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CONVERGENCE IN MEASURE OF STRONG LOGARITHMIC MEANS OF DOUBLE FOURIER SERIES

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Abstract. Nörlund strong logarithmic means of double Fourier series acting from space $L\log L(\mathbb{T}^2)$ into space $L_p(\mathbb{T}^2)$, $0<p<1$, are studied. The maximal Orlicz space such that the Nörlund strong logarithmic means of double Fourier series for the functions from this space converge in two-dimensional measure is found.

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1. Introduction

It is known that the rectangular partial sums of double Fourier series $S_{n,m}(f;x,y)$ of a function $f\in L_p(\mathbb{T}^2)$, $\mathbb{T}:=[-\pi,\pi)$, $1<p<\infty$, converge in L_p norm to the function f as $n\to\infty$ (see [14]). In the case $L_1(\mathbb{T}^2)$ this result does not hold. But for $f\in L_1(\mathbb{T})$, the operator $S_n(f;x)$ is of weak type $(1,1)$ (see [16]). This fact implies convergence of $S_n(f;x)$ in measure on $\mathbb{T}$ to the function $f\in L_1(\mathbb{T})$. However, for double Fourier series this result does not hold (see [7, 9]). Moreover, it is proved that the quadratic partial sums $S_{n,n}(f;x,y)$ of double Fourier series do not converge in two-dimensional measure on $\mathbb{T}^2$ even for functions from Orlicz spaces wider than the Orlicz space $L\log L(\mathbb{T}^2)$. On the other hand, it is well-known that the rectangular partial sums $S_{n,m}(f;x,y)$ of a function $f\in L\log L(\mathbb{T}^2)$ converge in measure on $\mathbb{T}^2$.

Note that the classical regular summation methods often improve the convergence of Fourier series. For instance, the Fejér means of the double Fourier series of a function $f\in L_1(\mathbb{T}^2)$ converge in $L_1(\mathbb{T}^2)$ norm to the function f (see [14]). These means represent the particular case of the Nörlund means.

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The Nörlund logarithmic means of a double Fourier series are defined by

$$t_{n,m}(f;x,y) := \frac{1}{l_n l_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{S_{i,j}(f;x,y)}{(n-i+1)(m-j+1)},$$

where $l_n := \sum_{k=1}^{n+1}(1/k)$ and $S_{i,j}(f;x,y)$ denote the rectangular partial sums of the double Fourier series of the function f.

It is well known that the method of Nörlund logarithmic means of double Fourier series is weaker than the Cesбro method of any positive order. In [10] Tkebuchava proved that these means of double Fourier series in general do not converge in two-dimensional measure on $\mathbb{T}^2$ even for functions from Orlicz spaces wider than the Orlicz space $L\log L(\mathbb{T}^2)$. For Nörlund logarithmic means $t_{n,m}(f;x,y)$ of double Fourier series Tkebuchava [11] proved the following result.

Theorem 1.1. *Let $L_Q(\mathbb{T}^2)$ be an Orlicz space, such that*

$$L_Q(\mathbb{T}^2) \not\subseteq L\log L(\mathbb{T}^2).$$

Then the set of function from the Orlicz space $L_Q(\mathbb{T}^2)$ with logarithmic means of rectangular partial sums of double Fourier series, convergent in measure on $\mathbb{T}^2$, is of first Baire category in $L_Q(\mathbb{T}^2)$.

On the other hand, as it was noted in [1] Remark 1, the regularity of summation method does not allow to deduce the summability in measure of a functional sequence from its convergence in measure.

In this paper we consider the strong logarithmic means of rectangular partial sums of double Fourier series and prove that these means are acting from the space $L\log L(\mathbb{T}^2)$ into the space $L_p(\mathbb{T}^2), 0 < p < 1$ (Theorem 4.1). This fact implies convergence of strong logarithmic means of rectangular partial sums of double Fourier series in measure on $\mathbb{T}^2$ to the function $f \in L\log L(T^2)$ (Corollary 4.1). Uniting these results with Tkebuchava result from [10] (see Theorem 1.1), we prove that the rectangular partial sums of double Fourier series converge in measure for all functions from Orlicz space if and only if their Nörlund logarithmic means and strong Nörlund logarithmic means converge in measure (Theorem 4.3). Thus, not all classical regular summation methods can improve the convergence in measure of double Fourier series.

For the results on summability of logarithmic means of Walsh-Fourier series we refer the papers [3] - [5], [12, 13].

2. Double Fourier series and conjugate functions

We denote by $L_0 = L_0(\mathbb{T}^2)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{T}^2$.

Let $L_Q = L_Q(\mathbb{T}^2)$ be the Orlicz space (see [8]), generated by Young function Q, that is, Q is a convex continuous even function, such that $Q(0) = 0$ and

$$\lim_{u\to+\infty} \frac{Q(u)}{u} = +\infty, \quad \lim_{u\to 0} \frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_Q(\mathbb{T}^2)} = \inf\{k > 0 : \iint_{\mathbb{T}^2} Q(|f(x,y)|/k)dxdy \le 1\}.$$

In particular, if $Q(u) = u\log^+ u$, $\log^+ u := 1_{\{u>1\}} \log u$, then the corresponding space will be denoted by $L\log L(\mathbb{T}^2)$.

Given a function $f \in L_1(\mathbb{T}^2)$, its double Fourier series is defined by

$$\sum_{(n,m)\in\mathbb{Z}^2} \widehat{f}(m,n)\, e^{imx} e^{iny}, \tag{2.1}$$

where $\mathbb{Z}$ is the set of integers and

$$\widehat{f}(m,n) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(x,y) e^{-imx} e^{-iny} dxdy \tag{2.2}$$

are the Fourier coefficients of f.

Denote by $S_{n,m}(f;x,y)$ the $(n,m)^{th}$ symmetric rectangular partial sums of series (2.1). As is it well-known, we have

$$S_{n,m}(f;x,y) = \frac{1}{\pi^2} \iint_{\mathbb{T}^2} f(s,t)\, D_n(x-s)\, D_m(y-t)\, dsdt,$$

where

$$D_n(u) := \frac{\sin((n+1/2)u)}{2\sin(u/2)}$$

is the Dirichlet kernel.

One can associate three conjugate series to the double Fourier series (2.1):

(a) conjugate with respect to the first variable:

$$\widetilde{f}^{(1,0)} \sim \sum_{(j,k)\in\mathbb{Z}^2} (-i\,\mathrm{sign} j)\, \widehat{f}(j,k)\, e^{i(jx+ky)} \tag{2.3}$$

(b) conjugate with respect to the second variable:

$$\widetilde{f}^{(0,1)} \sim \sum_{(j,k)\in\mathbb{Z}^2} (-i\,\mathrm{sign} k)\, \widehat{f}(j,k)\, e^{i(jx+ky)} \tag{2.4}$$

(c) conjugate with respect to both variables:

$$\widetilde{f}^{(1,1)} \sim \sum_{(j,k)\in\mathbb{Z}^2} (-i\,\mathrm{sign}j)\,(-i\,\mathrm{sign}k)\,\widehat{f}(j,k)\,e^{i(jx+ky)}. \tag{2.5}$$

It is well known that for an integrable function f we have

$$\widetilde{f}^{(1,0)}(x,y) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{T}} \frac{f(s,y)}{2\tan\left(\frac{x-s}{2}\right)}ds,$$

$$\widetilde{f}^{(0,1)}(x,y) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{T}} \frac{f(x,t)}{2\tan\left(\frac{y-t}{2}\right)}dt$$

and

$$\widetilde{f}^{(1,1)}(x,y) = \text{p.v.}\frac{1}{\pi^2}\iint_{\mathbb{T}^2} \frac{f(s,t)}{2\tan\left(\frac{x-s}{2}\right)2\tan\left(\frac{y-t}{2}\right)}dsdt.$$

Privalov's theorem (see, e.g., [16], vol. II, p. 121) immediately implies the a. e. existence of $\widetilde{f}^{(1,0)}$ and $\widetilde{f}^{(0,1)}$ under the assumption $f \in L_1(\mathbb{T}^2)$. The a. e. existence of $\widetilde{f}^{(1,1)}$ for $f \in L\log L(\mathbb{T}^2)$ was proved by Zygmund (see [15, 17]).

We consider the symmetric rectangular partial sums of series (2.3)-(2.5) defined by

$$\widetilde{S}^{10}_{n,m}(f;x,y) := \sum_{|j|\le n}\sum_{|k|\le m} (-i\mathrm{sign}j)\,\widehat{f}(j,k)\,e^{i(jx+ky)},$$

$$\widetilde{S}^{01}_{n,m}(f;x,y) := \sum_{|j|\le n}\sum_{|k|\le m} (-i\mathrm{sign}k)\,\widehat{f}(j,k)\,e^{i(jx+ky)}$$

and

$$\widetilde{S}^{11}_{n,m}(f;x,y) := \sum_{|j|\le n}\sum_{|k|\le m} (-i\mathrm{sign}j)\,(-i\mathrm{sign}k)\,\widehat{f}(j,k)\,e^{i(jx+ky)}.$$

It follows from (2.2) that

$$\widetilde{S}^{10}_{n,m}(f;x,y) = \frac{1}{\pi^2}\iint_{\mathbb{T}^2} f(s,t)\,\widetilde{D}_n(x-s)\,D_m(y-t)\,dsdt,$$

$$\widetilde{S}^{01}_{n,m}(f;x,y) = \frac{1}{\pi^2}\iint_{\mathbb{T}^2} f(s,t)\,D_n(x-s)\,\widetilde{D}_m(y-t)\,dsdt$$

and

$$\widetilde{S}^{11}_{n,m}(f;x,y) = \frac{1}{\pi^2}\iint_{\mathbb{T}^2} f(s,t)\,\widetilde{D}_n(x-s)\,\widetilde{D}_m(y-t)\,dsdt,$$

where

$$\widetilde{D}_m(u) := \frac{1}{2\tan(u/2)} - \frac{\cos((m+1)u)}{2\sin(u/2)}, \quad m = 1,2,... \tag{2.6}$$

is the conjugate Dirichlet kernel.

In this paper we also consider the following operators

$$\widetilde{\overline{S}}_{n,m}^{10}(f;x,y)=\frac{1}{\pi^2}\iint\limits_{\mathbb{T}^2} f(s,t)\widetilde{D}_n(x-s)\overline{D}_m(y-t)\,dsdt,$$

$$\widetilde{\overline{S}}_{n,m}^{01}(f;x,y)=\frac{1}{\pi^2}\iint\limits_{\mathbb{T}^2} f(s,t)\overline{D}_n(x-s)\widetilde{D}_m(y-t)\,dsdt$$

and

$$\overline{S}_{n,m}(f;x,y)=\frac{1}{\pi^2}\iint\limits_{\mathbb{T}^2} f(s,t)\overline{D}_n(x-s)\overline{D}_m(y-t)\,dsdt,$$

where $\overline{D}_n(u)$ is the modified Dirichlet kernel defined by

$$\overline{D}_n(u):=\frac{\sin(nu)}{2\tan(u/2)}.$$

3. Strong Riesz logarithmic and strong Nörlund logarithmic means

The strong Riesz logarithmic means, the strong Nörlund logarithmic means and the strong Fejér means of rectangular partial sums $\widetilde{S}_{i,j}^{ab}f$ are defined by the following formulas, respectively:

$$\widetilde{R}_{n,m}^{ab}(f;x,y):=\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\left|\widetilde{S}_{i,j}^{ab}(f;x,y)\right|}{(i+1)(j+1)},$$

$$\widetilde{\tau}_{n,m}^{ab}(f;x,y):=\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\left|\widetilde{S}_{i,j}^{ab}(f;x,y)\right|}{(n-i+1)(m-j+1)},$$

$$\widetilde{\sigma}_{n,m}^{ab}(f;x,y):=\frac{1}{(n+1)(m+1)}\sum_{i=0}^{n}\sum_{j=0}^{m}\left|\widetilde{S}_{i,j}^{ab}(f;x)\right|,\ a,b=0,1.$$

Denote

$$\widetilde{R}_{n,m}^{00}(f)=R_{n,m}(f),\ \widetilde{S}_{n,m}^{00}(f)=S_{n,m}(f),$$
$$\widetilde{\tau}_{n,m}^{00}(f)=\tau_{n,m}(f),\ \widetilde{\sigma}_{n,m}^{00}(f)=\sigma_{n,m}(f).$$

In [6], among others, it was proved the following result.

Theorem 3.1. *Let $f\in L\log L(\mathbb{T}^2)$ and $0<p<1$. Then for any $a,b=0,1$ the following inequality holds*

$$\left(\iint\limits_{\mathbb{T}^2}\left(\sup_{n,m}\widetilde{\sigma}_{n,m}^{(a,b)}(f;x,y)\right)^p dxdy\right)^{1/p}\leq c_1\iint\limits_{\mathbb{T}}|f(x,y)|\log^+|f(x,y)|\,dxdy+c_2.$$

Applying Hardy's transformation, we obtain

$$l_n l_m \widetilde{R}^{ab}_{n,m}(f;x,y)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{\widetilde{\sigma}^{ab}_{i,j}(f;x,y)}{(i+2)(j+2)}+ \tag{3.1}$$

$$+\sum_{j=0}^{m-1}\frac{1}{j+2}\widetilde{\sigma}^{ab}_{n,j}(f;x,y)+\sum_{i=0}^{n-1}\frac{1}{i+2}\widetilde{\sigma}^{ab}_{i,m}(f;x,y)+\widetilde{\sigma}^{ab}_{n,m}(f;x,y).$$

Consequently, for $f\in L\log L(\mathbb{T}^2)$ from Theorem 3.1 we obtain

$$\begin{aligned}
&\left(\iint_{\mathbb{T}^2}\left(\widetilde{R}^{ab}_{n,m}(f;x,y)\right)^p dxdy\right)^{1/p} \\
\le\;& 4\left(\iint_{\mathbb{T}^2}\left(\sup_{n,m}\widetilde{\sigma}^{(a,b)}_{n,m}(f;x,y)\right)^p dxdy\right)^{1/p} \\
\le\;& c_1\iint_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|\,dxdy+c_2.
\end{aligned} \tag{3.2}$$

Since

$$\left(\int_{\mathbb{T}}\left(\sup_n\sigma_n(f;x)\right)^p dx\right)^{1/p},\left(\int_{\mathbb{T}}\left(\sup_n\widetilde{\sigma}_n(f;x)\right)^p dx\right)^{1/p}$$

$$\le c_1\int_{\mathbb{T}}|f(x)|\,dx,\ f\in L_1(\mathbb{T}),\quad 0<p<1.$$

Similarly, for one dimensional case we can show that

$$\left(\int_{\mathbb{T}}\left(R_n(f;x)\right)^p dx\right)^{1/p},\left(\int_{\mathbb{T}}\left(\widetilde{R}_n(f;x)\right)^p dx\right)^{1/p} \tag{3.3}$$

$$\le c_1\int_{\mathbb{T}}|f(x)|\,dx,\ f\in L_1(\mathbb{T}),\quad 0<p<1,$$

where $\sigma_n(f;x)$, $\widetilde{\sigma}_n(f;x)$, $R_n(f;x)$ and $\widetilde{R}_n(f;x)$ are the strong Fejér and the strong Riesz means of Fourier series and conjugate Fourier series, respectively.

4. Main Results

Theorem 4.1. *Let $f\in L\log L(\mathbb{T}^2)$ and $0<p<1$. Then the following inequality holds*

$$\left(\iint_{\mathbb{T}^2}\left(\tau_{n,m}(f;x,y)\right)^p dxdy\right)^{1/p}\le c_1\iint_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|\,dxdy+c_2.$$

Theorem 4.2. *Let $f \in L\log L\left(\mathbb{T}^2\right)$ and $0 < p < 1$. Then*

$$\iint\limits_{\mathbb{T}^2} \left(\frac{1}{l_n l_m} \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{\left| S_{i,j}\left(f;x,y\right) - f\left(x,y\right)\right|}{\left(n-i+1\right)\left(m-j+1\right)} \right)^p dxdy \to 0$$

as $n, m \to \infty$.

Corollary 4.1. *Let $f \in L\log L\left(\mathbb{T}^2\right)$. Then*

$$\frac{1}{l_n l_m} \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{\left| S_{i,j}\left(f;x,y\right) - f\left(x,y\right)\right|}{\left(n-i+1\right)\left(m-j+1\right)} \to 0$$

in measure on $\mathbb{T}^2$ as $n, m \to \infty$.

Uniting these results with Tkebuchava theorem (Theorem 1.1), we can state the following result.

Theorem 4.3. *The following assertions are equivalent:*

(a) *the embedding $L_Q(\mathbb{T}^2) \subset L\log L(\mathbb{T}^2)$ holds;*
(b) *the strong Nörlund logarithmic means of double Fourier series for all functions from Orlicz space $L_Q(\mathbb{T}^2)$ converges in measure on $\mathbb{T}^2$;*
(c) *the Nörlund logarithmic means of double Fourier series for all functions from Orlicz space $L_Q(\mathbb{T}^2)$ converges in measure on $\mathbb{T}^2$.*

5. Proof of main results

Proof of Theorem 4.1. Setting $\alpha_n\left(t\right) := \sin\left(\left(n+1\right)t\right)$ and $\beta_n\left(t\right) := \cos\left(\left(n+1\right)t\right)$, we can write

$$S_{n-k}\left(f;x\right) = \frac{1}{\pi} \int\limits_{\mathbb{T}} f\left(t\right) \frac{\sin\left(\left(n-k+1/2\right)\left(x-t\right)\right)}{2\sin\left(\left(x-t\right)/2\right)} dt \tag{5.1}$$

$$:= \frac{1}{\pi} \int\limits_{\mathbb{T}} f\left(t\right) \sin\left(\left(n+1\right)\left(x-t\right)\right) \frac{\cos\left(\left(k+1/2\right)\left(x-t\right)\right)}{2\sin\left(\left(x-t\right)/2\right)} dt$$

$$-\frac{1}{\pi} \int\limits_{\mathbb{T}} f\left(t\right) \cos\left(\left(n+1\right)\left(x-t\right)\right) \frac{\sin\left(\left(k+1/2\right)\left(x-t\right)\right)}{2\sin\left(\left(x-t\right)/2\right)} dt$$

$$= \frac{1}{\pi} \int\limits_{\mathbb{T}} f\left(t\right) \sin\left(\left(n+1\right)\left(x-t\right)\right) \left(\frac{\cos\left(\left(k+1/2\right)\left(x-t\right)\right)}{2\sin\left(\left(x-t\right)/2\right)} - \frac{\cos\left(\left(x-t\right)/2\right)}{2\sin\left(\left(x-t\right)/2\right)} \right) dt$$

$$+\frac{1}{\pi}\int_{\mathbb{T}} f(t)\frac{\sin((n+1)(x-t))}{2\tan((x-t)/2)}dt$$
$$-\frac{1}{\pi}\int_{\mathbb{T}} f(t)\cos((n+1)(x-t))\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$
$$= -\frac{\alpha_n(x)}{\pi}\int_{\mathbb{T}} f(t)\beta_n(t)\widetilde{D}_k(x-t)\,dt+\frac{\beta_n(x)}{\pi}\int_{\mathbb{T}} f(t)\alpha_n(t)\widetilde{D}_k(x-t)\,dt$$
$$+\frac{1}{\pi}\int_{\mathbb{T}} f(t)\frac{\sin((n+1)(x-t))}{2\tan((x-t)/2)}dt$$
$$-\frac{\beta_n(x)}{\pi}\int_{\mathbb{T}} f(t)\beta_n(t)\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$
$$-\frac{\alpha_n(x)}{\pi}\int_{\mathbb{T}} f(t)\alpha_n(t)\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$
$$= -\alpha_n(x)\widetilde{S}_k(f\beta_n;x)+\beta_n(x)\widetilde{S}_k(f\alpha_n;x)$$
$$-\beta_n(x)S_k(f\beta_n;x)-\alpha_n(x)S_k(f\alpha_n;x)+\overline{S}_{n+1}(f;x).$$

Hence

$$\tau_n(f;x) \; : \; =\frac{1}{l_n}\sum_{k=0}^{n}\frac{|S_k(f;x)|}{n-k+1}\le \widetilde{R}_n(f\beta_n,x)+\widetilde{R}_n(f\alpha_n,x)$$
$$+R_n(f\beta_n,x)+R_n(f\alpha_n,x)+\overline{S}_{n+1}(f;x).$$

Since

$$\left(\int_{\mathbb{T}}|S_n(f;x)|^p\,dx\right)^{1/p}\le c_p\int_{\mathbb{T}}|f(x)|\,dx,$$

from (3.3) we conclude that for $0<p<1$ and $f\in L_1(\mathbb{T})$

$$\left(\int_{\mathbb{T}}(\tau_n(f;x))^p\,dx\right)^{1/p}\le c_p\int_{\mathbb{T}}|f(x)|\,dx, f\in L_1(\mathbb{T}). \tag{5.2}$$

Now we consider the rectangular partial sums of double Fourier series. In view of (5.1) we can write

$$\begin{aligned}S_{n-i,m-j}(f;x,y)&=S_{n-i}(S_{m-j}(f;y);x)=-\alpha_n(x)\widetilde{S}_i(S_{m-j}(f;y)\beta_n;x)\\&+\beta_n(x)\widetilde{S}_i(S_{m-j}(f;y)\alpha_n;x)-\beta_n(x)S_i(S_{m-j}(f;y)\beta_n;x)\\&-\alpha_n(x)S_i(S_{m-j}(f;y)\alpha_n;x)+\overline{S}_{n+1}(S_{m-j}(f,y);x)\\&=\sum_{s=1}^{4}I_s(i,j;x,y)+\overline{S}_{n+1}(S_{m-j}(f,y);x).\end{aligned} \tag{5.3}$$

Next, for $I_1(i,j;x,y)$, in view of (5.1) we have

$$\begin{aligned}(5.4)\quad I_1(i,j;x,y) &= -\alpha_n(x)\,S_{m-j}\left(\widetilde{S}_i(f\beta_n;x);y\right)\\ &= \alpha_n(x)\,\alpha_m(y)\,\widetilde{S}_j\left(\widetilde{S}_i(f\beta_n;x)\,\beta_m;y\right)-\alpha_n(x)\,\beta_m(y)\,\widetilde{S}_j\left(\widetilde{S}_i(f\beta_n;x)\,\alpha_m;y\right)\\ &\quad+\alpha_n(x)\,\beta_m(y)\,S_j\left(\widetilde{S}_i(f\beta_n;x)\,\beta_m;y\right)+\alpha_n(x)\,\alpha_m(y)\,S_j\left(\widetilde{S}_i(f\beta_n;x)\,\alpha_m;y\right)\\ &\quad-\alpha_n(x)\,\overline{S}_{m+1}\left(\widetilde{S}_i(f\beta_n;x);y\right)=\alpha_n(x)\,\alpha_m(y)\,\widetilde{S}^{11}_{ij}(f\beta_n\beta_m;x,y)\\ &\quad-\alpha_n(x)\,\beta_m(y)\,\widetilde{S}^{11}_{ij}(f\beta_n\alpha_m;x,y)+\alpha_n(x)\,\beta_m(y)\,\widetilde{S}^{10}_{ij}(f\beta_n\beta_m;x,y)\\ &\quad+\alpha_n(x)\,\alpha_m(y)\,\widetilde{S}^{10}_{ij}(f\beta_n\alpha_m;x,y)-\alpha_n(x)\,\widetilde{\overline{S}}^{01}_{i,m+1}(f\beta_n;x,y)\\ &= \sum_{l=1}^{4} I_{1l}(i,j;x,y)+I_{15}(i,m;x,y).\end{aligned}$$

From (3.2) we obtain

$$\begin{aligned}(5.5)\quad &\iint_{\mathbb{T}^2}\left(\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{|I_{11}(i,j;x,y)|}{(i+1)(j+1)}\right)^p dxdy\\ &\le \iint_{\mathbb{T}^2}\left|\widetilde{R}^{11}_{n,m}(f\beta_n\beta_m;x,y)\right|^p dxdy\\ &\le c_1\iint_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|\,dxdy+c_2.\end{aligned}$$

Similarly it can be shown that

$$\begin{aligned}(5.6)\quad &\iint_{\mathbb{T}^2}\left(\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{|I_{1l}(i,j;x,y)|}{(i+1)(j+1)}\right)^p dxdy\\ &\le c_1\iint_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|\,dxdy+c_2,\quad l=2,3,4.\end{aligned}$$

Now, we turn to $I_{15}(i,m;x,y)$. Taking into account that

$$\widetilde{\overline{S}}^{10}_{i,m+1}(f\beta_n;x,y)=\widetilde{S}_i\left(\overline{S}_{m+1}(f;y)\,\beta_n;x\right),$$

$$f(\cdot,y)\in L\log L(\mathbb{T})\ \text{for a.e. } y\in\mathbb{T}\ \text{and}\ f\in L\log L\left(\mathbb{T}^2\right),$$

and

$$\int_{\mathbb{T}}\left|\overline{S}_{m+1}(f;x,y)\right|dx\le c_1\int_{\mathbb{T}}|f(x,y)|\log^+|f(x,y)|\,dx+c_2,$$

from (3.3) we obtain

$$\left(\int_{\mathbb{T}}\left(\frac{1}{l_n}\sum_{i=0}^{n}\frac{\left|\widetilde{S}^{10}_{i,m+1}(f\beta_n;x,y)\right|}{i+1}\right)^p dx\right)^{1/p}$$
$$\leq \int_{\mathbb{T}}\left|\overline{S}_{m+1}(f;x,y)\right|dx \leq c_1\int_{\mathbb{T}}|f(x,y)|\log^+|f(x,y)|\,dx+c_2.$$

Consequently,

$$\iint_{\mathbb{T}^2}\left(\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{|I_{15}(i,m;x,y)|}{(i+1)(j+1)}\right)^p dxdy \tag{5.7}$$
$$\leq c_1\iint_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|\,dxdy+c_2.$$

A combination of (5.4)-(5.7) yields

$$\iint_{\mathbb{T}^2}\left(\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{|I_1(i,j;x,y)|}{(i+1)(j+1)}\right)^p dxdy \tag{5.8}$$
$$\leq c_1\iint_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|\,dxdy+c_2.$$

Similarly, we can prove that

$$\iint_{\mathbb{T}^2}\left(\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{|I_s(i,j;x,y)|}{(i+1)(j+1)}\right)^p dxdy \tag{5.9}$$
$$\leq c_1\iint_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|\,dxdy+c_2,\quad s=2,3,4,$$

and

$$\left(\iint_{\mathbb{T}^2}\left(\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\left|\overline{S}_{n+1}(S_{m-j}(f,y);x)\right|}{(i+1)(j+1).}\right)^p dxdy\right)^{1/p} \tag{5.10}$$
$$=\left(\iint_{\mathbb{T}^2}\left(\frac{1}{l_nl_m}\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\left|S_{m-j}\left(\overline{S}_{n+1}(f,x);y\right)\right|}{(i+1)(j+1)}\right)^p dxdy\right)^{1/p}$$
$$=\left(\iint_{\mathbb{T}^2}\left(\frac{1}{l_m}\sum_{j=0}^{m}\frac{\left|S_{m-j}\left(\overline{S}_{n+1}(f,x);y\right)\right|}{j+1}\right)^p dxdy\right)^{1/p}$$
$$\leq c_1\iint_{\mathbb{T}^2}|f(x,y)|\log^+|f(x,y)|\,dxdy+c_2.$$

Combining (5.3) and (5.8) - (5.10), we complete the proof of Theorem 4.1. □

Proof of Theorem 4.2. The result follows immediately from the density of polynomials and by virtue of standard arguments (see [16]). □

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