Известия НАН Армении. Математика, том 49, н. 6, 2014, стр. 16-28.

ON A q-FRACTIONAL VARIANT OF NONLINEAR LANGEVIN EQUATION OF DIFFERENT ORDERS

B. AHMAD, J. J. NIETO, A. ALSAEDI AND H. AL-HUTAMI

King Abdulaziz University, Saudi Arabia
Universidad de Santiago de Compostela, Santiago de Compostela, Spain

E-mails: *bashirahmad_qau@yahoo.com; juanjose.nieto.roig@usc.es*
aalsaedi@hotmail.com; hanno.1407@hotmail.com

Abstract. **In this paper we introduce a q-fractional variant of nonlinear Langevin equation of different orders with q-fractional antiperiodic boundary conditions. The nonlinearity in the proposed problem involves an integral term (a Riemann-Liouville type q-integral) and a non-integral term. Some existence results for solutions of the given problem are established by means of classical tools of fixed point theory. An illustrative example is also presented.**

MSC2010 numbers: 34A08, 34B10, 34B15.

Keywords: Langevin equation; q-fractional; antiperiodic; existence; fixed point.

1. Introduction

Nonlinear boundary value problems of fractional differential equations have received considerable attention in the preceding decades. In the literature one can easily find a variety of results on the topic, ranging from theoretical analysis to asymptotic behavior and numerical methods for fractional equations.

An important feature of a fractional order differential operator, distinguishing it from an integer-order differential operator, is that it is of nonlocal nature and takes into account memory and hereditary properties of some important and useful materials and processes.

The fractional calculus has evolved as an effective mathematical modeling tool in several real world phenomena occurring in physical and technical sciences (see [1]). More details and examples on the topic can be found in the papers [2] and [3] and references therein.

The subject of q-difference equations has gained considerable attention over the years since its inception by Jackson [4]. One of the advantages to consider q-difference equations is that these equations are always completely controllable and appear in the q-optimal control problems (see [5]). For further details, we refer the reader to references [6] and [7].

Fractional q-difference (or q-fractional) equations, regarded as fractional counterparts of q-difference equations, have been studied by a number of authors (see [8] - [10]). For some earlier work on the topic, we refer to [11] and [12], whereas the basic concepts on q-fractional calculus can be found in the recent text [7].

Antiperiodic boundary conditions occur in the mathematical modeling of a variety of problems of applied nature. An account of classical and fractional antiperiodic boundary conditions can be found in the papers [13] and [14] and references therein. However, the concept of fractional q-difference antiperiodic boundary conditions has not been introduced yet.

The Langevin equation involving fractional derivatives of different non-integer orders provides a more flexible model for fractal processes. Some recent results on Langevin equation can be found in the papers [15] and [16]. We recall that the ordinary Langevin equation does not provide correct description of the dynamics of systems in complex media. Notice that Langevin equation involving q-fractional derivatives of different orders has not been studied so far.

The objective of the present paper is to study a new boundary value problem for the q-fractional nonlinear Langevin equation of different orders involving an integral term (a Riemann-Liouville type q-integral) and a non-integral term, with q-fractional antiperiodic boundary conditions. More precisely, for given numbers $0 < \beta < 1$ and $0 < \gamma < 1$, we consider a full q-fractional antiperiodic boundary value problem for the Langevin equation given by

$$^cD_q^\beta(^cD_q^\gamma + \lambda)x(t) = \rho f(t, x(t)) + \delta I_q^\zeta g(t, x(t)), \quad 0 \le t \le 1, \ 0 < q < 1, \tag{1.1}$$

$$x(0) = -x(1), \quad ^cD_q^\gamma x(0) = -^cD_q^\gamma x(1), \tag{1.2}$$

where $^cD_q^\beta$ and $^cD_q^\gamma$ denote the Caputo type fractional q-derivative, $I_{q,0}^\zeta(.) = I_q^\zeta(.)$ denotes the Riemann-Liouville integral with $0 < \zeta < 1$, f, g are given continuous functions, $\lambda \neq 0$, and ρ, δ are real constants.

The paper is organized as follows. Section 2 deals with some general concepts and results from q-fractional calculus, as well as an auxiliary lemma for a linear variant of the problem (1.1), (1.2). In Section 3, we present some existence results for solutions of the problem (1.1), (1.2) by applying Krasnoselskii's fixed point theorem, Leray-Schauder alternative and Banach's contraction mapping principle.

2. Preliminaries on fractional q-calculus

In this section we discuss some general concepts and results from q-fractional calculus. We first recall the necessary notation and definitions, and introduce the terminology of q-fractional calculus (see [7, 17, 18]).

For a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, a q-real number denoted by $[a]_q$ is defined by

$$[a]_q = \frac{1-q^a}{1-q}, \; a \in \mathbb{R}.$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a;q)_0 = 1, \;\; (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i), \; k \in \mathbb{N} \cup \{\infty\}.$$

The q-analogue of the exponent $(x-y)^k$ is defined as

$$(x-y)^{(0)} = 1, \;\; (x-y)^{(k)} = \prod_{j=0}^{k-1}(x-yq^j), \; k \in \mathbb{N}, \; x, y \in \mathbb{R}.$$

The q-gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y) = \frac{(1-q)^{(y-1)}}{(1-q)^{y-1}},$$

where $y \in \mathbb{R} \setminus \{0, -1, -2, ...\}$. Observe that $\Gamma_q(y+1) = [y]_q \Gamma_q(y)$.

Definition 2.1. *Let f be a function defined on $[0,b]$, $b > 0$ and let $a \in (0,b)$ be an arbitrary fixed point. The Riemann-Liouville type fractional q-integral is defined by*

$$(I_{q,a}^{\beta} f)(t) = \int_a^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)} f(s) d_q(s), \;\; \beta > 0,$$

provided that the integral exists.

Remark 2.1. *The q-fractional integration possesses the semigroup property:*

$$(I_{q,a}^{\gamma} I_{q,a}^{\beta} f)(t) = (I_{q,a}^{\beta+\gamma} f)(t); \;\; \gamma, \beta \in \mathbb{R}^+, \;\; a \in (0,b).$$

Before giving the definition of fractional q-derivative, we recall the concept of q-derivative. We know that the q-derivative of a function $f(t)$ is defined as

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt}, \;\; t \neq 0, \;\; (D_q f)(0) = \lim_{t \to 0} (D_q f)(t).$$

Furthermore, we define $D_q^0 f = f$, $D_q^n f = D_q(D_q^{n-1} f)$, $n = 1, 2, \ldots$.

Definition 2.2. *([17]) The Riemann-Liouville type fractional q-derivative of order β of a function $f(t)$ is defined by*

$$(D^{\beta}_{q,a}f)(t)=\begin{cases}(I^{-\beta}_{q,a}f)(t), & \beta<0,\\ f(x), & \beta=0,\\ (D^{\lceil\beta\rceil}_{q}I^{\lceil\beta\rceil-\beta}_{q,a}f)(t), & \beta>0,\end{cases} \tag{2.1}$$

where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

Remark 2.2. *The following relations hold (see [18], Lemma 6):*

(i) $(D^{\beta}_{q,a}I^{\beta}_{q,a}f)(t)=f(t),\ 0<a<t.$

(ii) $I^{\beta}_{q,a}\left((x-a)^{(\lambda)}\right)=\frac{\Gamma_q(\lambda+1)}{\Gamma_q(\beta+\lambda+1)}(x-a)^{(\beta+\lambda)},\ 0<a<x<b,\beta\in\mathbb{R}^+,\lambda\in(-1,\infty).$

Definition 2.3. *([17]) The Caputo type fractional q-derivative of order $\beta\in\mathbb{R}^+$ of a function $f(t)$ is defined by $({}^cD^{\beta}_{q,a}f)(t)=(I^{\lceil\beta\rceil-\beta}_{q,a}D^{\lceil\beta\rceil}_{q}f)(t)$.*

Remark 2.3. *For $0<a<t$ and $\beta\in\mathbb{R}\setminus\mathbb{N}$, the following relations hold (see [17]):*

(a): $({}^cD^{\beta+1}_{q,a}f)(t)=({}^cD^{\beta}_{q,a}D_qf)(t)$;

(b): $({}^cD^{\beta}_{q,a}I^{\beta}_{q,a}f)(t)=f(t)$;

(c): $(I^{\beta}_{q,a}\,{}^cD^{\beta}_{q,a}f)(t)=f(t)-\sum_{k=0}^{\lceil\beta\rceil-1}\frac{(D^k_qf)(a)}{\Gamma_q(k+1)}t^k(a/t;q)_k$;

To define the solution of the problem (1.1), (1.2), we need the following lemma.

Lemma 2.1. *For a given $h\in C([0,1],\mathbb{R})$, the unique solution of the boundary value problem*

$$\begin{cases}{}^cD^{\beta}_q({}^cD^{\gamma}_q+\lambda)x(t)=h(t), & 0\le t\le 1,\ 0<q<1,\\ x(0)=-x(1), & {}^cD^{\gamma}_qx(0)=-{}^cD^{\gamma}_qx(1)\end{cases} \tag{2.2}$$

is given by

$$\begin{aligned}x(t)&=\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}h(m)d_qm-\lambda x(u)\Big)d_qu\\ &+\frac{(1-2t^{\gamma})}{4\Gamma_q(\gamma+1)}\int_0^1\frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)}h(u)d_qu\\ &-\frac{1}{2}\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}h(m)d_qm-\lambda x(u)\Big)d_qu.\end{aligned} \tag{2.3}$$

Proof. Applying the operator I^{β}_q to the q-fractional Langevin equation in (2.2), we get

$$ {}^cD^{\gamma}_qx(t)=I^{\beta}_qh(t)-\lambda x(t)-b_0. \tag{2.4}$$

Next, we apply the operator I_q^γ to the both sides of (2.4) with $t \in [0,1]$ to obtain

$$x(t) = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} h(m) d_q m - \lambda x(u)\Big) d_q u - \frac{t^\gamma}{\Gamma_q(\gamma+1)} b_0 - b_1. \tag{2.5}$$

Using the boundary conditions (2.2) in (2.5) and solving the resulting system of equations for b_0 and b_1, we get

$$b_0 = \frac{1}{2}\int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} h(u) d_q u,$$

$$b_1 = \frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} h(m) d_q m - \lambda x(u)\Big) d_q u - \frac{1}{4\Gamma_q(\gamma+1)}\int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} h(u) d_q u.$$

Substituting the obtained values of b_0 and b_1 into (2.5) we get (2.3). This completes the proof of Lemma 2.1. □

3. The main results

Let $\mathfrak{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$, endowed with the usual norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$. We use Lemma 2.1 to define an operator $\mathfrak{U} : \mathfrak{C} \to \mathfrak{C}$ by

$$\begin{aligned}
(\mathfrak{U}x)(t) &= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\rho\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m)) d_q m \\
&\quad +\delta\int_0^u \frac{(u-qm)^{(\beta+\varsigma-1)}}{\Gamma_q(\beta+\varsigma)} g(m,x(m)) d_q m - \lambda x(u)\Big) d_q u \\
&\quad +\frac{(1-2t^\gamma)}{4\Gamma_q(\gamma+1)}\Big(\rho\int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} f(u,x(u)) d_q u \\
&\quad +\delta\int_0^1 \frac{(1-qu)^{(\beta+\varsigma-1)}}{\Gamma_q(\beta+\varsigma)} g(u,x(u)) d_q u\Big) \\
&\quad -\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\rho\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m,x(m)) d_q m \\
&\quad +\delta\int_0^u \frac{(u-qm)^{(\beta+\varsigma-1)}}{\Gamma_q(\beta+\varsigma)} g(m,x(m)) d_q m - \lambda x(u)\Big) d_q u.
\end{aligned} \tag{3.1}$$

Observe that the problem (1.1), (1.2) has solutions if and only if the operator equation $x = \mathfrak{U}x$ has fixed points. In the sequel, we need the following assumptions:

(A_1) $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that $|f(t,x) - f(t,y)| \le L_1|x-y|$ and $|g(t,x) - g(t,y)| \le L_2|x-y|$, $\forall t \in [0,1]$, $L_1, L_2 > 0$, $x, y \in \mathbb{R}$;

(A_2) There exist $\vartheta_1, \vartheta_2 \in C([0,1], \mathbb{R}^+)$ with $|f(t,x)| \leq \vartheta_1(t)$, $|g(t,x)| \leq \vartheta_2(t)$, $\forall (t,x) \in [0,1] \times \mathbb{R}$, where $\|\vartheta_i\| = \sup_{t\in[0,1]} |\vartheta_i(t)|$, $i = 1, 2$.

For the sake of brevity, we introduce the following quantities:

$$\mu_1 = \frac{3}{2\Gamma_q(\beta+\gamma+1)} + \frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+1)},$$
$$(3.2) \qquad \mu_2 = \frac{3}{2\Gamma_q(\beta+\zeta+\gamma+1)} + \frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+\zeta+1)},$$
$$\mu_3 = \frac{3}{2\Gamma_q(\gamma+1)},$$

$$(3.3) \qquad \Omega = L\Big[|\rho|\Big(\frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+1)} + \frac{1}{2\Gamma_q(\beta+\gamma+1)}\Big) + |\delta|\Big(\frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+\zeta+1)} + \frac{1}{2\Gamma_q(\beta+\zeta+\gamma+1)}\Big)\Big] + \frac{|\lambda|}{2\Gamma_q(\gamma+1)},$$

where $L = \max\{L_1, L_2\}$.

Our first existence result is based on the Krasnoselskii's fixed point theorem ([19]).

Lemma 3.1 (Krasnoselskii). *Let Y be a closed, convex, bounded and nonempty subset of a Banach space X, and let $\mathcal{S}_1, \mathcal{S}_2$ be operators such that*

(i) $\mathcal{S}_1 x + \mathcal{S}_2 y \in Y$ whenever $x, y \in Y$;

(ii) $\mathcal{S}_1$ is compact and continuous;

(iii) $\mathcal{S}_2$ is a contraction mapping.

Then there exists $z \in Y$ such that $z = \mathcal{S}_1 z + \mathcal{S}_2 z$.

Theorem 3.1. *Let $f, g : [0,1]\times\mathbb{R} \to \mathbb{R}$ be continuous functions satisfying assumptions (A_1) and (A_2). Furthermore let $\Omega < 1$, where Ω is given by (3.3). Then the problem (1.1), (1.2) has at least one solution on $[0,1]$.*

Proof. With μ_1, μ_2, μ_3 given by (3.2), we consider the set $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$, where

$$r \;\geq\; \frac{|\rho|\|\vartheta_1\|\mu_1 + |\delta|\|\vartheta_2\|\mu_2}{1 - |\lambda|\mu_3}.$$

Now we show that the conditions of Lemma 3.1 are satisfied. To this end, we define the operators $\mathcal{U}_1$ and $\mathcal{U}_2$ on B_r by

$$(\mathcal{U}_1 x)(t) = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\rho\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m, x(m))d_q m$$
$$+\delta\int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} g(m, x(m))d_q m - \lambda x(u)\Big)d_q u, \quad t \in [0,1],$$
$$(\mathcal{U}_2 x)(t) = \frac{(1-2t^\gamma)}{4\Gamma_q(\gamma+1)}\Big(\rho\int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} f(u, x(u))d_q u$$

$$+\delta\int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}g(u,x(u))d_qu\Big)$$

$$-\frac{1}{2}\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(\rho\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m,x(m))d_qm$$

$$+\delta\int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}g(m,x(m))d_qm-\lambda x(u)\Big)d_qu,\quad t\in[0,1].$$

For $x, y \in B_r$, we find that

$$\|\mathcal{U}_1x+\mathcal{U}_2y\| \le |\rho|\|\vartheta_1\|\mu_1+|\delta|\|\vartheta_2\|\mu_2+|\lambda|r\mu_3 \le r,$$

implying that $\mathcal{U}_1x+\mathcal{U}_2y \in B_r$. It is clear that the continuity of the operator $\mathcal{U}_1$ follows from that of f and g. Also, observe that $\mathcal{U}_1$ is uniformly bounded on B_r since

$$\|\mathcal{U}_1x\| \le \frac{|\rho|\|\vartheta_1\|}{\Gamma_q(\beta+\gamma+1)}+\frac{|\delta|\|\vartheta_2\|}{\Gamma_q(\beta+\zeta+\gamma+1)}+\frac{|\lambda|r}{\Gamma_q(\gamma+1)}.$$

Next, we show the compactness of the operator $\mathcal{U}_1$. In view of (A_1), we set

$$\sup_{(t,x)\in[0,1]\times B_r}|f(t,x)|=f_1,\qquad \sup_{(t,x)\in[0,1]\times B_r}|g(t,x)|=g_1.$$

Hence, we have

$$\|(\mathcal{U}_1x)(t_2)-(\mathcal{U}_1x)(t_1)\| \le \int_0^{t_1}\frac{(t_2-qu)^{(\gamma-1)}-(t_1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|\rho|f_1\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm$$

$$+|\delta|g_1\int_0^u\frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}d_qm+|\lambda|r\Big)d_qu+\int_{t_1}^{t_2}\frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\times$$

$$\times\Big(|\rho|f_1\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}d_qm+|\delta|g_1\int_0^u\frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}d_qm+|\lambda|r\Big)d_qu,$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{U}_1$ is relatively compact on B_r. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{U}_1$ is compact on B_r.

Now, we show that $\mathcal{U}_2$ is a contraction. In view of (A_1), for $x, y \in B_r$ we can write

$$\|\mathcal{U}_2 x - \mathcal{U}_2 y\| \leq \sup_{t\in[0,1]} \Big\{ \frac{1}{4\Gamma_q(\gamma+1)} \Big(|\rho| \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(u,x(u)) - f(u,y(u)) \Big| d_q u$$
$$+ |\delta| \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(u,x(u)) - g(u,y(u)) \Big| d_q u \Big)$$
$$+ \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m,x(m)) - f(m,y(m)) \Big| d_q m$$
$$+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(m,x(m)) - g(m,y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Big\}$$
$$\leq \sup_{t\in[0,1]} \Big\{ \frac{1}{4\Gamma_q(\gamma+1)} \Big(|\rho| \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} L_1 |x(u) - y(u)| d_q u$$
$$+ |\delta| \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} L_2 |x(u) - y(u)| d_q u \Big)$$
$$+ \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} L_1 |x(m) - y(m)| d_q m$$
$$+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} L_2 |x(m) - y(m)| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Big\}$$
$$\leq \Big[L \Big[|\rho| \Big(\frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+1)} + \frac{1}{2\Gamma_q(\beta+\gamma+1)} \Big)$$
$$+ |\delta| \Big(\frac{1}{4\Gamma_q(\gamma+1)\Gamma_q(\beta+\zeta+1)} + \frac{1}{2\Gamma_q(\beta+\zeta+\gamma+1)} \Big) \Big] + \frac{|\lambda|}{2\Gamma_q(\gamma+1)} \Big] \|x - y\|$$
$$= \Omega \|x - y\|,$$

where we have used (3.3). Hence, taking into account that by our assumption $\Omega < 1$, we conclude that $\mathcal{U}_2$ is a contraction mapping. Thus all the assumptions of Lemma 3.1 are satisfied. So the conclusion of Lemma 3.1 applies and the problem (1.1), (1.2) has at least one solution on $[0, 1]$. This completes the proof of Theorem 3.1. □

The second existence result is based on the Leray-Schauder alternative (see [20]).

Lemma 3.2. *(A nonlinear alternative for single valued maps). Let E be a Banach space, C be a closed, convex subset of E, and V be an open subset of C with $0 \in V$. Suppose that $\mathcal{U} : \overline{V} \to C$ is a continuous, compact (that is, $\mathcal{U}(\overline{V})$ is a relatively compact subset of C) map. Then either $\mathcal{U}$ has a fixed point in $\overline{V}$, or there is a $x \in \partial V$ (the boundary of V in C) and $\kappa \in (0,1)$ with $x = \kappa \mathcal{U}(x)$.*

Theorem 3.2. *Let $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions and let the following assumptions be fulfilled:*

(A_3) *There exist functions $\nu_1, \nu_2 \in C([0,1], \mathbb{R}^+)$, and nondecreasing functions $\psi_1, \psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,x)| \leq \nu_1(t)\psi_1(\|x\|)$ and $|g(t,x)| \leq \nu_2(t)\psi_2(\|x\|)$ $\forall (t,x) \in [0,1] \times \mathbb{R}$.*

(A_4) *There exists a constant $\omega > 0$ such that*

$$\omega > \frac{|\rho|\|\nu_1\|\psi_1(\omega)\mu_1 + |\delta|\|\nu_2\|\psi_2(\omega)\mu_2}{1 - |\lambda|\mu_3}, \quad \text{where} \quad |\lambda| \neq \frac{1}{\mu_3}.$$

Then the boundary value problem (1.1), (1.2) has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathcal{U} : \mathcal{C} \to \mathcal{C}$ defined by (3.1). It is easy to show that $\mathcal{U}$ is continuous. We complete the proof in the following steps.

(i) $\mathcal{U}$ maps bounded sets into bounded sets in $C([0,1], \mathbb{R})$. Indeed, for a positive number ε, let $B_\varepsilon = \{x \in \mathcal{C} : \|x\| \leq \varepsilon\}$ be a bounded set in $C([0,1], \mathbb{R})$. Since $|f(m, x(m))| \leq \nu_1(m) \cdot \psi_1(\|x\|)$ and $|g(m, x(m))| \leq \nu_2(m) \cdot \psi_2(\|x\|)$, we have

$$\|(\mathcal{U}x)\| \leq \sup_{t \in [0,1]} \Bigg\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m))| d_q m$$

$$+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} |g(m, x(m))| d_q m + |\lambda||x(u)| \Big) d_q u + \frac{1}{4\Gamma_q(\gamma+1)} \times$$

$$\times \Big(|\rho| \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} |f(u, x(u))| d_q u + |\delta| \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} |g(u, x(u))| d_q u \Big) +$$

$$\frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} |f(m, x(m))| d_q m \tag{3.4}$$

$$+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} |g(m, x(m))| d_q m + |\lambda||x(u)| \Big) d_q u \Bigg\}$$

$$\leq |\rho|\|\nu_1\|\psi_1(\|x\|)\mu_1 + |\delta|\|\nu_2\|\psi_2(\|x\|)\mu_2 + |\lambda|\|x\|\mu_3,$$

and the result follows.

(ii) $\mathcal{U}$ maps bounded sets into equicontinuous sets of $C([0,1], \mathbb{R})$.

Indeed, let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\varepsilon$, where B_ε is a bounded set of

$C([0,1],\mathbb{R})$. Then we can write

$$\begin{aligned}
&\|(\mathcal{U}x)(t_2)-(\mathcal{U}x)(t_1)\| \\
\leq\ & \left|\int_0^{t_1}\frac{(t_2-qu)^{(\gamma-1)}-(t_1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|\rho|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\nu_1(m)\psi_1(\varepsilon)d_qm\right. \\
+\ & |\delta|\int_0^u\frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}\nu_2(m)\psi_2(\varepsilon)d_qm+|\lambda|\varepsilon\Big)d_qu \\
+\ & \int_{t_1}^{t_2}\frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(|\rho|\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\nu_1(m)\psi_1(\varepsilon)d_qm \\
+\ & \left.|\delta|\int_0^u\frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}\nu_2(m)\psi_2(\varepsilon)d_qm+|\lambda|\varepsilon\Big)d_qu\right| \\
+\ & \frac{(t_2^\gamma-t_1^\gamma)}{2\Gamma_q(\gamma+1)}\Big(|\rho|\int_0^1\frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)}\nu_1(u)\psi_1(\varepsilon)d_qu \\
+\ & |\delta|\int_0^1\frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)}\nu_2(u)\psi_2(\varepsilon)d_qu\Big).
\end{aligned}$$

It is clear that the right hand side of the above inequality tends to zero independently of $x\in B_\varepsilon$ as $t_2-t_1\to 0$. Since $\mathcal{U}$ satisfies the above assumptions, it follows from the Arzelá-Ascoli theorem that $\mathcal{U}:\mathcal{C}\to\mathcal{C}$ is completely continuous.

(iii) Let x be a solution and $x=\kappa\mathcal{U}x$ for $\kappa\in(0,1)$. Using the arguments of the proof of boundedness of $\mathcal{U}$, for $t\in[0,1]$ we can write

$$|x(t)|=|\kappa(\mathcal{U}x)(t)|\leq|\rho|\|\nu_1\|\psi_1(\|x\|)\mu_1+|\delta|\|\nu_2\|\psi_2(\|x\|)\mu_2+|\lambda|\|x\|\mu_3.$$

Consequently, we have

$$\|x\|\leq\frac{|\rho|\|\nu_1\|\psi_1(\|x\|)\mu_1+|\delta|\|\nu_2\|\psi_2(\|x\|)\mu_2}{1-|\lambda|\mu_3}.$$

In view of (A_4), there exists ω such that $\|x\|\neq\omega$. We set

$$V=\{x\in\mathcal{C}:\|x\|<\omega\},$$

and observe that the operator $\mathcal{U}:\overline{V}\to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of V, there is no $x\in\partial V$ such that $x=\kappa\mathcal{U}(x)$ for some $\kappa\in(0,1)$. Consequently, we can apply Lemma 3.2, a nonlinear Leray-Schauder type alternative, to conclude that $\mathcal{U}$ has a fixed point $x\in\overline{V}$ which is a solution of the problem (1.1), (1.2). This completes the proof of Theorem 3.2. □

Now we are going to prove the uniqueness of solutions of problem (1.1), (1.2), using Banach's contraction principle (that is, Banach fixed point theorem).

Theorem 3.3. *Suppose that the assumption (A_1) holds and that*

$$\overline{\Omega} = (L\Lambda + |\lambda|\mu_3) < 1, \quad \Lambda = |\rho|\mu_1 + |\delta|\mu_2, \tag{3.5}$$

where μ_1, μ_2 and μ_3 are given by (3.2) and $L = \max\{L_1,\ L_2\}$. Then the boundary value problem (1.1), (1.2) has a unique solution.

Proof. Define $N = \max\{N_1,\ N_2\}$, where N_1 and N_2 are finite numbers given by $N_1 = \sup_{t\in[0,1]} |f(t,0)|$ and $N_2 = \sup_{t\in[0,1]} |g(t,0)|$. Selecting $\sigma \geq \dfrac{N\Lambda}{1-\overline{\Omega}}$, we show that $\mathcal{U}B_\sigma \subset B_\sigma$, where $B_\sigma = \{x \in \mathcal{C} : \|x\| \leq \sigma\}$. Indeed, using the inequalities

$$|f(s,x(s)) \leq |f(s,x(s)) - f(s,0)| + |f(s,0)| \leq L_1\sigma + N_1,$$

and

$$|g(s,x(s))| \leq |g(s,x(s)) - g(s,0)| + |g(s,0)| \leq L_2\sigma + N_2$$

for $x \in B_\sigma$, we can write (see (3.4))

$$\begin{aligned}
\|(\mathcal{U}x)\| \leq\ & |\rho|(L_1\sigma + N_1) \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_qm \Big) d_qu \\
& + \frac{1}{4\Gamma_q(\gamma+1)} \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} d_qu \\
& + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} d_qm \Big) d_qu \Big\} \\
& + |\delta|(L_2\sigma + N_2) \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} d_qm \Big) d_qu \\
& + \frac{1}{4\Gamma_q(\gamma+1)} \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} d_qu \\
& + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(\int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} d_qm \Big) d_qu \Big\} \\
& + |\lambda|\sigma \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_qu + \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} d_qu \Big\} \\
\leq\ & (L\sigma + N)\Lambda + |\lambda|\sigma\mu_3 \leq \sigma,
\end{aligned}$$

showing that $\mathcal{U}B_\sigma \subset B_\sigma$.

Next, for $x, y \in \mathcal{C}$, we have

$$
\begin{aligned}
&\|\mathcal{U}x - \mathcal{U}y\| \\
&\leq \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m \\
&+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(m, x(m)) - g(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \\
&+ \frac{1}{4\Gamma_q(\gamma+1)} \Big(|\rho| \int_0^1 \frac{(1-qu)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(u, x(u)) - f(u, y(u)) \Big| d_q u \\
&+ |\delta| \int_0^1 \frac{(1-qu)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(u, x(u)) - g(u, y(u)) \Big| d_q u \Big) \\
&+ \frac{1}{2} \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \Big(|\rho| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \Big| f(m, x(m)) - f(m, y(m)) \Big| d_q m \\
&+ |\delta| \int_0^u \frac{(u-qm)^{(\beta+\zeta-1)}}{\Gamma_q(\beta+\zeta)} \Big| g(m, x(m)) - g(m, y(m)) \Big| d_q m + |\lambda| |x(u) - y(u)| \Big) d_q u \Bigg\} \\
&\leq \overline{\Omega} \|x - y\|.
\end{aligned}
$$

Taking into account that by our assumption $\overline{\Omega} < 1$,, we conclude that the operator $\mathcal{U}$ is a contraction. Therefore, by Banach's contraction principle, the problem (1.1), (1.2) has a unique solution. This completes the proof of Theorem 3.3. □

4. An Example

Consider a boundary value problem for integro-differential equations of fractional order given by

$$
(4.1) \qquad \begin{cases} {}^cD_q^{1/2}({}^cD_q^{1/2} + \frac{1}{16})x(t) = \frac{1}{3}f(t, x(t)) + \frac{1}{7}I_q^{1/2}g(t, x(t)), \quad t, q \in (0,1), \\ x(0) = -x(1), \quad {}^cD_q^{\gamma}x(0) = -{}^cD_q^{\gamma}x(1), \end{cases}
$$

where $f(t,x) = \frac{1}{(4+t^2)^2}\Big(\sin t + \frac{|x|}{1+|x|} + |x|\Big)$ and $g(t,x) = \frac{1}{4}\tan^{-1} x + t^3 + 6$.

It is clear that

$$
|f(t,x) - f(t,y)| \leq \frac{1}{8}|x-y|, \; |g(t,x) - g(t,y)| \leq \frac{1}{4}|x-y|.
$$

With $\beta = \gamma = \zeta = q = 1/2$, $\lambda = 1/16$, $p = 1/3$, $k = 1/7$, $L_1 = 1/8$, $L_2 = 1/4$, we find that $\overline{\Omega} \simeq 0.2905925472 < 1$.

Clearly $L = \max\{L_1, L_2\} = 1/4$. Thus all the assumptions of Theorem 3.3 are satisfied. Hence, by the conclusion of Theorem 3.3, the problem (4.1) has a unique solution.

Список литературы

[1] D. Baleanu, K. Diethelm, E. Scalas, J. J.Trujillo, *Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos* (World Scientific, Boston, 2012).
[2] R.P. Agarwal, Y. Zhou, J.R. Wang, X. Luo, "Fractional functional differential equations with causal operators in Banach spaces", Math. Comput. Modelling, **54**, 1440-1452, 2011.
[3] B. Ahmad, J.J. Nieto, "Sequential fractional differential equations with three-point boundary conditions", Comput. Math. Appl., **64**, 3046-3052, 2012.
[4] F.H. Jackson, "On q-difference equations", American J. Math., **32**, 305-314, 1910.
[5] G. Bangerezako, "q-difference linear control systems", J. Difference Equ. Appl., **17**, 1229-1249, 2011.
[6] V. Kac and P. Cheung, *Quantum Calculus* (Springer, New York, 2002).
[7] M.H. Annaby, Z.S. Mansour, *q-Fractional Calculus and Equations* (Lecture Notes in Mathematics 2056, Springer-Verlag, Berlin, 2012).
[8] R. Ferreira, "Nontrivial solutions for fractional q-difference boundary value problems", Electron. J. Qual. Theory Differ. Equ., **70**, 1-10, 2010.
[9] J. R. Graef and L. Kong, "Positive solutions for a class of higher order boundary value problems with fractional q-derivatives", Appl. Math. Comput., **218**, 9682-9689, 2012.
[10] B. Ahmad, S. K. Ntouyas, I. K. Purnaras, "Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations", Adv. Difference Equ., **140**, 1-15, 2012.
[11] W. A. Al-Salam, "Some fractional q-integrals and q-derivatives", Proc. Edinb. Math. Soc., **15**, 135-140, 1966-1967.
[12] R. Agarwal, "Certain fractional q-integrals and q-derivatives", Proc. Cambridge Philos. Soc., **66**, 365-370, 1969.
[13] Y. Chen, J.J. Nieto, D. O'Regan, "Anti-periodic solutions for evolution equations associated with maximal monotone mappings", Appl. Math. Lett., **24**, 302-307, 2011.
[14] B. Ahmad, J.J. Nieto, "Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory", Topol. Methods Nonlinear Anal., **35**, 295-304, 2010.
[15] S.C. Lim, M. Li, L.P. Teo, "Langevin equation with two fractional orders", Phys. Lett. A, **372**, 6309-6320, 2008.
[16] B. Ahmad, J.J. Nieto, A. Alsaedi, M. El-Shahed, "A study of nonlinear Langevin equation involving two fractional orders in different intervals", Nonlinear Anal. Real World Appl., **13**, 599-606, 2012.
[17] P.M. Rajkovic, S.D. Marinkovic, M.S. Stankovic, "On q-analogues of Caputo derivative and Mittag-Leffler function", Fract. Calc. Appl. Anal., **10**, 359-373, 2007.
[18] P. M. Rajkovic, S. D. Marinkovic, M. S. Stankovic, "Fractional integrals and derivatives in q-calculus", Appl. Anal. and Discrete Math., **1**, 311-323, 2007.
[19] D.R. Smart, *Fixed Point Theorems* (Cambridge University Press, Cambridge, 1980).
[20] A. Granas, J. Dugundji, *Fixed Point Theory* (Springer-Verlag, New York, 2003).

Поступила 10 января 2014