{j=2^m}^{2^{m+1}-1} \int_{j2^{-2N}}^{(j+1)2^{-2N}} \left|K_{2^{2N}}^{-\alpha}(x)\right| dx$$

$$=\frac{1}{2}\sum_{m=0}^{2N-1}\int_{2^{m-2N}}^{2^{m+1-2N}}\left|K_{2^{2N}}^{-\alpha}(x)\right|dx\geq c(\alpha)\sum_{m=0}^{2N-1}2^{m\alpha}\geq c(\alpha)\,2^{2N\alpha}.$$

Similarly, we can prove that

$$\int_I\varphi_N(x)\left|K_{2^{2N}}^{-\beta}(x)\right|dx\geq c(\beta)\,2^{2N\beta},\quad N\in\mathbb{N},\quad 0<\beta<1. \tag{4.4}$$

Combining (4.3) and (4.4) we get

$$\left|\sigma_{2^{2N},2^{2N}}^{-\alpha,-\beta}(0,0;H_N)\right|\geq\frac{c(\alpha,\beta)}{\eta_N}\to\infty\text{ as }N\to\infty. \tag{4.5}$$

Applying the Banach-Steinhaus theorem, from (4.1) and (4.5) we infer that there exists a continuous function $f\in\Lambda^{\#}BV$ such that

$$\sup_N\left|\sigma_{N,N}^{-\alpha,-\beta}(0,0,;f)\right|=+\infty.$$

Theorem 4.5 is proved. □

Taking into account the embedding $\Lambda^{*}BV\subset\Lambda^{\#}BV$, from Theorem 4.5 we obtain the following result.

Corollary 4.1. *Let $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$ and $f\in\left\{n^{1-(\alpha+\beta)}\right\}^{*}BV$. Then the means $\sigma_{n,m}^{-\alpha,-\beta}(x,y;f)$ converge to $f(x,y)$, if f is continuous at (x,y).*

A combination of Theorems 1.3 and 4.5 yields the following result.

Theorem 4.6. *Let $\alpha,\beta\in(0,1)$, $\alpha+\beta<1$ and*

$$\sum_{j=1}^{\infty}\frac{v_s^{\#}(f;2^j)}{2^{j(1-(\alpha+\beta))}}<\infty\quad\text{for}\quad s=1,2.$$

Then the means $\sigma_{n,m}^{-\alpha,-\beta}(x,y;f)$ converge to $f(x,y)$, if f is continuous at (x,y).

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