Известия НАН Армении. Математика, том 49, н. 4, 2014, стр. 61-76.

SOME PROPERTIES OF m-TH ROOT FINSLER METRICS

A. TAYEBI, A. NANKALI AND E. PEYGHAN

University of Qom, Qom, Iran
Arak University, Arak, Iran

E-mails: *akbar.tayebi@gmail.com, ali.nankali2327@yahoo.com, epeyghan@gmail.com*

Abstract. We prove that every m-th root metric with isotropic mean Berwald curvature reduces to a weakly Berwald metric. Then we show that an m-th root metric with isotropic mean Landsberg curvature is a weakly Landsberg metric. We find necessary and sufficient condition under which conformal β-change of an m-th root metric is locally dually flat. Finally, we prove that the conformal β-change of locally projectively flat m-th root metrics are locally Minkowskian.

MSC2010 numbers: 53C60, 53C25.

Keywords: Conformal change; m-th root metric; β-change; locally dually flat metric; projectively flat metric.

1. Introduction

Let (M, F) be a Finsler manifold of dimension n, TM be its tangent bundle and (x^i, y^i) be the coordinates in a local chart on TM.

An m-th root Finsler metric on M, denoted by F, is defined to be $F = \sqrt[m]{A}$, where A is given by $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 \dots i_m}$ symmetric in all its indices (see [4], [9], [14] – [16]).

The theory of m-th root metrics has been developed by Shimada [14], and applied to Biology as an ecological metric [2]. It can be regarded as a direct generalization of Riemannian metric in the sense that the second root metric is a Riemannian metric.

Let (M, F) be a Finsler manifold of dimension n. Denote by $\tau(x, y)$ the distortion of the Minkowski norm F_x on $T_x M_0$, and let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$. The rate of change of $\tau(x, y)$ along Finslerian geodesics $\sigma(t)$ is called S-curvature. The Finsler metric F is said to have isotropic S-curvature and almost isotropic S-curvature if $\mathbf{S} = (n+1)cF$ and $\mathbf{S} = (n+1)cF + dh$, respectively, where $c = c(x)$ and $h = h(x)$ are scalar functions defined on M and $dh = h_{x^i}(x) y^i$ is the differential of h [19]. Taking twice vertical covariant derivatives of the S-curvature gives rise to the E-curvature. The Finsler metric F is called weakly Berwald metric if $\mathbf{E}=0$ and is said to have isotropic mean Berwald curvature if $\mathbf{E}=\frac{n+1}{2}cF\mathbf{h}$, where $c=c(x)$ is a scalar function defined on M and $\mathbf{h}=h_{ij}dx^i dx^j$ is the angular metric.

Theorem 1.1. *Let $F=\sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U\subseteq\mathbb{R}^n$.*

(i) For a scalar function $c=c(x)$ on M, the following are equivalent:

: *(ia)* $\mathbf{S}=(n+1)cF+\eta$;

: *(ib)* $\mathbf{S}=\eta$.

(ii) For a scalar function $c=c(x)$ on M, the following are equivalent:

: *(iia)* $\mathbf{E}=\frac{n+1}{2}cF\mathbf{h}$;

: *(iib)* $\mathbf{E}=0$.

Let (M,F) be a Finsler manifold. There are two basic tensors on Finsler manifolds: the fundamental metric tensor $\mathbf{g}_y$ and the Cartan torsion $\mathbf{C}_y$, which are the second and the third order derivatives of $\frac{1}{2}F_x^2$ at $y\in T_xM_0$, respectively. Taking a trace of Cartan torsion $\mathbf{C}_y$ gives us the mean Cartan torsion $\mathbf{I}_y$. The rate of change of the Cartan torsion along the Finslerian geodesics, $\mathbf{L}_y$, is called Landsberg curvature (see [17], [18]). Taking a trace of Landsberg curvature $\mathbf{L}_y$ yields the mean Landsberg curvature $\mathbf{J}_y$. The metric F is called isotropic mean Landsberg curvature if $\mathbf{J}=cF\mathbf{I}$, where $c=c(x)$ is a scalar function on M.

Theorem 1.2. *Let (M,F) be a non-Riemannian m-th root Finsler manifold. For a scalar function $c=c(x)$ defined on M, the following are equivalent:*

(ia): $\mathbf{J}+cF\mathbf{I}=0$;

(ib): $\mathbf{J}=0$.

There are two important transformation in Finsler geometry: the conformal change and the β-change. Two metric functions F and $\bar{F}$ defined on a manifold M are called conformal if the length of an arbitrary vector in the one is proportional to the length in the other, that is, if $\bar{g_{ij}}=\varphi g_{ij}$. Here the length of a vector ε means the fact that

φg_{ij}, as well as g_{ij}, must be Finsler metric tensors and showed that φ falls into a point function.

A change of Finsler metric $F \to \bar{F}$ is called a β-change of F, if $\bar{F}(x, y) = F(x, y) + \beta(x, y)$, where $\beta(x, y) = b_i(x)y^i$ is an 1-form on a smooth manifold M. It is easy to see that, if $\sup_{F(x,y)=1} |b_i(x)y^i| < 1$, then $\bar{F}$ is again a Finsler metric. The notion of a β-change has been proposed by Matsumoto, named by Hashiguchi-Ichijyō, and was studied in detail by Shibata (see [6], [8], [13]). If the Finsler metric F reduces to a Riemannian metric, then $\bar{F}$ reduces to a Randers metric. So, the β-change is also called the Randers change of Finsler metric.

Let (M, F) be a Finsler manifold. We consider the conformal β-changes of Finsler metrics $\bar{F} = e^{\alpha(x)}F + \beta$, where $\beta(x, y) = b_i(x)y^i$ is an 1-form on a smooth manifold M and $\alpha = \alpha(x)$ is the conformal factor. It is easy to see that, if $\sup_{F(x,y)=1} ||\beta|| < 1$, then $\bar{F}$ is again a Finsler metric.

Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Put

$$A_i = \frac{\partial A}{\partial y^i}, \ \ A_{ij} = \frac{\partial^2 A}{\partial y^j \partial y^j}, \ \ A_{x^i} = \frac{\partial A}{\partial x^i}, \ \ A_0 = A_{x^i} y^i.$$

Suppose that A_{ij} defines a positive definite tensor and let A^{ij} denote its inverse. The following equalities hold:

$$\begin{aligned}
g_{ij} &= \frac{A^{\frac{2}{m}} - 2}{m^2}[mAA_{ij} + (2 - m)A_i A_j], \\
g^{ij} &= A^{-\frac{2}{m}}[mAA^{ij} + \frac{m-2}{m-1}y^i y^j], \\
y^i A_i &= mA, \ \ y^i A_{ij} = (m-1)A_j, \ \ A^{ij}A_i = \frac{1}{m-1}y^j, \\
y_i &= \frac{1}{m}A^{\frac{2}{m}-1}A_i, \ \ A_i A_j A^{ij} = \frac{m}{m-1}A.
\end{aligned}$$

In [1], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they studied the information geometry on Riemannian manifolds. A Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is called dually flat if it satisfies the equality $(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}$ (see [12], [19]).

We consider conformal β-changes of locally dually flat m-th root Finsler metrics and prove the following result.

Theorem 1.3. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = e^{\alpha}F + \beta$ is a conformal β-change of F, where $\beta = b_i(x)y^i$ and $\alpha = \alpha(x)$. Then $\bar{F}$ is locally dually flat if and only if there exists an 1-form $\theta = \theta_l(x)y^l$ on U such that the following equalities hold:*

$$(1.1)\quad \beta_{0l}\beta + \beta_l\beta_0 = 2\beta\beta_{x^l},$$

$$(1.2)\quad A_{x^l} = \frac{1}{3m}[mA\theta_l + 2\theta A_l + 2(\alpha_0 A_l - \alpha_{x^l}A)],$$

$$(1.3)\quad \beta[(\frac{1}{m} - 2)A_l A^{-1} A_0 - 4A_{x^l} + \alpha_0 A_l] + 2[A_l\beta_0 + (A_0\beta)_l] = -2me^{\alpha}A\Psi,$$

where $\beta_{0l} = \beta_{x^k y^l}y^k$, $\alpha_0 = \alpha_{x^l}y^l$, $\beta_{x^l} = (b_i)_{x^l}y^i$, $\beta_0 = \beta_{x^l}y^i$, $\beta_{0l} = (b_l)_0$ and $\Psi = \alpha_0\beta_l + \beta_{0l} - 2\beta_{x^l} - 2\alpha_{x^l}\beta$.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if $G^i = Py^i$, where $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$ (see [7]). Finally, we study conformal β-change of locally projectively flat m-th root metrics and prove the following result.

Theorem 1.4. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = e^{\alpha}F + \beta$ is a conformal β-change of F, where $\beta = b_i(x)y^i$ and $\alpha = \alpha(x)$. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.*

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M}T_xM$ the tangent bundle of M, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on TM_0;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;

(iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM:

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM$$

is positive definite.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u,v)\right]|_{t=0}, \quad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}$=0 if and only if F is Riemannian.

Given a Finsler manifold (M,F), then a global vector field G is induced by F on TM_0, which in standard coordinates (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(y)$ are local functions on TM. $\mathbf{G}$ is called the associated spray to (M,F). The projection of an integral curve of $\mathbf{G}$ is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy the equation $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

Define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u,v,w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u,v) := E_{jk}(y)u^j v^k$, respectively, where

$$B^i{}_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y), \quad E_{jk}(y) := \frac{1}{2}B^m{}_{jkm}(y),$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and the mean Berwald curvature, respectively. A Finsler metric is called Berwald metric and mean Berwald metric if $\mathbf{B} = 0$ or $\mathbf{E} = 0$, respectively.

A scalar function $\tau = \tau(x,y)$ on $TM \setminus \{0\}$

$$\tau(x,y) := \ln\Big[\frac{\sqrt{\det\Big(g_{ij}(x,y)\Big)}}{\mathrm{Vol}(\mathrm{B}^n(1))} \cdot \mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \Big| \, F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}\Big],$$

is called the distortion. Let

$$\mathbf{S}(x,y) := \frac{d}{dt}\Big[\tau\Big(\sigma(t), \dot{\sigma}(t)\Big)\Big]_{t=0},$$

where $\sigma(t)$ is the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. $\mathbf{S}$ is called the S-curvature. $\mathbf{S}$ said to be *isotropic* if there is a scalar functions $c(x)$ on M such that

$$\mathbf{S}(x,y) = (n+1)c(x)F(x,y).$$

3. Proof of Theorem 1.1

In local coordinates (x^i, y^i), the vector filed $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a global vector field on TM_0, where $G^i = G^i(x, y)$ are local functions on TM_0 given by

$$G^i := \frac{1}{4} g^{il} \Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \Big], \quad y \in T_x M.$$

By a simple calculation, we have the following result (see [22]).

Lemma 3.1. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Then the spray coefficients of F are given by*

$$G^i = \frac{1}{2}(A_{0j} - A_{x^j})A^{ij}.$$

Thus the spray coefficients of an m-th root Finsler metric are rational functions with respect to y.

Lemma 3.2. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Then the following are equivalent:*

a): $\mathbf{S} = (n+1)cF + \eta$;

b): $\mathbf{S} = \eta$,

where $c = c(x)$ is a scalar function and $\eta = \eta_i(x)y^i$ is an 1-form on M.

Proof. By Lemma 3.1, the E-curvature of an m-th root metric is a rational function in y. On the other hand, by taking twice vertical covariant derivatives of the S-curvature, we get the E-curvature. Thus the S-curvature is a rational function in y. Suppose that F has almost isotropic S-curvature, $\mathbf{S} = (n+1)c(x)F + \eta$, where $c = c(x)$ is a scalar function and $\eta = \eta_i(x)y^i$ is an 1-form on M. Then the left hand side of $\mathbf{S} - \eta = (n+1)c(x)F$ is a rational function in y, while the right hand side is an irrational function, implying that $c = 0$ and $\mathbf{S} = \eta$.

Lemma 3.3. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Then the following are equivalent:*

a): $\mathbf{E} = \frac{n+1}{2}cF\mathbf{h}$;

b): $\mathbf{E} = 0$,

where $c = c(x)$ is a scalar function on M.

Proof. Suppose that $F = \sqrt[m]{A}$ has an isotopic mean Berwald curvature:

$$\mathbf{E} = \frac{n+1}{2} cF\mathbf{h},$$

where $c = c(x)$ is a scalar function on M. The left hand side of $\mathbf{E} = \frac{n+1}{2} cF\mathbf{h}$ is a rational function in y, while the right hand side is an irrational function, implying that $c = 0$ and $\mathbf{E} = 0$.

Proof of Theorem 1.1 is an immediate consequence of Lemmas 3.2 and 3.3.

From Theorem 1.1 we infer the following result.

Corollary 3.1. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Suppose that F has isotropic S-curvature $\mathbf{S} = (n+1)cF$, for some scalar function $c = c(x)$ on M. Then $\mathbf{S} = 0$.*

A Finsler metric F satisfying $F_{x^k} = FF_{y^k}$ is called a Funk metric. The standard Funk metric on the Euclidean unit ball $B^n(1)$, denoted by Θ, is defined by

$$\Theta(x, y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - < x, y >^2)} + < x, y >}{1 - |x|^2}, \quad y \in T_xB^n(1) \simeq \mathbb{R}^n,$$

where $< \cdot, \cdot >$ and $|\cdot|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. In [5], Chen-Shen has introduced the notion of isotropic Berwald metrics. A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature has the following form:

$$B^i{}_{jkl} = c\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\}, \tag{3.1}$$

for some scalar function $c = c(x)$ on M. Berwald metrics are trivially isotropic Berwald metrics with $c = 0$. Funk metrics are also non-trivial isotropic Berwald metrics. In (3.1), putting $i = l$ we get

$$E_{ij} = \frac{n+1}{2} cF^{-1}h_{ij}.$$

Plugging it into (3.1) we obtain

$$B^i{}_{jkl} = \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{kl}\delta^i_j + E_{lj}\delta^i_k + E_{jk,l}y^i\}. \tag{3.2}$$

This means that every isotropic Berwald metric is a Douglas metric. For the definition of Douglas metrics we refer to [3].

Now, let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Suppose that F has isotropic Berwald curvature given by (3.1). By Lemma 3.1, the left hand side of (3.1) is a rational function in y, while the right hand side is an irrational function, implying that $c = 0$. Thus we have the following result.

Theorem 3.1. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Suppose that F has isotropic Berwald curvature. Then F is a Berwald metric.*

In [21], Tayebi-Rafie Rad proved that every isotropic Berwald metric (3.1) on a manifold M has isotopic S-curvature $\mathbf{S} = (n+1)cF$, for some scalar function $c = c(x)$ on M. Thus, as an immediate consequence of Theorem 3.1, we can state the following result.

Corollary 3.2. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Suppose that F has isotropic Berwald curvature. Then $\mathbf{S} = 0$.*

4. Proof of Theorem 1.2

The quotient $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of the mean Cartan torsion $\mathbf{I}$ along Finslerian geodesics. Then F is said to be isotropic mean Landsberg metric if $\mathbf{J} = cF\mathbf{I}$, where $c = c(x)$ is a scalar function on M. In this section, we are going to prove Theorem 1.2. More precisely, we show that every m-th root isotropic mean Landsberg metric reduces to a weakly Landsberg metric.

Proof of Theorem 1.2: The mean Cartan tensor of F is given by the following formula:

$$\begin{aligned} I_i &= g^{jk}C_{ijk} = \frac{1}{m}A^{-3}\big[mAA^{jk} + \frac{m-2}{m-1}y^jy^k\big] \\ &\quad \times\Big[A^2A_{ijk} + (\frac{2}{m}-1)\{(\frac{2}{m}-2)A_iA_jA_k + A[A_iA_{jk} + A_jA_{ki} + A_kA_{ij}]\}\Big]. \end{aligned}$$

The mean Landsberg curvature of F is given by

$$\begin{aligned} J_i &= g^{jk}L_{ijk} = A^{-\frac{2}{m}}\big[mAA^{jk} + \frac{m-2}{m-1}y^jy^k\big]\big[-\frac{1}{2m}A^{\frac{2}{m}-1}A_sG^s_{ijk}\big] \\ &= -\frac{1}{2m}A^{-1}A_sG^s_{ijk}\big[mAA^{jk} + \frac{m-2}{m-1}y^jy^k\big]. \end{aligned}$$

Since $\mathbf{J} = cF\mathbf{I}$, then

$$A_s G^s_{ijk} = -2cA^{\frac{1}{m}-2}\Big[A^2 A_{ijk} + (\frac{2}{m}-1)\{(\frac{2}{m}-2)A_i A_j A_k + A[A_i A_{jk} + A_j A_{ki} + A_k A_{ij}]\}\Big].$$

By Lemma 3.1, the left hand side is a rational function in y, while its right-hand side is an irrational function in y. Thus, either $c = 0$ or A satisfies the following PDE:

$$A^2 A_{ijk} + (\frac{2}{m}-1)(\frac{2}{m}-2)A_i A_j A_k + (\frac{2}{m}-1)A\{A_i A_{jk} + A_j A_{ki} + A_k A_{ij}\} = 0.$$

This implies that $C_{ijk} = 0$. Hence, by Deike's theorem, F is a Riemannian metric, which contradicts our assumption, and hence $c = 0$. This completes the proof. □

By the similar method can be proved the following result.

Theorem 4.1. *Let $F = \sqrt[m]{A}$ be an non-Riemannian m-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Suppose that F has an isotropic Landsberg curvature, that is, $\mathbf{L} = cF\mathbf{C}$, where $c = c(x)$ is a scalar function on M. Then F reduces to a Landsberg metric.*

5. Proof of Theorem 1.3

A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients have the form $G^i = -\frac{1}{2}g^{ij}H_{y^j}$, where $H = H(x, y)$ is a C^∞ scalar function on $TM_0 = TM\setminus\{0\}$ satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system (see [15]). Recently, Shen proved that the Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies $(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}$. In this case we have $H = -\frac{1}{6}[F^2]_{x^m} y^m$.

In this section, we prove an extended version of Theorem 1.3. More precisely, we find a necessary and sufficient condition under which a conformal β-change of a generalized m-th root metric is locally dually flat. Let F be a scalar function on TM defined by $F = \sqrt{A^{2/m} + B}$, where A and B are given by

$$A := a_{i_1 \cdots i_m}(x) y^{i_1} \ldots y^{i_m}, \quad B := b_{ij}(x) y^i y^j.$$

Then F is called generalized m-th root Finsler metric. Suppose that matrix (A_{ij}) defines a positive definite tensor and (A^{ij}) denotes its inverse. Now, we are going to prove the following result.

Theorem 5.1. *Let $F=\sqrt{A^{2/m}+B}$ be a generalized m-th root Finsler metric on an open subset $U\subset\mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F}=e^{\alpha}F+\beta$ is a conformal β-change of F, where $\beta=b_i(x)y^i$ and $\alpha=\alpha(x)$. Then $\bar{F}$ is locally dually flat if and only if there exists an 1-form $\theta=\theta_l(x)y^l$ on U such that the following equalities hold:*

$$e^{2\alpha}[2B_{x^l}+4\alpha_{x^l}B-B_{0l}-2\alpha_0B_l]=2(\beta_l\beta_0+\beta\beta_{0l}-2\beta\beta_{x^l}),\tag{5.1}$$

$$A_{x^l}=\frac{1}{3m}[mA\theta_l+2\theta A_l+2(\alpha_0A_l-\alpha_{x^l}A)],\tag{5.2}$$

$$\Upsilon_l\Upsilon_0\beta=2\Upsilon[((\Upsilon_0\beta)_l+\Upsilon_l\beta_0+\alpha_0\beta\Upsilon_l-2\Upsilon_{x^l}\beta)+2e^{\alpha}\Upsilon\Psi],\tag{5.3}$$

where $\Upsilon:=A^{\frac{2}{m}}+B$, $\beta_{0l}=\beta_{x^ky^l}y^k$, $\alpha_0=\alpha_{x^l}y^l$, $\beta_{x^l}=(b_i)_{x^l}y^i$, $\beta_0=(b_i)_0y^i$, $\beta_{0l}=(b_l)_0$, and

$$\Upsilon_p=\frac{2}{m}A^{\frac{2}{m}-1}A_p+B_p,$$
$$\Upsilon_{0p}=\frac{2}{m}A^{\frac{2}{m}-2}[(\frac{2}{m}-1)A_pA_0+AA_{0p}]+B_{0p},$$
$$\Psi=\alpha_0\beta_l+\beta_{0l}-2\beta_{x^l}-2\alpha_{x^l}\beta.$$

To prove Theorem 5.1, we need the following lemma.

Lemma 5.1. *Suppose that the equation $\Phi A^{\frac{2}{m}-2}+\Psi A^{\frac{1}{m}-1}+\Theta=0$ holds, where Φ,Ψ,Θ are polynomials in y and $m>2$. Then $\Phi=\Psi=\Theta=0$.*

Proof of Theorem 5.1: We have

$$\begin{aligned}\bar{F}^2&=e^{2\alpha}(A^{\frac{2}{m}}+B)+2e^{\alpha}\beta(A^{\frac{2}{m}}+B)^{1/2}+\beta^2,\\(\bar{F}^2)_{x^k}&=2\alpha_{x^k}e^{2\alpha}(A^{\frac{2}{m}}+B)+e^{2\alpha}(\frac{2}{m}A^{\frac{2}{m}-1}A_{x^k}+B_{x^k})+2\alpha_{x^k}e^{\alpha}\beta(A^{\frac{2}{m}}+B)^{\frac{1}{2}}\\&\quad+e^{\alpha}[(A^{\frac{2}{m}}+B)^{-1/2}(\frac{2}{m}A^{\frac{2}{m}-1}A_{x^k}+B_{x^k})\beta+2(A^{\frac{2}{m}}+B)^{1/2}\beta_{x^k}]+2\beta_{x^k}\beta.\end{aligned}$$

Then

$$\begin{aligned}[\bar{F}^2]_{x^ky^l}y^k=\;&2\alpha_0e^{2\alpha}\Upsilon_l+e^{2\alpha}\Upsilon_{0l}+2\alpha_0e^{\alpha}\beta_l\Upsilon^{\frac{1}{2}}+\alpha_0e^{\alpha}\beta\Upsilon^{-\frac{1}{2}}\Upsilon_l+2e^{\alpha}\beta_{0l}\Upsilon^{\frac{1}{2}}\\&+\;e^{\alpha}\beta_0\Upsilon^{-\frac{1}{2}}\Upsilon_l+e^{\alpha}\beta_l\Upsilon^{-\frac{1}{2}}\Upsilon_0-\frac{1}{2}e^{\alpha}\beta\Upsilon^{-\frac{3}{2}}\Upsilon_l\Upsilon_0+e^{\alpha}\beta\Upsilon^{-\frac{1}{2}}\Upsilon_{0l}\\&+\;2\beta_l\beta_0+2\beta\beta_{0l}.\end{aligned}$$

Since $\bar{F}$ is a locally dually flat metric, we can write

$$\begin{aligned}
&e^{\alpha}\Upsilon^{-\frac{3}{2}}\Big[-\frac{1}{2}\beta\Upsilon_l\Upsilon_0+\Upsilon(\beta\Upsilon_{0l}+\beta_l\Upsilon_0+\beta_0\Upsilon_l+\alpha_0\beta\Upsilon_l-2\beta\Upsilon_{x^l})\\
&\quad+2e^{\alpha}\Upsilon^2(\alpha_0\beta_l+\beta_{0l}-2\alpha_{x^l}\beta-2\beta_{x^l})\Big]\\
&\quad+\frac{2}{m}e^{2\alpha}A^{\frac{2}{m}-2}\Big[2\alpha_0AA_l+(\frac{2}{m}-1)A_lA_0+AA_{0l}-2\alpha_{x^l}A^2-2AA_{x^l}\Big]\\
&\quad+e^{2\alpha}\Big[2\alpha_0B_l+B_{0l}-4\alpha_{x^l}B-2B_{x^l}\Big]\\
&\quad-4\beta\beta_{x^l}+2\beta_l\beta_0+2\beta\beta_{0l}=0.
\end{aligned}$$

By Lemma 5.1, we have

$$2\alpha_0AA_l+(\frac{2}{m}-1)A_lA_0+AA_{0l}-2\alpha_{x^l}A^2=2AA_{x^l}, \tag{5.4}$$

$$\frac{1}{2}\beta\Upsilon_l\Upsilon_0=\Upsilon[(\beta\Upsilon_0)_l+\beta_0\Upsilon_l+\alpha_0\beta\Upsilon_l-2\beta\Upsilon_{x^l}+2e^{\alpha}\Upsilon\Psi], \tag{5.5}$$

$$e^{2\alpha}\big[2\alpha_0B_l+B_{0l}-4\alpha_{x^l}B-2B_{x^l}\big]=2(2\beta\beta_{x^l}-\beta_l\beta_0-\beta\beta_{0l}). \tag{5.6}$$

The equality (5.4) can be written as follows

$$A(2A_{x^l}-A_{0l}+2\alpha_{x^l}A)=((\frac{2}{m}-1)A_0+2\alpha_0A)A_l. \tag{5.7}$$

Irreducibility of A and $deg(A_l)=m-1$ imply that there exists an 1-form $\theta=\theta_ly^l$ on U such that

$$A_0=\theta A. \tag{5.8}$$

By (5.8) we get

$$A_{0l}=A\theta_l+\theta A_l-A_{x^l}. \tag{5.9}$$

Substituting (5.8) and (5.9) into (5.7) we get (5.2). The converse assertion can be obtained by a direct computation. This completes the proof. □

6. Proof of Theorem 1.4

It is known that a Finsler metric $F(x,y)$ on $\mathcal{U}$ is projective if and only if its geodesic coefficients G^i have the form $G^i(x,y)=P(x,y)y^i$, where $T\mathcal{U}=\mathcal{U}\times\mathbb{R}^n\to\mathbb{R}$ is positively homogeneous with degree one, while $P(x,\lambda y)=\lambda P(x,y)$, $\lambda>0$. We call $P(x,y)$ the *projective factor* of $F(x,y)$. The following lemma plays an important role.

Lemma 6.1. (Rapcsák) *Let $F(x,y)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Then $F(x,y)$ is projective on $\mathcal{U}$ if and only if it satisfies*

$$F_{x^k y^l} y^k = F_{x^l}. \tag{6.1}$$

In this case, the projective factor $P(x,y)$ is given by

$$P = \frac{F_{x^k} y^k}{2F}. \tag{6.2}$$

Much earlier, G. Hamel proved that a Finsler metric $F(x,y)$ on $\mathcal{U} \subset \mathbb{R}^n$ is projective if and only if

$$F_{x^k y^l} = F_{x^l y^k}. \tag{6.3}$$

Thus (6.1) and (6.2) are equivalent.

In this section, we prove an extended version of Theorem 1.4. Specifically, we study the conformal β-change of a generalized m-th root metric $F = \sqrt{A^{\frac{2}{m}} + B}$, where A is irreducible, and prove the following result.

Theorem 6.1. *Let $F = \sqrt{A^{2/m} + B}$ be a generalized m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = e^{\alpha} F + \beta$ is a conformal β-change of F, where $\beta = b_i(x) y^i$, $\alpha = \alpha(x)$. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.*

To prove Theorem 1.4, we need the following lemma.

Lemma 6.2. *Let (M,F) be a Finsler manifold. Suppose that $\bar{F} = e^{\alpha} F + \beta$ is a conformal β-change of F. Then $\bar{F}$ is a projectively flat Finsler metric if and only if the following holds:*

$$e^{\alpha}(F_{0l} - F_{x^l}) = e^{\alpha}(\alpha_{x^l} F - \alpha_0 F_l) + (b_i)_{x^l} y^i - (b_l)_0. \tag{6.4}$$

Proof. We have

$$\begin{aligned}
\bar{F} &= e^{\alpha} F + \beta, \\
\bar{F}_{x^k} &= \alpha_{x^k} e^{\alpha} F + e^{\alpha} F_{x^k} + (b_i)x^k y^i, \\
\bar{F}_0 &= \alpha_0 e^{\alpha} F + e^{\alpha} F_0 + (b_i)_0 y^i, \\
\bar{F}_{0l} &= \alpha_0 e^{\alpha} F_l + e^{\alpha} F_0 + (b_l)_0,
\end{aligned}$$

and the result follows. This completes the proof.

Proposition 6.1. *Let $F=\sqrt{A^{2/m}+B}$ be a generalized m-th root Finsler metric on an open subset $U\subset\mathbb{R}^n$, where A is irreducible, $m>4$ and $B\neq 0$. Suppose that $\bar{F}=e^{\alpha}F+\beta$ is a conformal β-change of F, where $\beta=b_i(x)y^i,\alpha=\alpha(x)$. If $\bar{F}$ is a projectively flat metric, then F reduces to a Berwald metric.*

Proof. By Lemma 6.2 we have

$$F_{x^l}=\frac{2A^{2/m}A_{x^l}+mAB_{x^l}}{2mA\sqrt{A^{\frac{2}{m}}+B}},$$

Therefore

$$\begin{aligned}F_{x^ky^l}y^k=\ &(A^{\frac{2}{m}}+B)^{-1/2}\Big[\frac{1}{4}(\frac{2A^{2/m}A_0}{mA}+B_0)(\frac{2A^{2/m}A_l}{mA}+B_l)(A^{\frac{2}{m}}+B)^{-1}\\&+\frac{1}{2}(\frac{4A^{2/m}A_0A_l}{m^2A^2}+\frac{2A^{2/m}A_{0l}}{mA}-\frac{2A^{2/m}A_0A_l}{mA^2}+B_{0l})\Big].\end{aligned}$$

Thus

$$\begin{aligned}F_{0l}-F_{x^l}=e^{\alpha}\frac{(A^{\frac{2}{m}}+B)^{-\frac{3}{2}}}{m^2A^2}&\Big[A^{\frac{4}{m}}(mAA_l\alpha_0+(1-m)A_lA_0+mAA_{0l}-mAA_{x^l})\\&+A^{\frac{2}{m}}(mAA_lB\alpha_0+\frac{1}{2}m^2A^2B_l\alpha_0+(2-m)A_lA_0B+mAA_{0l}B)\\&+\frac{1}{2}mA^{\frac{2}{m}+1}(mAB_{0l}-A_0B_l-A_lB_0-A_{x^l}B-mAB_{x^l})\\&+\frac{1}{2}m^2A^2(BB_l\alpha_0+B_{0l}B-\frac{1}{2}B_0B_l-BB_{x^l})\Big].\end{aligned}$$

By (6.4) we obtain $\Phi A^{\frac{2}{m}}+\Psi A^{\frac{4}{m}}+\Theta=0$, where

$$\begin{aligned}\Phi=&-\frac{mA}{2}\Big[A_0B_l+B_oA_l+2B(A_{x^l}-A_l\alpha_0-A_{0l})+mA(B_{x^l}-B_l\alpha_0-B_{0l})\Big]\\&-\ (m-2)A_0A_lB,\\\Psi=&\ mA(A_{0l}+A_l\alpha_0-A_{x^l})-(m-1)A_0A_l,\\\Theta=&\ \frac{1}{4}m^2A^2\Big[2BB_l\alpha_0-2B_{0l}B+B_0B_l+2B_{x^l}B\Big],\\&+m^2A^2(A^{\frac{2}{m}}+B)^{\frac{3}{2}}e^{-2\alpha}\Big[(b_l)_0-(b_i)_{x^l}y^i+e^{\alpha}(\alpha_{x^l}A^{\frac{1}{m}}-\frac{1}{m}\alpha_0A^{\frac{1}{m}-1}A_l)\Big].\end{aligned}$$

By Lemma 5.1 we have

$$\Phi=0,\tag{6.5}$$

$$\Psi=0,\tag{6.6}$$

$$\Theta=0.\tag{6.7}$$

It follows from (6.6) that

$$mA(A_l\alpha_0 + A_{0l} - A_{x^l}) = (m-1)A_0A_l. \tag{6.8}$$

The irreducibility of A and $deg(A_l) = m-1 < deg(A)$ imply that A_0 is divisible by A. This means that there is an 1-form $\theta = \theta_l y^l$ on U, such that

$$A_0 = 2mA\theta. \tag{6.9}$$

Substituting (6.9) into (6.8), we obtain

$$A_{0l} = A_{x^l} - A_l\alpha_0 + 2(m-1)\theta A_l. \tag{6.10}$$

Plugging (6.9) and (6.10) into (6.5), we get

$$mA(2\theta B_l - B_{0l} - B_l\alpha_0 + B_{x^l}) = A_l(4B\theta - B_0). \tag{6.11}$$

Clearly, the right-hand side of (6.11) is divisible by A. Since A is irreducible, and both $\deg(A_l)$ and $\deg(2\theta B - \frac{1}{2}B)$ are less than $\deg(A)$, we have

$$B_0 = 4B\theta. \tag{6.12}$$

By (6.9) and (6.12), we get the spray coefficients $G^i = Py^i$ with $P = \theta$, showing that F is a Berwald metric. This completes the proof.

The Riemann curvature

$$\mathbf{K}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$$

is a family of linear maps on tangent spaces, defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For a flag $P = \mathrm{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{K}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2},$$

When F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P$, which is precisely the sectional curvature of P in Riemannian geometry. We say that a Finsler metric F is of scalar curvature if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent bundle TM_0. One of the important problems in Finsler geometry is to characterize the Finsler manifolds of scalar flag curvature (see

[10], [11]). If $\mathbf{K} = constant$, then the Finsler metric F is said to be of constant flag curvature.

Proof of Theorem 6.1. By Proposition 6.1, F is a Berwald metric. On the other hand, according to Numata's theorem, every Berwald metric of non-zero scalar flag curvature $\mathbf{K}$ must be Riemaniann. This contradicts to our assumption. Therefore $\mathbf{K} = 0$, showing that F reduces to a locally Minkowskian metric. □

Список литературы

[1] S.-I. Amari, Differential-Geometrical Methods in Statistics, Springer Lecture Notes in Statistics, Springer-Verlag (1985).

[2] P. L. Antonelli, R. Ingarden and M. Matsumoto, The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, Kluwer Acad. publ., Netherlands (1993).

[3] S. Bácsó and M. Matsumoto, "On Finsler spaces of Douglas type, A generalization of notion of Berwald space", Publ. Math. Debrecen, **51**, 385 – 406 (1997).

[4] V. Balan and N. Brinzei, "Einstein equations for (h, v)-Berwald-Moór relativistic models", Balkan. J. Geom. Appl., **11**(2), 20 – 26 (2006).

[5] X. Chen and Z. Shen, "On Douglas metrics", Publ. Math. Debrecen, **66**, 503 – 512 (2005).

[6] H. Hashiguchi and Y. Ichijyo, "Randers spaces with rectilinear geodetics", Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.), **13**, 33 – 40 (1980).

[7] B. Li and Z. Shen, "On projectively flat fourth root metrics", Canad. Math. Bull., **55**, 138 – 145 (2012).

[8] M. Matsumoto, "On Finsler spaces with Randers metric and special forms of important tensors", J. Math Kyoto Univ., **14**, 477 – 498 (1975).

[9] M. Matsumoto and H. Shimada, "On Finsler spaces with 1-form metric. II. Berwald-Moór's metric $L = \left(y^1 y^2 ... y^n\right)^{1/n}$", Tensor N. S., **32**, 275 – 278 (1978).

[10] B. Najafi, Z. Shen and A. Tayebi, "On Finsler metrics of scalar curvature with some non-Riemannian curvature properties", Geom. Dedicata, **131**, 87 – 97 (2008).

[11] B. Najafi and A. Tayebi, Finsler metrics of scalar flag curvature and projective invariants, Balkan. J. Geom. Appl, **15**, 90 – 99 (2010).

[12] Z. Shen, Riemann-Finsler geometry with applications to information geometry, Chin. Ann. Math., **27**, 73 – 94 (2006).

[13] C. Shibata, "On invariant tensors of β-changes of Finsler metrics, J. Math. Kyoto Univ., **24**, 163 – 188 (1984).

[14] H. Shimada, "On Finsler spaces with metric $L = \sqrt[m]{a_{i_1 i_2 \ldots i_m} y^{i_1} y^{i_2} \ldots y^{i_m}}$", Tensor, N.S., **33**, 365 – 372 (1979).

[15] A. Tayebi and B. Najafi, "On m-th root Finsler metrics", J. Geom. Phys. **61**, 1479 – 1484 (2011).

[16] A. Tayebi and B. Najafi, "On m-th root metrics with special curvature properties", C. R. Acad. Sci. Paris, Ser. I, **349**, 691 – 693 (2011).

[17] A. Tayebi and E. Peyghan, "Finsler metrics with special Landsberg curvature", Iran. J. Sci. Tech, Trans A, **33**, no. A3, 241 – 248 (2009).

[18] A. Tayebi and E. Peyghan, "On a special class of Finsler metrics", Iran. J. Sci. Tech, Trans A, **33**, no. A2, 179 – 186 (2009).

[19] A. Tayebi, E. Peyghan and H. Sadeghi, "On a class of locally dually flat Finsler metrics with isotropic S-curvature", Iran. J. Sci. Tech, Trans A, **36**, accepted (2012).

[20] A. Tayebi, E. Peyghan and M. Shahbazi, "On generalized m-th root Finsler metrics", Linear Algebra. Appl, **437**, 675 – 683 (2012).

[21] A. Tayebi and M. Rafie Rad, "S-curvature of isotropic Berwald metrics", Science in China, Series A: Mathematics, 51, 2198 – 2204 (2008).
[22] Y. Yu and Y. You, "On Einstein m-th root metrics", Diff. Geom. Appl., 28, 290 – 294 (2010).

Поступила 7 августа 2013