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POWER SERIES WITH H.-O. GAPS; TAUBERIAN THEOREMS

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Abstract.[1] Let $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence 1, and let $s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ denote its partial sums. For a given triangular matrix $A = [\alpha_{n\nu}]$ we consider the A-transforms $\sigma_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_\nu(z)$, and prove two Tauberian theorems of the following type: from certain summability properties of $\{\sigma_n(z)\}$ outside the unit disk and a condition on the entries $\alpha_{n\nu}$ the convergence of a subsequence $\{s_{n_k}(z)\}$ is concluded.

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1. Introduction

1.1. Overconvergence and H.-O. gaps. Let be given a power series with radius of convergence 1:

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \quad \overline{\lim_{\nu\to\infty}} |a_\nu|^{1/\nu} = 1, \tag{1.1}$$

which represents a holomorphic function in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. As usual, we denote its partial sums by

$$s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu. \tag{1.2}$$

Such a series is called overconvergent if there exists a domain G which is not contained in $\mathbb{D}$ and a subsequence $\{p_k\}$ of natural numbers such that $\{s_{p_k}(z)\}$ converges compactly on G. Then $\{s_{p_k}(z)\}$ is called an overconvergent subsequence of (1.1). If G intersects $\mathbb{D}$, then $\{s_{p_k}(z)\}$ generates an analytic continuation of f. (Note that there are other definitions of overconvergence.)

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The phenomenon of overconvergence was discovered by Porter [16] more than a century ago and thoroughly has been investigated by Ostrowski in [11] - [14]. For a good treatise on the theory of overconvergence we refer to Hille's book [5, Section 16.7]. One of Ostrowski's main results is an interdependence between overconvergence and existence of certain gaps in the sequence of coefficients $\{a_\nu\}$.

We say that the power series (1.1) has a sequence $\{p_k, q_k\}$ of H.-O. gaps (short for Hadamard-Ostrowski gaps) if p_k and q_k are natural numbers satisfying

$$p_1 < q_1 < p_2 < q_2 < \dots, \qquad \lim_{k\to\infty} \frac{q_k}{q_k} > 1$$

and

$$\overline{\lim_{\substack{\nu\to\infty \\ \nu\in J}}} |a_\nu|^{1/\nu} < 1 \qquad \text{for} \qquad J = \bigcup_{k=1}^{\infty} \{p_k, \dots, q_k\}.$$

We summarize the main results on overconvergence in the following theorem.

Theorem O (Ostrowski [11], [13]).

(a) *If the power series* (1.1) *possesses H.-O. gaps* $\{p_k, q_k\}$, *then any sequence* $\{s_{n_k}(z)\}$ *with* $n_k \in [p_k, q_k]$ *converges compactly in a domain which contains every point on* $|z| = 1$ *in which* f *is holomorphic.*

(b) *Every overconvergent power series possesses H.-O. gaps.*

1.2. Summability of power series. Let $A = [\alpha_{n\nu}]_{\nu,n=0}^{\infty}$ be an infinite triangular matrix with complex entries $\alpha_{n\nu}$, where $\alpha_{n\nu} = 0$ for $\nu > n$. Such a matrix generates a transformation of a power series. The A-transforms of the series (1.1) are given by

$$\sigma_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_\nu(z). \tag{1.3}$$

The matrix A is called p-regular ("regular for power series") if for all series of type (1.1) the sequence $\{\sigma_n(z)\}$ converges compactly in $\mathbb{D}$. This property can be characterized by the entries of A only. The following conditions are necessary and sufficient for p-regularity (see [7]):

$$\lim_{n\to\infty} \alpha_{n\nu} = 0 \qquad \text{for all } \nu \in \mathbb{N}_0, \tag{1.4}$$

$$\lim_{n\to\infty} \sum_{\nu=0}^{n} \alpha_{n\nu} = 1, \tag{1.5}$$

$$\sup_{n} \sum_{\nu=0}^{n} |\alpha_{n\nu}| r^\nu < \infty \qquad \text{for all } r \in (0,1). \tag{1.6}$$

In common use of summability theory the matrix A is regular if and only if the conditions (1.4), (1.5) and instead of (1.6) the following stronger property

$$\sup_n \sum_{\nu=0}^{n} |\alpha_{n\nu}| < \infty$$

hold. Observe that if A is regular, then A is also p-regular, but not conversely.

If A is p-regular, then the following properties of the sequence (1.2) can easily be obtained by straightforward estimates:

$$\overline{\lim_{n\to\infty}} \Big\{ \max_{|z|=R} |\sigma_n(z)| \Big\}^{1/n} \leq R \quad \text{for all } R \geq 1; \tag{1.7}$$

if (1.1) has H.-O. gaps $\{p_k, q_k\}$, then

$$\overline{\lim_{k\to\infty}} \Big\{ \max_{|z|=R} |\sigma_{q_k}(z)| \Big\}^{1/q_k} < R \quad \text{for all } R > 1. \tag{1.8}$$

2. A Tauberian Theorem

The following Theorem is our main result.

Theorem 2.1. *Suppose that $A = [\alpha_{n\nu}]$ is a p-regular matrix with the property that there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ and a constant $\gamma \in (0,1)$ such that*

$$\lim_{\substack{\nu\to\infty \\ \nu\in J}} \Big| \sum_{\mu=\nu}^{n_k} \alpha_{n_k\mu} \Big|^{1/\nu} = 1, \quad \text{where} \quad J = \bigcup_{k=1}^{\infty} \{[\gamma n_k], \ldots, n_k\}. \tag{2.1}$$

Let a power series of type (1.1) be given and $\sum_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_\nu(z)$ be its A-transforms. Suppose that for an $R > 1$ there exists a closed arc $\Gamma \subset \{z : |z| = R\}$ with

$$\overline{\lim_{k\to\infty}} \Big\{ \max_{\Gamma} |\sigma_{n_k}(z)| \Big\}^{1/n_k} < R. \tag{2.2}$$

Then the considered power series has H.-O. gaps of the type $\{[\delta n_k], n_k\}$ for some $\delta \in (0,1)$. If f has an analytic continuation, then $\{s_{n_k}(z)\}$ is overconvergent.

Remark 2.1

(1) If the sequence $\{\sigma_{n_k}(z)\}$ converges compactly in a domain, which is not contained in $\mathbb{D}$, then (2.2) is trivially satisfied for suitably chosen $R > 1$ and arcs $\Gamma \subset \{z : |z| = R\}$. In the case where the matrix A has the property that there are constants $c > 0$ and $\gamma \in (0,1)$ with

$$\Big| \sum_{\mu=\nu}^{n} \alpha_{n\nu} \Big| \geq c \quad \text{for all } \nu \text{ with } [\gamma n] \leq \nu \leq n \tag{2.3}$$

and all sufficiently large n, then (2.1) is satisfied. In section 3 we list a number of well known summability methods which are generated by matrices and satisfy (2.3).

(2) Suppose that the condition (2.1) is satisfied. Then the matrix A is not efficient for the analytic continuation of *all* function elements of type (1.1). More precisely, there exist power series (1.1) such that $\sigma_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_\nu(z)$ converges compactly in $\mathbb{D}$, but not in any bigger domain (however, a subsequence $\{\sigma_{n_k}(z)\}$ may have this property). If $\lim_{n\to\infty} |a_n|^{1/n} = 1$, then $\{\sigma_n(z)\}$ and all its subsequences are compactly convergent only in $\mathbb{D}$. For a detailed discussion of this problem we refer to [1], [2], [7], [8] (see, also, [3], [4]).

(3) Theorem 2.1 may be considered as a Tauberian theorem: From summability properties (here condition (2.2)) together with a so-called Tauberian condition (here (2.1)) convergence properties are derived. However, in contrast to classical Tauberian results, in Theorem 2.1 convergence of a subsequence is concluded.

Proof of Theorem 2.1. Suppose that an $\varepsilon > 0$ is given and consider the circle $|z| = R$ and its closed subarc Γ. Then, by (1.7) there exists an n_0 such that for all $n \geq n_0$

$$\max_{|z|=1} \left| \frac{\sigma_n(z)}{z^n} \right| \leq e^{\varepsilon n}, \quad \max_{|z|=R} \left| \frac{\sigma_n(z)}{z^n} \right| \leq e^{\varepsilon n}.$$

In addition, by (2.2) there exists a $k_0 \geq n_0$ such that for all $k \geq k_0$

$$\max_{\Gamma} \left| \frac{\sigma_{n_k}(z)}{z^{n_k}} \right| \leq \frac{e^{\varepsilon n_k}}{R^{n_k}}.$$

Let r with $1 < r < R$ be given. Then, according to Nevanlinna's N-constants theorem (see Hille, [5, p. 409]), there exists a universal $\Theta \in (0,1)$, which depends only on the geometrical configuration, such that for all $k \geq k_0$

$$\max_{|z|=r} \left| \frac{\sigma_{n_k}(z)}{z^{n_k}} \right| \leq e^{(1-\Theta)\varepsilon n_k} \cdot \frac{e^{\Theta \varepsilon n_k}}{R^{\Theta n_k}} = \left(\frac{e^{\varepsilon}}{R^{\Theta}} \right)^{n_k}.$$

Therefore, if $\varepsilon > 0$ is chosen sufficiently small, we obtain for those k

$$\max_{|z|=r} |\sigma_{n_k}(z)| \leq (qr)^{n_k},$$

where $0 < q < 1$ (but $qr > 1$). We have

$$\sigma_{n_k}(z) = \sum_{\nu=0}^{n_k} a_\nu z^\nu \cdot \sum_{\mu=\nu}^{n_k} \alpha_{n_k \mu},$$

and Cauchy's inequality gives for $0 \leq \nu \leq n_k$ and all $k \geq k_0$

$$|a_\nu| \cdot \left| \sum_{\mu=\nu}^{n_k} \alpha_{n_k \mu} \right| \leq \left(q \cdot r^{1-\nu/n_k} \right)^{n_k}.$$

If we now choose δ with $\gamma \le \delta < 1$ so near to 1 that $r^{1-\delta^*} < \frac{1}{q}$, then for all ν with $[\delta n_k] \le \nu \le n_k$ and $k \ge k_0$ we get the estimate

$$|a_\nu|^{1/\nu} \cdot \Big| \sum_{\mu=\nu}^{n_k} \alpha_{n_k\mu} \Big|^{1/\nu} \le q \cdot r^{1-\delta} < 1.$$

But then (2.2) implies

$$\overline{\lim_{\substack{\nu\to\infty \\ \nu\in J}}} |a_\nu|^{1/\nu} < 1 \quad \text{for} \quad J = \bigcup_{k=1}^{\infty} \{[\delta n_k], \dots, n_k\}.$$

Therefore the power series under consideration has H.-O. gaps of the type $\{[\delta n_k], n_k\}$. In the case where f has an analytic continuation, Theorem 0 implies that $\{s_{n_k}(z)\}$ is an overconvergent subsequence of the series. Theorem 2.1 is proved. □

As a corollary of Theorem 2.1 we have the following result.

Theorem 2.2. *Let a power series of type (1.1) be given which has an analytic continuation. Suppose that $A = [\alpha_{n\nu}]$ is a p-regular triangular matrix and that for a sequence $\{p_k\}_{k=0}^{\infty}$ with $\underline{\lim}_{k\to\infty} \frac{p_{k+1}}{p_k} > 1$ transformations*

$$\tau_n(z) = \sum_{\nu=0}^{n} \alpha_{n\nu} s_{p_\nu}(z)$$

are compactly convergent in a domain which is not contained in the unit disk. If

$$\text{(2.4)} \qquad \lim_{\substack{\nu\to\infty \\ \nu\in J}} |\alpha_{kk}|^{1/\nu} = 1 \quad \textit{for} \quad J = \bigcup_{k=1}^{\infty} \{p_k + 1, \dots, p_{k+1}\}$$

is satisfied, then $\{s_{p_k}(z)\}$ is an overconvergent subsequence of the considered power series.

Proof. Without loss of generality we can assume that $p_{k+1}/p_k \ge \lambda > 1$ for all $k \in \mathbb{N}_0$. We define a triangular matrix $B = [\beta_{n\nu}]$ in the following way:

$$\text{for } n \ne p_k: \qquad \beta_{n\nu} := \begin{cases} 0 & \text{if } \nu \ne n \\ 1 & \text{if } \nu = n \end{cases},$$

$$\text{for } n = p_k: \qquad \beta_{p_k\nu} := \begin{cases} 0 & \text{if } \ \nu \ne p_\mu \\ \alpha_{k\mu} & \text{if } \ \nu = p_\mu \end{cases} \quad (\mu = 0, \dots, k).$$

The matrix B is obviously p-regular. We consider

$$\sigma_n(z) = \sum_{\nu=0}^{n} \beta_{n\nu} s_\nu(z)$$

and obtain

$$\sigma_{p_k}(z) = \sum_{\mu=0}^{k} \beta_{p_k p_\mu} s_{p_\mu}(z) = \sum_{\mu=0}^{k} \alpha_{k\mu} s_{p_\mu}(z) = \tau_k(z)$$

as well as $\sum_{\mu=\nu}^{p_k} \beta_{p_k\mu} = \alpha_{kk}$ for all ν with $p_{k-1} < \nu \le p_k$.

Therefore, the matrix B satisfies a condition of type (2.1), while the sequence $\{\sigma_{p_k}(z)\}$ has property (2.2) for a suitably chosen $R > 1$ and an arc $\Gamma \subset \{z : |z| = R\}$. It follows that $\{s_{p_k}(z)\}$ is an overconvergent subsequence. Theorem 2.2 is proved. □

3. Examples

We discuss some examples of well-known summability methods that are defined by triangular matrices and satisfy the requirements of Theorem 2.1. Especially we are interested whether the property (2.1), which acts as a Tauberian condition in this result, can be realized. Whenever in addition a power series (1.1) is considered, for which the corresponding transformations satisfy a property of type (2.2), then a Tauberian result as in Theorem 2.1 can be concluded for this series.

1. Nørlund means $\mathbf{N_c}$. Let $c = \{c_n\}$ be a sequence of real numbers with $c_0 > 0$ and $c_n \ge 0$ for $n \ge 1$, and let $C_n = \sum_{\nu=0}^{n} c_\nu$. Then the Nørlund means are generated by the triangular matrix $A = [\alpha_{n\nu}]$ given by

$$\alpha_{n\nu} = \frac{c_{n-\nu}}{C_n} \quad \text{if } 0 \le \nu \le n.$$

The condition $\lim_{n\to\infty} \frac{c_n}{C_n} = 0$ is necessary and sufficient for the regularity of N_c, and it is also well-known that all Nørlund methods are ineffective for analytic continuations of any power series. For $0 \le \nu \le n$ we get

$$\frac{c_0}{C_n} \le \sum_{\mu=\nu}^{n} \alpha_{n\nu} \le 1,$$

and by the regularity condition we have $\lim_{n\to\infty} \frac{C_{n-1}}{C_n} = 1$. Hence $\lim_{n\to\infty}(C_n)^{1/n} = 1$, which implies that for all Nørlund means the condition (2.1) is satisfied for all subsequences $\{n_k\}$ of $\mathbb{N}$.

Hence a Tauberian result as in Theorem 2.1 holds for all power series whose N_c transformations satisfy condition (2.2).

(Actually the N_c method was first introduced by Russian mathematician Voronoi in 1902 (see [17]); independently of Voronoi the definition was given by Nørlund in 1920 (see [10]).)

2. Cesàro means $\mathbf{C_\alpha}$. These are special regular Nørlund means which for $\alpha \ge 0$ are generated by the sequence $c_n = \binom{n+\alpha-1}{n}$.

3. Weighted means $\mathbf{R_c}$. (Also known as Riesz means or Nørlund-type means.) Let $c = \{c_n\}$ be again a sequence of real numbers with $c_0 > 0$ and $c_n \ge 0$ for $n \ge 1$, and let $C_n = \sum_{\nu=0}^{n} c_\nu$. Then the R_c means are generated by the triangular matrix $A = [\alpha_{n\nu}]$ given by

$$\alpha_{n\nu} = \frac{c_\nu}{C_n} \quad \text{if} \quad 0 \le \nu \le n.$$

Here the condition $\lim_{n\to\infty} C_n = \infty$ is necessary and sufficient for the regularity of R_c. As in the case of Nørlund means we have for $0 \le \nu \le n$

$$\frac{c_0}{C_n} \le \sum_{\mu=\nu}^{n} \alpha_{n\nu} \le 1.$$

Therefore condition (2.1) is satisfied if $\{C_n\}$ is not "too fast" increasing sequence, that is, if $\lim_{n\to\infty}(C_n)^{1/n} = 1$. In this case Theorem 2.1 applies also to this method.

4. Hausdorff means $\mathbf{H}_\chi$. This is a wide class of summability methods, containing many well-known methods as special cases.

Let χ be a real-valued function of bounded variation on $[0,1]$ satisfying

$$\chi(t) = \chi(t+) \quad \text{for all } \ t \in [0,1).$$

The H_χ means are generated by the triangular matrix $A = [\alpha_{n\nu}]$ with

$$\alpha_{n\nu} = \binom{n}{\nu} \int_0^1 t^\nu (1-t)^{n-\nu} d\chi(t)$$

and the regularity conditions are $\chi(0) = 0$, $\chi(1) = 1$ (see [15]). The best known Hausdorff means are the Cesàro means C_α $(\alpha > 0)$, where

$$\chi(t) = 1 - (1-t)^\alpha,$$

the Hölder means H_α $(\alpha > 0)$, where (Γ denotes the Gamma function)

$$\chi(t) = \frac{1}{\Gamma(\alpha)} \cdot \int_0^t \left(\ln \frac{1}{s}\right)^{\alpha-1} ds,$$

and the Euler means E_r $(0 < r < 1)$, where

$$\chi(t) = \begin{cases} 0 & \text{for } \ 0 \le t < r \\ 1 & \text{for } \ r \le t \le 1. \end{cases}$$

The (upper) order of a regular Hausdorff mean is defined as

$$\rho = \rho(\chi) = \inf\{s : \chi(t) = 1 \text{ for all } \ t \in [s,1]\}.$$

Obviously C_α and H_α have order $\rho = 1$, while $\rho = r < 1$ for the Euler means E_r.

If a power series has an analytic continuation, then all H_χ means with $\rho < 1$ are efficient for those series and also an estimate (depending on ρ) for the summability domain can be given. On the other hand, all H_χ means of order $\rho = 1$ are inefficient for analytic continuation (for details see [9, section 2], [15, chapter IV, 2]). Especially for Cesàro and Hölder means we have inefficiency for any power series.

If H_χ has order $\rho = 1$, then there exist constants $\gamma \in (0,1)$ and $c > 0$ such that for all sufficiently large n $\left|\sum_{\mu=\nu}^{n} \alpha_{n\nu}\right| \geq c$ for all $\nu \in [\gamma n, n]$.

This estimate is a special case of a result on the distribution of Hausdorff elements (see [9], Lemma 1), which was proved by probabilistic methods.

It follows that H_χ means of order $\rho = 1$ satisfy condition (2.1) for any subsequence of N, and under the additional assumption (2.2) on the behavior of a power series a Tauberian result as in Theorem 2.1 holds.

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