Известия НАН Армении. Математика, том 49, н. 4, 2014, стр. 47-60.

ON A CLASS OF ELLIPTIC SYSTEMS INVOLVING THE $p(x)$-LAPLACIAN AND NONLINEAR BOUNDARY CONDITIONS

S. H. RASOULI AND K. FALLAH

Babol University of Technology, Babol, Iran

E-mail: *s.h.rasouli@nit.ac.ir*

Abstract. In this paper we study an elliptic system involving the $p(x)$-Laplacian and nonlinear boundary conditions. We prove that there exist at least two positive solutions by using the Nehari manifold and the fibrering maps associated with the energy functional.

MSC2010 numbers: 35J05; 35J66

Keywords: Elliptic system; Nehari manifold; weight function.

1. Introduction

We are concerned with the existence and multiplicity of nontrivial nonnegative solutions of the following problem:

$$(1.1)\qquad \begin{cases} -\Delta_{p(x)}u = \frac{\alpha(x)}{\alpha(x)+\beta(x)} f(x)|u|^{\alpha(x)-2}u|v|^{\beta(x)} & \text{in } \Omega, \\ -\Delta_{p(x)}v = \frac{\beta(x)}{\alpha(x)+\beta(x)} f(x)|u|^{\alpha(x)}|v|^{\beta(x)-2}v & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = \lambda g(x)|u|^{q(x)-2}u, & \\ |\nabla v|^{p(x)-2}\frac{\partial v}{\partial n} = \mu h(x)|v|^{q(x)-2}v & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain; p, q, α and β belong to $C(\overline{\Omega})$ and satisfy: $1 < q(x) < p(x) < \alpha(x)+\beta(x) < p^*(x)$ ($p^*(x) = \frac{Np(x)}{N-p(x)}$ if $N > p(x)$, $p^*(x) = \infty$ if $N \leq p(x)$), $1 < p^- := ess\,inf_{x\in\Omega}\, p(x) \leq p(x) \leq p^+ := ess\,sup_{x\in\Omega}\, p(x) < \infty$, $1 < q^- \leq q^+ < p^- \leq p^+ < \alpha^- + \beta^- < \alpha^+ + \beta^+$, $(\lambda,\mu) \in \mathbb{R}^2\backslash(0,0)$, and the weight functions f, g, h satisfy the following conditions:

(A) $f \in C(\overline{\Omega})$ with $\|f\|_\infty = 1$ and $f^+ = \max\{f, 0\} \not\equiv 0$;

(B) $g, h \in C(\partial\Omega)$ with $\|g\|_\infty = \|h\|_\infty = 1$, $g^\pm = \max\{\pm g, 0\} \not\equiv 0$ and $h^\pm = \max\{\pm h, 0\} \not\equiv 0$.

The main interest in studying such problems is motivated by the presence of the $p(x)$-Laplace operator, $div(|\nabla u|^{p(x)-2}\nabla u)$ in (1.1). This is a generalization of the classical p-Laplace operator $div(|\nabla u|^{p-2}\nabla u)$, obtained in the case when p is a positive constant. We point out that problems involving elliptic equations with $p(x)$-Laplace operators are not trivial generalizations of the similar problems, studied in the constant case, since the $p(x)$- Laplace operator is not homogeneous, and thus, some techniques, such as the Lagrange multiplier theorem, used in the classical case are no longer applicable in the general setting involving $p(x)$-Laplace operators. On the other hand, stimulated by the development of the study of elastic mechanics, interest in variational problems and differential equations with $p(x)$-growth conditions has grown in recent decades (see, e.g., [3, 4, 8]), and systematic discussion of the spaces $W^{k,p(x)}(\Omega)$ becomes necessary. Also, note that the study of Lebesgue spaces $L^{p(x)}$ and Sobolev spaces $W^{1,p(x)}$ has been a subject of active research area (see, e.g., [7, 9]).

In a recent paper [1], Brown and Wu considered the corresponding semilinear elliptic system. They showed that the above problem has at least two nonnegative solutions if the pair (λ, μ) belongs to a certain subset of $\mathbb{R}^2$. In [10], the authors extended the results of [1], to the corresponding p-Laplacian system. The main purpose of this paper is to develop the approach used in [10], and to extend the results obtained in [10] to the case of $p(x)$-Laplacian with multiple parameters. This extension is nontrivial and requires more detailed analysis of the nonlinearity and an application of the variational methods under certain conditions.

2. Notations and preliminaries

In this section we discuss some basic properties of the Sobolev spaces $W^{1,p(x)}(\Omega)$, which will be used later (for details we refer to [5, 7, 9]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real-valued functions defined on Ω. Two functions from $\mathbf{S}(\Omega)$ will be identified, when they are equal almost everywhere. We set

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$

$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}),$$

and define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \text{ for } p \in C_-(\overline{\Omega})\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\, dx \le 1\},$$

and $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$,
with the norm $\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}$. Observe that $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces (see [7]).

Setting $\rho(u) = \int_\Omega |u(x)|^{L^{p(x)}(\Omega)}\,dx$, for any $u \in L^{p(x)}(\Omega)$, we can write

$$|u|_{L^{p(x)}(\Omega)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1 \tag{2.1}$$

$$|u|_{L^{p(x)}(\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\Omega)} \le \rho(u) \le |u|^{p^+}_{L^{p(x)}(\Omega)} \tag{2.2}$$

$$|u|_{L^{p(x)}(\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\Omega)} \le \rho(u) \le |u|^{p^-}_{L^{p(x)}(\Omega)}. \tag{2.3}$$

With $\rho(u) = \int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |u|^{p(x)}\,dx$, we have similar to (2.1)-(2.3) relations with $\|u\|$ instead of $|u|_{L^{p(x)}(\Omega)}$.

In the spaces $W^{1,p(x)}(\Omega)$ the Poincaré inequality holds, that is, there exists a positive constant C_0 such that

$$|u|_{L^{p(x)}(\Omega)} \le C_0 |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W^{1,p(x)}(\Omega),$$

implying that $|\nabla u|_{L^{p(x)}(\Omega)}$ is a norm equivalent to the norm $\|u\|$ in the space $W^{1,p(x)}(\Omega)$. Later we will use this equivalence, and for simplicity, we will write $\|u\|_p = |\nabla u|_{L^{p(x)}(\Omega)}$.

The embedding $W^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ is compact and continuous ($p^* = \frac{Np(x)}{N-p(x)}$ if $p(x) \le N$ and is $p^* = \infty$ if $p(x) > N$) (see [7]).

If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then there exists a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ for $1 \le q(x) \le p^\partial$, where p^∂ is $\frac{(N-1)p(x)}{N-p(x)}$, when $p(x) < N$, and is ∞, when $p(x) > N$ (see [5]).

3. The main results

Let $W^{1,p(x)} = W^{1,p(x)}(\Omega)$ b the usual Sobolev space. In the Banach space $W = W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega)$ we introduce the norm

$$\|(u,v)\|_W = \Big(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_\Omega |\nabla v|^{p(x)}\,dx\Big)^{\frac{1}{p(x)}},$$

and for $(u,v) \in W$ denote by $\mathfrak{J}_{\lambda,\mu}(u)$ the energy functional associated with problem (1.1), defined by the formula:

$$\mathfrak{J}_{\lambda,\mu}(u,v) := \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\,dx$$
$$-\frac{1}{\alpha(x)+\beta(x)}\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx - \frac{1}{q(x)}K_{\lambda,\mu}(u,v),$$

where

$$K_{\lambda,\mu}(u,v) = \lambda \int_{\partial\Omega} g(x)|u|^{q(x)}\, ds + \mu \int_{\partial\Omega} h(x)|v|^{q(x)}\, ds.$$

Denote by $\mathcal{N}_{\lambda,\mu}(\Omega)$ the Nehari manifold, defined by

$$\mathcal{N}_{\lambda,\mu}(\Omega) := \{(u,v) \in W\backslash\{(0,0)\} : \langle \mathcal{J}'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\}.$$

The energy functional $\mathcal{J}_{\lambda,\mu}$ is not bounded below on the whole space W, implying that $(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega)$ if and only if

$$\mathcal{I}(u,v) = \langle \mathcal{J}'_{\lambda,\mu}(u,v),(u,v)\rangle = \int_\Omega (|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\, dx$$

$$- \int_\Omega f(x)|u|^{\alpha(x)}|v|^{\alpha(x)}\, dx - K_{\lambda,\mu}(u,v) = 0.$$

Therefore for $(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega)$ we have

$$\langle \mathcal{I}'(u,v),(u,v)\rangle = \int_\Omega p(x)(|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\, dx - \tag{3.1}$$

$$- \int_\Omega (\alpha(x)+\beta(x)) f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\, dx - q(x)K_{\lambda,\mu}(u,v) \le$$

$$\le (p^+ - (\alpha^- + \beta^-)) \int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\, dx + (p^+ - q^-)K_{\lambda,\mu}(u,v).$$

We split $\mathcal{N}_{\lambda,\mu}$ into three parts:

$$\mathcal{N}^+_{\lambda,\mu} = \{(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega) : \langle \mathcal{I}'(u,v),(u,v)\rangle > 0\},$$

$$\mathcal{N}^0_{\lambda,\mu} = \{(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega) : \langle \mathcal{I}'(u,v),(u,v)\rangle = 0\}.$$

$$\mathcal{N}^-_{\lambda,\mu} = \{(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega) : \langle \mathcal{I}'(u,v),(u,v)\rangle < 0\}.$$

Let $C_0 = (\frac{q^+}{p^-})^{\frac{p^-}{(p^- - q^+)}} C(\alpha,\beta,q,S,\overline{S})$ be a positive number, where

$$C(\alpha,\beta,q,S,\overline{S}) = (\frac{\alpha^+ + \beta^+ - q^+}{p^- - q^+} S^{\alpha^+ + \beta^+})^{\frac{p^-}{(p^- - \alpha^+ - \beta^+)}} (\frac{\alpha^- + \beta^- - p^+}{\alpha^- + \beta^- - q^-} \overline{S}^{-q^+})^{\frac{p^-}{p^- - q^+}}.$$

Theorem 3.1. *If the parameters λ and μ satisfy $0 < |\lambda|^{\frac{p^-}{p^- - q^+}} + |\mu|^{\frac{p^-}{p^- - q^+}} < C_0$, then the problem (1.1) has at least two solutions (u_0^+, v_0^+) and (u_0^-, v_0^-) such that $u_0^\pm \ge 0, v_0^\pm \ge 0$ in Ω and $u_0^\pm \ne 0, v_0^\pm \ne 0$. And, if $f \ge 0$, then $u_0^\pm > 0, v_0^\pm > 0$ in Ω.*

Lemma 3.1. *The energy functional $\mathcal{J}_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}(\Omega)$.*

Proof. Let $(u,v)\in \mathcal{N}_{\lambda,\mu}(\Omega)$ and $\|(u,v)\|_W>1$. By the Sobolev embedding theorem we have

$$\begin{aligned}
\mathcal{J}_{\lambda,\mu}(u,v) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx-\int_\Omega \frac{1}{\alpha(x)+\beta(x)}f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx-\\
&- \frac{1}{q(x)}K_{\lambda,\mu}(u,v)\geq \frac{1}{p^+}\int_\Omega(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx-\frac{1}{q^-}K_{\lambda,\mu}(u,v)\\
&- \frac{1}{\alpha^-+\beta^-}\Big((|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx-K_{\lambda,\mu}(u,v)\Big)\\
&\geq (\frac{1}{p^+}-\frac{1}{\alpha^-+\beta^-})\int_\Omega(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx\\
&+ (\frac{1}{\alpha^-+\beta^-}-\frac{1}{q^-})K_{\lambda,\mu}(u,v)\geq(\frac{\alpha^-+\beta^--p^+}{(\alpha^-+\beta^-)p^+})\|(u,v)\|_W^{p^-}-\\
&- \overline{S}^{q^+}(\frac{\alpha^-+\beta^--q^-}{(\alpha^-+\beta^-)q^-})(|\lambda|^{\frac{p^-}{p^--q^+}}+c_{12}|\mu|^{\frac{p^-}{p^--q^+}})\|(u,v)\|_W^{q^+}.
\end{aligned}$$

Since $p^->q^+$, we have $\mathcal{J}_{\lambda,\mu}(u,v)\to\infty$ as $\|(u,v)\|_W\to\infty$, implying that $\mathcal{J}_{\lambda,\mu}(u,v)$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}(\Omega)$.

Lemma 3.2. *If $0<|\lambda|^{\frac{p^-}{p^--q^+}}+|\mu|^{\frac{p^-}{p^--q^+}}<C(\alpha,\beta,q,S,\overline{S})$, then $\mathcal{N}^0_{\lambda,\mu}(\Omega)=\varnothing$.*

Proof. Assume the opposite, that is, $\mathcal{N}^0_{\lambda,\mu}(\Omega)\neq\varnothing$. Let $(u,v)\in\mathcal{N}^0_{\lambda,\mu}(\Omega)$ be such that $\|(u,v)\|_W>1$. Then for $0<|\lambda|^{\frac{p^-}{p^--q^+}}+|\mu|^{\frac{p^-}{p^--q^+}}<C(\alpha,\beta,q,S,\overline{S})$ we can write

$$\begin{aligned}
0=\langle \mathcal{J}'_{\lambda,\mu}(u,v),(u,v)\rangle &= \int_\Omega p(x)(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx-\\
&- (\alpha(x)+\beta(x))\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx-q(x)K_{\lambda,\mu}(u,v)\\
&\geq p^-\int_\Omega(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx\\
&- q^+\Big((|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx-\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx\Big)\\
&- (\alpha^++\beta^+)\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx\\
&\geq (p^--q^+)\int_\Omega p(x)(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})\,dx\\
&+ (q^+-(\alpha^++\beta^+))\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx.
\end{aligned}$$

Therefore

$$0\geq(p^--q^+)\|(u,v)\|_W^{p^-}+c_{10}(q^+-(\alpha^++\beta^+))S^{\alpha^++\beta^+}\|(u,v)\|_W^{\alpha^++\beta^+},$$

implying

$$\|(u,v)\|_W \geq c_{14}\Big(\frac{p^- - q^+}{\alpha^+ + \beta^+ - q^+} S^{\alpha^+ + \beta^+}\Big)^{\frac{1}{\alpha^+ + \beta^+ - p^-}}. \tag{3.2}$$

Similarly, we obtain

$$\begin{aligned}
&0 = \langle \mathcal{J}'_{\lambda,\mu}(u,v),(u,v)\rangle \\
&\leq p^+ \int_\Omega (|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\,dx - (\alpha^- + \beta^-)\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx - q^- K_{\lambda,\mu}(u,v) \\
&\leq p^+ \int_\Omega (|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\,dx - (\alpha^- + \beta^-)\big(\int_\Omega (|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\,dx - K_{\lambda,\mu}(u,v)\big) \\
&\quad -q^- K_{\lambda,\mu}(u,v).
\end{aligned}$$

Therefore

$$0 \leq (p^+ - (\alpha+\beta))\|(u,v)\|_W^{p^-} + (\alpha^- + \beta^- - q^-)K_{\lambda,\mu}(u,v),$$

implying

$$\|(u,v)\|_W \leq c_{15}\Big(\frac{\alpha^- + \beta^- - q^-}{\alpha^- + \beta^- - p^+}\Big)^{\frac{1}{p^- - q^+}} \overline{S}^{\frac{q^+}{p^- - q^+}} (|\lambda|^{\frac{p^-}{p^- - q^+}} + |\mu|^{\frac{p^-}{p^- - q^+}}). \tag{3.3}$$

From (3.2) and (3.3) we get $\|(u,v)\|_W < 1$ and $|\lambda|^{\frac{p^-}{p^- - q^+}} + |\mu|^{\frac{p^-}{p^- - q^+}} \geq C(\alpha,\beta,q,S,\overline{S})$. The obtained contradiction implies that $\mathcal{N}^0_{\lambda,\mu}(\Omega) = \varnothing$.

Lemma 3.3. *Suppose that (u_0,v_0) is a local minimizer for $\mathcal{J}_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}(\Omega)$, and that $(u_0,v_0) \notin \mathcal{N}^0_{\lambda,\mu}(\Omega)$. Then $\mathcal{J}'_{\lambda,\mu}(u_0,v_0) = 0$ in $W^{-1,p}$.*

Proof. The result can be obtained by the arguments similar to that of used in Brown and Zhang [2], and so is omitted.

Lemma 3.4. *We have*

- *if $(u_0,v_0) \in \mathcal{N}^+_{\lambda,\mu}$, then $K_{\lambda,\mu}(u,v) > 0$;*
- *if $(u_0,v_0) \in \mathcal{N}^0_{\lambda,\mu}$, then $K_{\lambda,\mu}(u,v) > 0$ and $\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx > 0$;*
- *if $(u_0,v_0) \in \mathcal{N}^-_{\lambda,\mu}$, then $\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx > 0$.*

Proof. The proof is an immediate consequence of $(u,v) \in \mathcal{N}_{\lambda,\mu}$ and (3.3).

We write $\mathcal{N}_{\lambda,\mu}(\Omega) = \mathcal{N}^+_{\lambda,\mu}(\Omega) \cup \mathcal{N}^-_{\lambda,\mu}(\Omega)$, and define

$$\theta^+_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}(\Omega)} \mathcal{J}_{\lambda,\mu}(u,v), \quad \theta^-_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}(\Omega)} \mathcal{J}_{\lambda,\mu}(u,v).$$

Theorem 3.2. *If $0 < |\lambda|^{\frac{p^-}{p^- - q^+}} + |\mu|^{\frac{p^-}{p^- - q^+}} < C_0$, then*

- *(i)* $\theta^+_{\lambda,\mu} < 0$;
- *(ii)* $\theta^-_{\lambda,\mu} > d_0$ *for some* $d_0 = d_0(\alpha, \beta, q, \overline{S}, S, \lambda, \mu) > 0$.

Proof. (i) Let $(u, v) \in \mathcal{N}^+_{\lambda,\mu}$. From (3.2) we have

$$\mathcal{J}_{\lambda,\mu}(u,v) \le \frac{1}{p^-}\int_\Omega (|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\,dx \tag{3.4}$$

$$-\frac{1}{\alpha^+ + \beta^+}\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx - \frac{1}{q^+}K_{\lambda,\mu}(u,v).$$

Since $(u, v) \in \mathcal{N}^+_{\lambda,\mu}(\Omega)$, we can write

(3.5)

$$p^+\int_\Omega (|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\,dx - (\alpha^- + \beta^-)\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx - q^- K_{\lambda,\mu}(u,v) > 0.$$

Next, by (3.4) we have

$$\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx < \frac{p^+ - q^-}{\alpha^- + \beta^- - q^-}\int_\Omega (|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\,dx, \tag{3.6}$$

and by (3.5) we can write

$$\mathcal{J}_{\lambda,\mu}(u,v) \le (\frac{1}{p^-} - \frac{1}{q^+})\int_\Omega (|\nabla u|^{p(x)} + |\nabla v|^{p(x)})\,dx \tag{3.7}$$

$$+(\frac{1}{q^+} - \frac{1}{\alpha^+ + \beta^+})\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx.$$

It follows from (3.6) and (3.7) that

$$\mathcal{J}_{\lambda,\mu}(u,v) < -\frac{(p^- - q^+)(\alpha^+ + \beta^+ - p^-)}{(\alpha^+ + \beta^+)p^- q^+}\|(u,v)\|_W^{p^-} < 0.$$

Thus, $\alpha^+_\lambda = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}}(\Omega)\mathcal{J}_\lambda(u,v) < 0$.

(ii) Let $(u, v) \in \mathcal{N}_{\lambda,\mu}(\Omega)$. By (3.2) we have

$$\frac{(p^- - q^+}{(\alpha^+ + \beta^+ - q^+)}\|(u,v)\|_W^{p^+} < \int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx.$$

By the Sobolev embedding theorem we obtain

$$\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx \le S^{\alpha^+ + \beta^+}\|(u,v)\|_W^{\alpha^+ + \beta^+}.$$

This implies

$$\|(u,v)\|_W > \Big(\frac{p^- - q^+}{\alpha^+ + \beta^+ - q^+}S^{\alpha^+ + \beta^+}\Big)^{\frac{1}{\alpha^+ + \beta^+ - p^-}}.$$

Therefore

$$\begin{aligned}
\mathcal{J}_{\lambda,\mu}(u,v) &\geq \|(u,v)\|_W^{q^+}\Big[\frac{\alpha^-+\beta^--p^+}{p^+(\alpha^-+\beta^-)}\|(u,v)\|_W^{p^--q^+} \\
&- \overline{S}^{q+}\frac{\alpha^-+\beta^--q^-}{q^-(\alpha^-+\beta^-)}(|\lambda|^{\frac{p^-}{p^--q^+}}+|\mu|^{\frac{p^-}{p^--q^+}}\Big] \\
&> \Big(\frac{p^--q^+}{\alpha^++\beta^+-q^+}S^{\alpha^++\beta^+}\Big)^{\frac{q^+}{\alpha^++\beta^+-p^-}} \\
&\times \Big[\frac{\alpha^-+\beta^--p^+}{p^+(\alpha^-+\beta^-)}\Big(\frac{p^--q^+}{\alpha^++\beta^+-q^+}S^{\alpha^++\beta^+}\Big)^{\frac{p^--q^+}{\alpha^++\beta^+-p^-}} \\
&-\overline{S}^{q+}\frac{\alpha^-+\beta^--q^-}{q^-(\alpha^-+\beta^-)}(|\lambda|^{\frac{p^-}{p^--q^+}}+|\mu|^{\frac{p^-}{p^--q^+}}\Big]
\end{aligned}$$

Choosing $0<|\lambda|^{\frac{p^-}{p^--q^+}}+|\mu|^{\frac{p^-}{p^--q^+}}<C_0$, we get $\mathcal{J}_{\lambda,\mu}(u,v)>d_0$ for all $(u,v)\in\mathcal{N}_{\lambda,\mu}^-$ and for some $d_0=d_0(\alpha,\beta,q,\overline{S},S,\lambda,\mu)>0$. This completes the proof.

We put

$$t_{\max}=\Big(\frac{(p^--q^+)\|(u,v)\|_W^{p^-}}{(\alpha^++\beta^+-q^+)\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx}\Big)^{\frac{1}{\alpha^++\beta^+-p^-}}.$$

Lemma 3.5. *For each $(u,v)\in W$ with $\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx>0$ we have*

(i) *if $K_{\lambda,\mu}(u,v)\leq 0$, then there is a unique $t^->t_{\max}$ such that $(t^-u,t^-v)\in\mathcal{N}_{\lambda,\mu}^-$ and*

$$\mathcal{J}_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}(tu,tv);$$

(ii) *if $K_{\lambda,\mu}(u,v)>0$, then there is a unique $0<t^+<t_{\max}<t^-$ such that $(t^+u,t^+v)\in\mathcal{N}_{\lambda,\mu}^+$, $(t^-u,t^-v)\in\mathcal{N}_{\lambda,\mu}^-$ and*

$$\mathcal{J}_{\lambda,\mu}(t^+u,t^+v)=\inf_{0\leq t\leq t_{\max}}\mathcal{J}_{\lambda,\mu}(tu,tv);\ \mathcal{J}_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}(tu,tv).$$

Proof. Fix $(u,v)\in W$ with $\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx>0$. Let

$$m(t)=t^{p^--q^+}\|(u,v)\|_W^{p^-}-t^{\alpha^++\beta^+-q^+}\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx \text{ for } t\geq 0.$$

Clearly, $m(0)=0$, $m(t)\to-\infty$ as $t\to\infty$. Moreover

$$m'(t)=(p^--q^-)t^{p^--q^+-1}\|(u,v)\|_W^{p^-}-(\alpha^++\beta^+-q)t^{\alpha^++\beta^+-q^+-1}\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx.$$

Hence we have $m'(t)=0$ at $t=t_{\max}, m'(t)>0$ for $t\in[0,t_{\max})$ and $m'(t)<0$ for $t\in(t_{\max},\infty)$. Thus, $m(t)$ achieves its maximum at t_{max}, increases for $t\in[0,t_{\max})$ and decreases for $t\in(t_{\max},\infty)$.

Furthermore, we have

$$m(t_{\max}) = \|(u,v)\|_W^{q^+}\Big[\big(\frac{p^- - q^+}{\alpha^+ + \beta^+ - q^+}\big)^{\frac{p^- - q^+}{\alpha^+ + \beta^+ - p^-}} - \big(\frac{p^- - q^+}{\alpha^+ + \beta^+ - q^+}\big)^{\frac{\alpha^+ + \beta^+ - q}{\alpha^+ + \beta^+ - p^-}}\Big]$$

$$\times\Big(\frac{\|(u,v)\|_W^{\alpha^+ + \beta^+}}{\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx}\Big)^{\frac{p^- - q^+}{\alpha^+ + \beta^+ - p^-}}$$

$$\geq \|(u,v)\|_W^{q^+}\big(\frac{\alpha^+ + \beta^+ - p^-}{\alpha^+ + \beta^+ - q^+}\big)\big(\frac{\alpha^+ + \beta^+ - q^+}{p^- - q^+}S^{\alpha^+ + \beta^+}\big)^{\frac{p^- - q^-}{p^- - \alpha^- - \beta^-}}. \tag{3.8}$$

(*i*) Let $K_{\lambda,\mu}(u,v) \leq 0$. There is a unique $t^- > t_{\max}$ such that $m(t^-) = K_{\lambda,\mu}(u,v)$ and $m'(t^-) < 0$. Then we have

$$(p^- - q^+)(t^-)^{p^-}\|(u,v)\|_W^{p^-} - (\alpha^+ + \beta^+ - q^+)(t^-)^{\alpha^+ + \beta^+}\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx =$$

$$= (t^-)^{1+q^+}m'(t^-) < 0,$$

and hence, we can write

$$\langle \mathcal{J}'_{\lambda,\mu}(t^-u, t^-v)\rangle = (t^-)^{q^+}[m(t^-) - K_{\lambda,\mu}(u,v)] = 0,$$

implying that $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$. Thus, for $t > t_{\max}$ we have

$$(p^- - q^+)\|(tu,tv)\|_W^{p^-} - (\alpha^+ + \beta^+ - q^+)\int_\Omega f(x)|tu|^{\alpha(x)}|tv|^{\beta(x)}\,dx < 0, \quad \frac{d^2}{dt^2}\mathcal{J}_{\lambda,\mu}(tu,tv) < 0,$$

and

$$\frac{d}{dt}\mathcal{J}_{\lambda,\mu}(tu,tv) = t\|(u,v)\|_W^{p^-} - t^{q^+}K_{\lambda,\mu}(u,v) - t^{\alpha^+ + \beta^+}\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx = 0,$$

for $t = t^-$. Thus, $\mathcal{J}_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}(tu,tv)$.

(*ii*) Let $K_{\lambda,\mu}(u,v) > 0$. Using (3.8) and

$$\begin{aligned} m(0) &= 0 < K_{\lambda,\mu}(u,v) \leq \overline{S}^{q^+}\big(|\lambda|^{\frac{p^-}{p^- - q^+}} + |\mu|^{\frac{p^-}{p^- - q^+}}\big)\|(u,v)\|_W^{q^+} \\ &< \|(u,v)\|_W^{q^+}\big(\frac{\alpha^+ + \beta^+ - p^-}{\alpha^+ + \beta^+ - q^+}\big)\big(\frac{\alpha^+ + \beta^+ - q^+}{p^- - q^+}S^{\alpha^+ + \beta^+}\big)^{\frac{p^- - q^+}{p^- - \alpha^- - \beta^-}} \leq m(t_{\max}), \end{aligned}$$

for $0 < |\lambda|^{\frac{p^-}{p^- - q^+}} + |\mu|^{\frac{p^-}{p^- - q^+}} < C(\alpha, \beta, q, S, \overline{S})$, we conclude that there are unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$, and

$$m(t^+) = K_{\lambda,\mu}(u,v) = m(t^-), \quad m'(t^+) > 0 > m'(t^-).$$

Then $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}, (t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$, and $\mathcal{J}_{\lambda,\mu}(t^-u, t^-v) \geq \mathcal{J}_{\lambda,\mu}(tu,tv) \geq \mathcal{J}_{\lambda,\mu}(t^+u, t^+v)$ for each $t \in [t^+, t^-]$ and $\mathcal{J}_{\lambda,\mu}(t^+u, t^+v) \leq \mathcal{J}_{\lambda,\mu}(tu,tv)$ for each $t \in$

$[0, t^+]$. Thus,

$$\mathcal{J}_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le t_{\max}} \mathcal{J}_{\lambda,\mu}(tu, tv) \quad \text{and} \quad \mathcal{J}_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge 0} \mathcal{J}_{\lambda,\mu}(tu, tv).$$

This completes the proof.

We choose $\bar{t}_{\max} = \left(\frac{(\alpha^-+\beta^--q^-)K_{\lambda,\mu}(u,v)}{(\alpha^-+\beta^--p^+)\|(u,v)\|_W^{p^-}}\right)^{\frac{1}{p^--q^+}} > 0$ for $(u, v) \in W$ and $K_{\lambda,\mu}(u, v) > 0$.

Theorem 3.3. *For each $(u, v) \in W$ with $K_{\lambda,\mu}(u, v) > 0$, we have*

- *if $\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx \le 0$, then there is a unique $0 < t^+ < \bar{t}_{\max}$ such that $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ and*

$$\mathcal{J}_{\lambda,\mu}(t^+u, t^+v) = \inf_{t\ge 0} \mathcal{J}_{\lambda,\mu}(tu, tv);$$

- *if $\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx > 0$, then there is a unique $0 < t^+ < \bar{t}_{\max} < t^-$ such that $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$, and $\mathcal{J}_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le \bar{t}_{\max}} \mathcal{J}_{\lambda,\mu}(tu, tv)$; $\mathcal{J}_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge 0} \mathcal{J}_{\lambda,\mu}(tu, tv)$.*

Proof. Fix $(u, v) \in W$ and $K_{\lambda,\mu}(u, v) > 0$. Let

$$\overline{m}(t) = t^{p^+-\alpha^--\beta^-}\|(u,v)\|_W^{p^-} - t^{q^--\alpha^--\beta^-}K_{\lambda,\mu}(u,v) \quad \text{for } t > 0. \tag{3.9}$$

Clearly, $\overline{m} \to -\infty$ as $t \to 0^+$ and $\overline{m}(t) \to 0$ as $t \to \infty$. Now the proof can be completed using the arguments of the proof of Lemma 3.5.

Theorem 3.4. *If $0 < |\lambda|^{\frac{p^-}{p^--q^+}} + |\mu|^{\frac{p^-}{p^--q^+}} < C_0$, then the functional $\mathcal{J}_{\lambda,\mu}$ has a minimizer (u_0^+, v_0^+) in $\mathcal{N}^+_{\lambda,\mu}$ and satisfies:*

(i) $\mathcal{J}_{\lambda,\mu}(u_0^+, v_0^+) = \theta^+_{\lambda,\mu}$;

(ii)(u_0^+, v_0^+) is a nontrivial nonnegative solution of problem (1.1), such that $u_0^+ \ge 0, v_0^+ \ge 0$ in Ω and $u_0^+ \ne 0, v_0^+ \ne 0$.

Proof. Let $\{u_n, v_n\}$ be the minimizing sequence for $\mathcal{J}_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$. Then in view of Lemma 3.1 and compact embedding theorem, there exist a subsequence $\{u_n, v_n\}$ and $(u_0^+, v_0^+) \in W$, such that (u_0^+, v_0^+) is a solution of problem (1.1) and

$$u_n^+ \rightharpoonup u_0^+ \text{ weakly in } W_0^{1,p}(\Omega),$$

$$u_n^+ \to u_0^+ \text{ strongly in } L^{q(x)}_{g(x)}(\partial\Omega) \text{ and in } L^{\alpha+\beta}_{f(x)}(\Omega),$$

$$v_n^+ \rightharpoonup v_0^+ \text{ weakly in } W_0^{1,p(x)}(\Omega),$$

$$v_n^+ \to v_0^+ \text{ strongly in } L^{q(x)}_{h(x)}(\partial\Omega) \text{ and in } L^{\alpha+\beta}_{f(x)}(\Omega),$$

and we have

$$K_{\lambda,\mu}(u_n^+, v_n^+) \to K_{\lambda,\mu}(u_0^+, v_0^+) \text{ as } n \to \infty,$$

$$\int_\Omega f(x)|u_n^+|^{\alpha(x)}|v_n^+|^{\beta(x)}\,dx \to \int_\Omega f(x)|u_0^+|^{\alpha(x)}|v_0^+|^{\beta(x)}\,dx \text{ as } n \to \infty.$$

We can write

$$\mathcal{J}_{\lambda,\mu}(u_n^+, v_n^+) = \frac{\alpha^- + \beta^- - p^+}{p^+(\alpha^- + \beta^-)}\|(u_n^+, v_n^+)\|_W^{p^-} - \frac{\alpha^- + \beta^- - q}{q^-(\alpha^- + \beta^- - p^+)}K_{\lambda,\mu}(u_n^+, v_n^+).$$

Therefore

$$\mathcal{J}_{\lambda,\mu}(u_n^+, v_n^+) \to \theta^+_{\lambda,\mu} < 0 \text{ as } n \to \infty.$$

It is easy to see that $K_{\lambda,\mu}(u_0^+, v_0^+) > 0$ as $n \to \infty$. We prove

$$u_n^+ \to u_0^+ \text{ strongly in } W^{1,p(x)}(\Omega),$$

$$v_n^+ \to v_0^+ \text{ strongly in } W^{1,p(x)}(\Omega).$$

Otherwise, suppose

$$u_n^+ \nrightarrow u_0^+ \text{ in } W^{1,p(x)}(\Omega) \text{ and } v_n^+ \nrightarrow v_0^+ \text{ in } W^{1,p(x)}(\Omega).$$

Then we have

$$\|u_0^+\|_{W^{1,p(x)}} < lim_{n\to\infty} \inf \|u_n^+\|_{W^{1,p(x)}} \text{ or } \|v_0^+\|_{W^{1,p(x)}} < lim_{n\to\infty} \inf \|v_n^+\|_{W^{1,p(x)}}.$$

Let $K_{\lambda,\mu}(u_n, v_n) > 0$ and

$$I_{(u,v)}(t) = \overline{m}(t) - \int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx,$$

where $\overline{m}(t)$ is as in (3.9). We have $I_{(u,v)}(t) \to -\infty$ as $t \to 0^+$ and

$$I_{(u,v)}(t) \to -\int_\Omega f(x)|u|^{\alpha(x)}|v|^{\beta(x)}\,dx \quad \text{as } t \to \infty.$$

We have $I'_{(u,v)}(t) = \overline{m}'(t)$. Using the argument of the proof of Lemma 3.5, we conclude that $\bar{t}_{\max}(u, v)$ is a maximum of $I_{(u,v)}(t)$, for $t \in (0, \bar{t}_{\max}(u, v))$, $I_{(u,v)}(t)$ increases and for $t \in (\bar{t}_{\max}(u, v), \infty)$ it decreases, so that

$$\bar{t}_{\max} = \Big(\frac{(\alpha^- + \beta^- - q^-)K_{\lambda,\mu}(u, v)}{(\alpha^- + \beta^- - p^+)\|(u, v)\|_W^{p^-}}\Big)^{\frac{1}{p^- - q^+}}.$$

Since $K_{\lambda,\mu}(u_0^+, v_0^+) > 0$, there is a unique $0 < t_0^+ < \bar{t}_{\max}(u_0^+, v_0^+)$ such that $(t_0^+ u_0^+, t_0^+ v_0^+) \in \mathcal{N}^+_{\lambda,\mu}$, Therefore

$$\mathcal{J}_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) = \inf_{0 \le t \le \bar{t}_{\max}} (u_0^+, v_0^+)\mathcal{J}_{\lambda,\mu}(t u_0^+, t v_0^+),$$

and we can write

$$I_{(u_0^+,v_0^+)}(t_0^+) = (t_0^+)^{-(\alpha+\beta)}\Big(\|(t_0^+u_0^+, t_0^+v_0^+)\|_W^p - K_{\lambda,\mu}(t_0^+u_0^+, t_0^+v_0^+) - \\ - \int_\Omega f(x)|t_0^+u_0^+|^{\alpha(x)}|t_0^+v_0^+|^{\beta(x)}\,dx\Big) = 0.$$

Moreover, for sufficiently large n we have

$$I_{(u_n,v_n)}(t_0^+) > 0,$$

implying that $\bar{t}_{\max}(u_n, v_n) > 1$. Furthermore, we have

$$I_{(u_n,v_n)}(1) = \|u_n, v_n)\|_W^p - K_{\lambda,\mu}(u_n, v_n) - \int_\Omega f(x)|u_n|^{\alpha(x)}|v_n|^{\beta(x)}\,dx = 0.$$

Since $I_{(u_n,v_n)}(t)$ increases for $t \in (0, \bar{t}_{\max}(u_n, v_n))$, for all $t \in (0,1]$ and for sufficiently large n we have $I_{(u_n,v_n)}(t) \le 0$. Therefore $1 < t_0^+ \le \bar{t}_{\max}(u_0^+, v_0^+)$.

On the other hand, $(t_0^+u_0^+, t_0^+v_0^+) \in \mathcal{N}_{\lambda,\mu}^+$ and

$$\mathcal{J}_{\lambda,\mu}(t_0^+u_0^+, t_0^+v_0^+) = \inf_{0\le t\le \bar{t}_{\max}} (u_0^+, v_0^+)\mathcal{J}_{\lambda,\mu}(tu_0^+, tv_0^+).$$

Therefore $\mathcal{J}_{\lambda,\mu}(t_0^+u_0^+, t_0^+v_0^+) < \mathcal{J}_{\lambda,\mu}(u_0^+, v_0^+) < \lim_{n\to\infty} \mathcal{J}_{\lambda,\mu}(u_n^+, v_n^+) = \theta_{\lambda,\mu}^+$, which is a contradiction. Thus both

$$u_n^+ \to u_0^+ \quad \text{and} v_n^+ \to v_0^+ \quad \text{strongly in } W_0^{1,p(x)}(\Omega),$$

and we obtain

$$\mathcal{J}_{\lambda,\mu}(u_n^+, v_n^+) \to \mathcal{J}_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}^+ \quad \text{as } n \to \infty.$$

Hence, (u_0^+, v_0^+) is a minimizer. With $\mathcal{J}_{\lambda,\mu}(u_0^+, v_0^+) = \mathcal{J}_{\lambda,\mu}(|u_0^+|, |v_0^+|)$, $(|u_0^+|, |v_0^+|) \in \mathcal{N}_{\lambda,\mu}^+(\Omega)$ and Lemma 3.2, (u_0^+, v_0^+) is a nonnegative solution of the problem (1.1). Now, we prove that $u_0^+ \ne 0, v_0^+ \ne 0$. We assume that $v_0^+ \equiv 0$. Then, since u_0^+ is a nonzero solution of the problem

$$\begin{cases} -\Delta_{p(x)}u = 0, & x \in \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = \lambda g(x)|u|^{q(x)-2}u, & x \in \partial\Omega, \end{cases}$$

we have

$$\|u_0^+\|_W^{p(x)} = \lambda \int_{\partial\Omega} g(x)|u_0^+|^{q(x)}\,ds > 0.$$

With $w \in W^{1,p(x)}(\Omega) \setminus \{0\}$

$$\|w\|_W^{p(x)} = \mu \int_{\partial\Omega} h(x)|w|^{q(x)}\,ds \ge 0,$$

we have

$$K_{\lambda,\mu}(u_0^+, w) = \int_{\partial\Omega} g(x)|u_0^+|^{q(x)}\, ds + \mu \int_{\partial\Omega} h(x)|w|^{q(x)}\, ds > 0,$$

implying that there is a unique $0 < t^+ < \bar{t}_{\max}$ such that $(t^+u_0^+, t^+w) \in \mathcal{N}^+_{\lambda,\mu}(\Omega)$.

We have

$$\bar{t}_{\max} = \Big(\frac{(\alpha^- + \beta^- - q^-)K_{\lambda,\mu}(u_0^+, w)}{(\alpha^- + \beta^- - p^+)\|(u_0^+, w)\|_W^{p^-}}\Big)^{\frac{1}{p^- - q^+}} = \Big(\frac{\alpha^- + \beta^- - q^-}{\alpha^- + \beta^- - p^+}\Big)^{\frac{1}{p^- - q^+}} > 1,$$

and

$$\mathfrak{J}_{\lambda,\mu}(t^+u_0^+, t^+w) = \inf_{0 \le t \le \bar{t}_{\max}} \mathfrak{J}_{\lambda,\mu}(tu_0^+, tw).$$

Therefore

$$\mathfrak{J}_{\lambda,\mu}(t^+u_0^+, t^+w) \le \mathfrak{J}_{\lambda,\mu}(u_0^+, w) < \mathfrak{J}_{\lambda,\mu}(u_0^+, 0) = \theta^+_{\lambda,\mu},$$

which is a contradiction, and the result follows.

Theorem 3.5. *If* $0 < (|\lambda|^{\frac{p^-}{p^- - q^+}} + |\mu|^{\frac{p^-}{p^- - q^+}} < C_0$, *then the functional* $\mathfrak{J}_{\lambda,\mu}$ *has a minimizer* (u_0^-, v_0^-) *in* $\mathcal{N}^-_{\lambda,\mu}$ *and satisfies:*

(i) $\mathfrak{J}_{\lambda,\mu}(u_0^-, v_0^-) = \theta^-_{\lambda,\mu}$;

(ii)(u_0^-, v_0^-) *is a nontrivial nonnegative solution of the problem* (1), *such that* $u_0^- \ge 0, v_0^- \ge 0$ *in* Ω *and* $u_0^- \ne 0, v_0^- \ne 0$.

Proof. The proof is similar to that of Theorem 3.4, and so is omitted.

The Proof of Theorem 3.1. In view of Theorems 3.4 and 3.5, we conclude that there exist $(u_0^+, v_0^+) \in N^+_{\lambda,\mu}$ and $(u_0^-, v_0^-) \in \mathcal{N}^-_{\lambda,\mu}$, such that $u_0^\pm \ge 0, v_0^\pm \ge 0$ in Ω and $u_0^\pm \ne 0, v_0^\pm \ne 0$. We have $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, implying that (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct. If $f \ge 0$, then by the maximum principle we have $u_0^\pm > 0, v_0^\pm > 0$.

Список литературы

[1] K. J. Brown and T.F. Wu, "A semilinear elliptic system involving nonlinear boundary condition and sign-changing weight function", J. Math. Anal.Appl., **337**, 1326 – 1336 (2008).

[2] K. J. Brown and Y. Zhang, "The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function", J. Differential Equations, **193**, no. 2, 481 – 499 (2003).

[3] Y. Chen, S. Levine, M. Rao, "Variable exponent, linear growth functionals in image restoration", SIAM J. Appl. Math. **66**, no. 4, 1383 – 1406 (2006).

[4] G. Dai, "Infinitely many solutions for a Neumann-type differential inclusion problem involving the $p(x)$-Laplacian", Nonlinear Anal. **70**, 2297 – 2305 (2009).

[5] S. G. Dend, "Eigenvalues of the p(x)-Laplacian *Steklov* problem", J. Mat. Anal. Appl., **339**, 925 – 937 (2008).

[6] D. Edmunds and J. Rakosnik, "Soboleve mbeddings with variable exponent", Studia Math., **143**, no. 3, 267 – 293 (2000).

[7] X.L. Fan, D. Zhao, "On the spaces $L^{p(x)}$ and $W^{k,p(x)}$", J. Math. Anal. Appl., **263**, 424 – 446 (2001).
[8] X. L. Fan, "Solutions for p(x)-Laplacian Dirichlet problems with singular coefficients", J. Math.Anal. Appl., **312**, no. 2, 464 – 477 (2005).
[9] X. L. Fan, J. S. Shen and D. Zhao, "Sobolev embedding theorems for spaces", J. Math. Anal. Appl., **262**, no. 2, 749 – 760 (2001).
[10] S. H. Rasouli and G. A. Afrouzi, "The Nehari manifold for a class of concave-convex elliptic systems involving the p-Laplacian and nonlinear boundary condition", Nonlinear Anal. **73**, 3390 – 3401 (2010).

Поступила 2 июня 2013