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CONSTRUCTING POLYNOMIALS OF MINIMAL GROWTH

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Abstract. In [2] a cyclic diagonal operator on the space of functions analytic on the unit disk with eigenvalues (λ_n) is shown to admit spectral synthesis if and only if for each j there is a sequence of polynomials (p_n) such that $\lim_{n\to\infty} p_n(\lambda_k) = \delta_{j,k}$ and $\limsup_{n\to\infty} \sup_{k>j} |p_n(\lambda_k)|^{1/k} \leq 1$. The author also shows, through contradiction, that certain classes of cyclic diagonal operators are synthetic. It is the intent of this paper to use the aforementioned equivalence to constructively produce examples of synthetic diagonal operators. In particular, this paper gives two different constructions for sequences of polynomials that satisfy the required properties for certain sequences to be the eigenvalues of a synthetic operator. Along the way we compare this to other results in the literature connecting polynomial behavior ([4] and [9]) and analytic continuation of Dirichlet series ([1]) to the spectral synthesis of diagonal operators.

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1. Introduction

A vector x in a complete, metrizable, topological, vector space $\mathcal{X}$ is said to be *cyclic* for a continuous, linear operator $T : \mathcal{X} \to \mathcal{X}$ if the closed, linear span of the orbit $\{T^n x : n \geq 0\}$ of x under T is all of $\mathcal{X}$. Operators that have a cyclic vector are said to be *cyclic.* A vector x is said to be a *root vector* for T if there exist $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$ such that $(T - \lambda I)^n x = 0$. A continuous, linear operator $T : \mathcal{X} \to \mathcal{X}$ on a complete, metrizable, topological, vector space $\mathcal{X}$ is said to *admit spectral synthesis* or *be synthetic* if every closed invariant subspace $\mathcal{M}$ of T equals the closed linear span of the root vectors for T contained in $\mathcal{M}$. Cyclicity results yield interesting approximation results. For instance, the Weierstrass Approximation Theorem asserts that the function $f(x) \equiv 1$ on $[0, 1]$ is cyclic for the operator $T : g(x) \to xg(x)$ of multiplication by x on the Banach space $C([0, 1])$ of continuous functions on $[0, 1]$.

Results about polynomials are intimately related to the results about cyclicity and synthesis since the vector space generated by the set $\{T^n x : x \in X\}$ is equal to the set $\{p(T) : p \in \mathbb{C}[z]\}$, where $\mathbb{C}[z]$ is the set of polynomials with coefficients in $\mathbb{C}$. There are three recent results which demonstrate this connection. To state these results, we first present the notation, used in the original papers, and the necessary background.

The operator $J(\lambda_n, m_n)$ is a Jordan block acting on a finite dimensional Hilbert space $\mathcal{H}_n$. The space of functions analytic on the unit disk in $\mathbb{C}$ is denoted as H_1, and the space of functions analytic on $\mathbb{C}$ is denoted as $H(\mathbb{C})$. A linear operator $D : H_1 \to H_1$ or $D : H(\mathbb{C}) \to H(\mathbb{C})$ is called *diagonal* if it has as eigenvectors the monomials z^n for $n \geq 0$. Formally, these maps are given by $\sum_{n=0}^{\infty} a_n z^n \to \sum_{n=0}^{\infty} \lambda_n a_n z^n$. The sequence (λ_n) is called D's associated sequence. A diagonal operator D with associated sequence (λ_n) on H_1 or $H(\mathbb{C})$ is defined and continuous if and only if $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} \leq 1$ or $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} < \infty$, respectively (see Proposition 1 in [3] and Lemma 1 in [7], respectively). In either case, the operator D is cyclic if and only if the eigenvalues are distinct (see Theorem 1 in [3] and Proposition 3 in [7]). Note that the root vectors for a diagonal operator are precisely its eigenvectors. The aforementioned results are as follows.

Theorem 1.1 ([9], Theorem 3). *Let $\{\lambda_n\}$ be a bounded sequence of distinct complex numbers, let $\{m_n\}$ be a bounded sequence of positive integers, and let $J = \oplus J(\lambda_n, m_n)$ be a Jordan operator acting on a Hilbert space $\mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{H}_n$. If for each positive integer i, the orthogonal projection $P_{\mathcal{H}_i} : \mathcal{H} \to \mathcal{H}_i$ is in the weakly closed algebra, generated by J and the identity, then the Jordan operator $J = \oplus J(\lambda_n, m_n)$ admits spectral synthesis.*

Theorem 1.2 ([9], Theorem 4). *Let $\{\lambda_n\}$ be a bounded sequence of distinct complex numbers, let $\{m_n\}$ be a bounded sequence of positive integers, and let $J = \oplus J(\lambda_n, m_n)$ be a Jordan operator acting on a Hilbert space $\mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{H}_n$. Let i be any positive integer and $\{p_\alpha\}$ be a set of polynomials. Then $\{p_\alpha(J)\}$ converges in the weak operator topology to the projection operator $P_{\mathcal{H}_i}$ if and only if*

(1) $\lim_\alpha p_\alpha(\lambda_k) = \begin{cases} 0 & \text{if } k \neq i \\ 1 & \text{if } k = i, \end{cases}$

(2) $\lim_\alpha \hat{p}_\alpha^{(j)}(\lambda_k) = 0$ *for all* $j, k \geq 1$, *and*

(3) $\sup_{\alpha,k} |\hat{p}_\alpha^{(j)}(\lambda_k)| < \infty$ *for all* $j \geq 0$,

where

$$\hat{p}_{\alpha}^{(j)}(\lambda_k) = \begin{cases} 0 & \text{if } j \geq m_k \\ p_{\alpha}^{(j)}(\lambda_k) & \text{if } j < m_k. \end{cases}$$

Theorem 1.3 ([2], Theorem 8). *Let $D : H_1 \to H_1$ be a diagonal operator with distinct eigenvalues and let $\mathcal{D}$ be the algebra generated by D. The following statements are equivalent:*

(1) *In the SOT, $\pi_n \in \overline{\mathcal{D}}$ for all $n \geq 0$.*
(2) *D is synthetic.*
(3) *The function $f \in H_1$ is cyclic, where $f(z) = \frac{1}{1-z}$.*
(4) *For each $j \geq 0$ there is some sequence of polynomials $(p_n) \subset \mathbb{C}[z]$, depending on j, such that $\lim_{n\to\infty} p_n(\lambda_k) = \delta_{j,k}$ and*

$$\limsup_{n\to\infty} \sup_{k>j}(\{|p_n(\lambda_k)|^{\frac{1}{k}}\}) \leq 1.$$

Theorem 1.4 ([4], Theorem 3.1). *Let D be a cyclic diagonal operator on $\mathcal{H}(\mathbf{C})$ having eigenvalues $\{\lambda_n\}$. If for each $j \geq 0$ there exists a sequence $\{p_{j,n}(z)\}$ of polynomials for which $\lim_{n\to\infty} p_{j,n}(\lambda_k) = \delta_{j,k}$ and $\sup(\{|p_{j,n}(\lambda_k)|^{1/k} : k \geq 0, n \geq 1\} < \infty$, then D admits spectral synthesis.*

All three results have similar kinds of conditions which guarantee that an operator is synthetic. First, condition (1) in Theorem 1.2, the first condition on the sequences of polynomials in Theorem 1.4, and the first condition on the sequences of polynomials in part 4 of Theorem 1.3 can colloquially be thought of as separating a sequence of points. Second, condition (3) in Theorem 1.2, the second condition on the sequences of polynomials in Theorem 1.4, and the second condition on the sequences of polynomials in part 4 of Theorem 1.3 specifies a growth condition on the polynomials on the sequence of points. In particular, the growth condition in part 4 of Theorem 1.3 is the strictest it can be and still allow for unbounded eigenvalues. This will be colloquially referred to as satisfying a minimal growth condition.

Seubert and Deters present examples of synthetic diagonal operators (see [9], Theorem 5, [3], Corollary 1 and Theorem 5, [2], Theorem 6, [4], Theorem 3.2 and Corollary 3.3). In [9] and [4] the authors use the existence of nets and sequences of polynomials to demonstrate the synthesis of some subset of diagonal operators. While Deters does not do this in [3], Theorem 8 of [3] guarantees the existence of sequences of polynomials which separate eigenvalues and satisfy a minimal growth condition.

No sequences of polynomials are constructed in [9] or [3]. Rather, they are shown to exist through arguments based on contradiction or using existence theorems like Mergelyan's Theorem. In the proof of Theorem 3.3 in [4], Seubert and Deters construct sequences of polynomials simply by looking at the power series expansion of canonical products. For instance, for a diagonal operator on $H(\mathbb{C})$ with eigenvalues (λ_n) such that $\lambda_n = n^2$ for $n \geq 0$, the polynomials (p_n) defined by $p_n(z) = \prod_{k=1}^{n} \frac{k^2-z}{k^2}$ satisfy the conditions in Theorem 1.4 for λ_0. However, observe that the growth restriction on the polynomials in the conditions of part 4 of Theorem 1.3 is much more restrictive than the growth restriction in Theorem 1.4. The purpose of this paper is to construct polynomials which satisfy the conditions in part 4 of Theorem 1.3.

2. Constructing Polynomials Of Minimal Growth

For the sake of brevity, we enumerate the conditions in part 4 of Theorem 1.3 as follows:

$$\lim_{n\to\infty} p_n(\lambda_k) = \delta_{j,k}, \tag{2.1}$$

$$\limsup_{n\to\infty} \sup_{k>j}(\{|p_n(\lambda_k)|^{\frac{1}{k}}\}) \leq 1. \tag{2.2}$$

Constructing polynomials which satisfy conditions (1) and (2) appears to be non-trivial. For instance, consider the diagonal operator with eigenvalues $\lambda_n = n$ for $n \geq 0$. Such an operator is synthetic by Theorem 6 in [2]. A natural choice for the polynomials corresponding to λ_0 would be $p_n(z) = \prod_{k=1}^{n} \frac{z-k}{-k}$. However, note that

$$|p_n(2n)|^{\frac{1}{2n}} = \left(\prod_{k=1}^{n} \frac{2n-k}{k}\right)^{\frac{1}{2n}} = \left(\prod_{k=1}^{n} 1 + \frac{n-1}{k}\right)^{\frac{1}{2n}} \geq \sqrt{\frac{3}{2}}.$$

Hence, while the sequence clearly satisfies condition (1) for $j = 0$, it does not satisfy condition (2).

It may also be thought that Theorem 3.2 in [4] would provide some insight into such constructions. Following the proof of Theorem 3.2 in [4], if D is a diagonal operator on H_1 with associated sequence (λ_n), then the equivalent condition to Theorem 3.2 in [4] would be that there is some non-trivial, entire function E of order ρ and type τ such that $E(\lambda_n) = 0$ for all $n \geq 0$ and $\lim_{n\to\infty} |\lambda_n|^{\rho}/n = 0$. However, if $|\lambda_n| \leq |\lambda_{n+1}|$ for $n \geq 0$ and $n(r) = |\{z : E(z) = 0, |z| \leq r\}|$, then, for sufficiently large k, Theorem

4.5.1 in [5] would imply that

$$1 \leq \frac{n(|\lambda_k|)}{k} \leq \frac{-\ln|E(0)| + (\tau+1)(3|\lambda_k|)^\rho}{k \ln 2} \to 0 \text{ as } k \to \infty.$$

Thus, the "obvious" candidates for sequences of polynomials do not work and more creativity must be used. We shall first construct polynomials which satisfy (1) and (2) for a particular family of bounded eigenvalues. To accomplish this we shall make use of polynomial approximations of Blaschke products.

Theorem 2.1. *Let $(\lambda_n) \subset \mathbb{C}$ be a sequence such that $|\lambda_n| < 1$ and $\sum_{n=0}^{\infty}(1 - |\lambda_n|) < \infty$, then for each $j \geq 0$, there is some sequence $(p_n) \subset \mathbb{C}[z]$ such that $\lim_{n\to\infty} p_n(\lambda_k) = \delta_{j,k}$ and $\limsup_{n\to\infty} \sup_{k>j} |p_n(\lambda_k)|^{\frac{1}{k}} \leq 1$.*

Proof. Fix $j \geq 0$, and for $n > j$ define

$$A_n = \max(\{|\lambda_k| : 1 \leq k \leq n\}) \quad \text{and} \quad B_n(z) = \prod_{k \neq j}^{n} \frac{(z-\lambda_k)|\lambda_k|}{(1-\overline{\lambda_k}z)\lambda_k}.$$

Choose M_n such that $(2/(1-A_n))^{n-1}(2nA_n^{M_n+1}/(1-A_n)) < 1/n$, and for $n \geq 1$ define

$$q_n(z) = \prod_{k \neq j}^{n} \frac{(z-\lambda_k)|\lambda_k|}{\lambda_k} \sum_{m=0}^{M_n} (\overline{\lambda_k}z)^m.$$

For $|z| \leq 1$ and $0 \leq k \leq n$ observe that

$$|(z-\lambda_k)(|\lambda_k|/\lambda_k) \sum_{m=0}^{M_n} (\overline{\lambda_k}z)^m| \leq 2 \sum_{m=0}^{M_n} A_n^m \leq 2/(1-A_n).$$

Since $|(z-\lambda_k)(|\lambda_k|/\lambda_k)|/|1-\overline{\lambda_k}z| \leq 1$ for $|z| \leq 1$, we have that for $n > \max(\{1,j\})$ and $|z| \leq 1$

$$\begin{aligned} |q_n(z) - B_n(z)| &\leq \left(\frac{2}{1-A_n}\right)^{n-1} \sum_{k \neq j}^{n} |z-\lambda_k| \sum_{m=M_n+1}^{\infty} |\overline{\lambda_k}z|^m \\ &\leq \left(\frac{2}{1-A_n}\right)^{n-1} \frac{2nA_n^{M_n+1}}{1-A_n} < \frac{1}{n}. \end{aligned}$$

Next, since $\sum_{k \neq j}^{\infty}(1-|\lambda_k|) < \infty$, there is some $B \in H_1$ such that B_n converges to B in H_1, $B(\lambda_k) = 0$ for $k \neq j$, and $B(\lambda_j) \neq 0$ (see Theorem 15.21 in [8]). Note that $\lim_{n\to\infty} q_n = B$ in H_1. Define $p_n = q_n/B(\lambda_j)$. Since

$$|p_n(\lambda_k)|^{\frac{1}{k}} \leq (|(q_n(\lambda_k) - B_n(\lambda_k)|/|B(\lambda_j)|)^{\frac{1}{k}} + |B_n(\lambda_k)/B(\lambda_j)|)^{\frac{1}{k}},$$

the sequence (p_n) clearly possesses the desired properties. □

Corollary 2.1. *If $D : H_1 \to H_1$ is a diagonal operator with associated sequence $(\lambda_n) \subset \mathbb{C}$ such that $|\lambda_n| < 1$ and $\sum_{n=0}^{\infty}(1 - |\lambda_n|) < \infty$, then D is synthetic.*

A more general, but non-constructive version of this result is known from Corollary 1 in [3]. However, the proof of Corollary 1 in [3] relies on Proposition 2 of [12] whose proof is nontrivial.

We now are going to construct polynomials that satisfy conditions (1) and (2) for a particular collection of sequences (λ_n) such that $\limsup_{n\to\infty} |\lambda_n|^{\frac{1}{n}} \leq 1$ and $\limsup_{n\to\infty} |\lambda_n| = \infty$. Although there will be many details in what follows, the main spirit of the approach will be to consider sequences (λ_n) such that $\sum_{n=0}^{\infty} \frac{1}{|\lambda_n|} = \infty$. Informally, the sequence (λ_n) does not grow too quickly. We will then find a sequence (z_n) of positive numbers such that $\lim_{n\to\infty} \frac{|\lambda_n|}{z_n} = 0$ and $\sum_{n=0}^{\infty} \frac{1}{z_n} = \infty$. Informally, the sequence (z_n) will grow faster than the sequence $(|\lambda_n|)$, but not too fast. The desired sequence (p_n) of polynomials will then look like $p_n(z) = \prod_{k=1}^{n} \frac{z-z_k}{\lambda_\ell - z_k}$. The first step will be the following lemma which follows directly from elementary entire function theory.

Lemma 2.1. *Let $z, z_0 \in \mathbb{C}$ such that $\operatorname{Re} z > \operatorname{Re} z_0$ and a sequence of positive numbers (z_n) such that $z_n \uparrow \infty$ be given. If $z \notin \{z_n : n \geq 1\}$, then $p_n(z) = \prod_{k=1}^{n} \frac{z-z_n}{z_0 - z_n} \to 0$ if and only if $\sum_{n=1}^{\infty} \frac{1}{z_n} = \infty$.*

Proof. Since $z_k \uparrow \infty$, there is a sequence of numbers (u_n) such that $\lim_{n\to\infty} u_n = 1$ and $(|z - z_n|/|z_0 - z_n|)^2 = 1 + 2(\operatorname{Re} z_0 - \operatorname{Re} z)u_n/z_n$. Hence, by Theorem 15.5 in [8], the result follows. □

In light of the above proposition, we will concentrate our efforts on polynomials of the form $p_n(z) = \prod_{k=1}^{n} \frac{z-z_k}{\lambda_\ell - z_k}$ that satisfy conditions (1) and (2) for some sequence (z_n). To this end, we now develop some ideas to judiciously select sequences of zeros for the polynomials. Our definitions will be recursive. Define $a_0(x) = x$, $a_n(x) = \ln a_{n-1}(x)$, $e_0 = 1$, and $e_n = e^{e_{n-1}}$ for $n \geq 1$. Also, define $b_n(x) = \prod_{k=0}^{n} a_k(x)$ for $n \geq 0$. We collect some basic results about these functions below. In particular, we will obtain an estimate for $\prod_{k=1}^{n} a_m(x + k - 1)$ similar in spirit to Stirling's formula.

Lemma 2.2. *Let a_n and b_n be defined as above. The following assertions hold.*

(1) *Let $m \geq 0$, $n \in \mathbb{N}$, and $x \geq e_m$ be given. There is a function $\varepsilon_{m,x} : \mathbb{N} \to \mathbb{R}$ such that $\prod_{k=1}^{n} a_m(x + k - 1) = a_m(x + n - 1)^{n\varepsilon_{m,x}(n)}$ and $\limsup_{n\to\infty}(1 - \varepsilon_{m,x}(n)) \ln n = 1$.*

(2) *Let* $m \geq 0$, $n \in \mathbb{N}$, *and* $x \geq e_m$ *be given. There is a function* $\varepsilon_{m,x} : \mathbb{N} \to \mathbb{R}$ *such that* $\prod_{k=1}^{n} b_m(x+k-1) = b_m(x+n-1)^{n\varepsilon_{m,x}(n)}$ *and* $\limsup_{n\to\infty}(1-\varepsilon_{m,x}(n))\ln n = 1$.

(3) *If* $n \geq 1$ *and* $0 < c < 1$, *then* $sup_x(ca_n(x))^{\frac{1}{x}} \leq e^{1/a_n^{-1}(1/c)}$.

(4) *For* $x, y \geq 2$ *and* $n \geq 0$, $a_n^{-1}(xy) \geq a_n^{-1}(x)a_n^{-1}(y)$.

(5) *For each* $\varepsilon \in (0,1)$ *there is some* M *such that* $a_n^{-1}(x^y) \geq (a_n^{-1}(x))^y$ *for* $x \geq M$, $y \in [\varepsilon, 1)$, *and* $n \geq 0$.

Proof. (1) Define a sequence $\{\varepsilon_{m,x}(n), n \in \mathbb{N}\}$ as follows: we put $\varepsilon_{m,x}(1) = 1$ and for $n \geq 2$ define

$$\varepsilon_{m,x}(n) = \frac{1}{na_{m+1}(x+n-1)}\sum_{k=1}^{n} a_{m+1}(x+k-1).$$

Observe that the only task is to prove the second property of $\varepsilon_{m,x}$. The proof will be by induction on m. The case $m = 0$ follows directly from Stirling's Formula (see [6], p. 313). Suppose that the result holds for some $m \geq 0$ and let $x \geq e_{m+1}$ be given. For $n \geq 2$ define

$$f_n(y) = \frac{a_{m+2}(y)}{a_{m+2}(x+n-1)} - \frac{a_{m+1}(y)}{a_{m+1}(x+n-1)}.$$

First, observe that

$$f_n(x+n-1) = 0, \qquad f_n(a_{m+1}^{-1}(a_{m+1}(x+n-1)^{\frac{1}{a_{m+1}(x+n-1)}})) < 0,$$

and

$$f_n(a_{m+1}^{-1}(a_{m+1}(x+n-1)^{\frac{2}{a_{m+1}(x+n-1)}})) > 0$$

for sufficiently large n. Write $y_n = a_{m+1}^{-1}(a_{m+1}(x+n-1)/a_{m+2}(x+n-1))$ and note that f_n has a unique maximum at y_n. Observe that for sufficiently large n, $x < y_n < x+n-1$. Define k_n to be the smallest k such that $|(x+k_n-1)-y_n| = \min(\{|(x+k-1)-y_n| : 1 \leq k \leq n\})$ and observe that $|y_n - (x+k_n-1)| \leq 1/2$ for sufficiently large n.

Next, by the Mean Value Theorem, there is some c_n between y_n and $x+k_n-1$ such that $|f_n(x+k_n-1) - f_n(y_n)| = |f_n'(c_n)||(x+k_n-1)-y_n|$. It is easy to see that $\lim_{n\to\infty} c_n = \infty$. Since $\lim_{n\to\infty} f_n'(c_n) = 0$, $\lim_{n\to\infty} f_n(y_n) = 1$, $\lim_{n\to\infty} |f_n(x+k_n-1) - f_n(y_n)| = 0$ and $\lim_{n\to\infty} f_n(x+k_n-1) = 1$. Hence, for sufficiently large n, we have

$$x+1 > a_{m+1}^{-1}\left(a_{m+1}(x+n-1)^{\frac{2}{a_{m+1}(x+n-1)}}\right)$$

and

$$n(\varepsilon_{m+1,x}(n)-\varepsilon_{m,x}(n)) \geq \frac{-a_{m+1}(x)}{a_{m+1}(x+n-1)}+\frac{a_{m+2}(x+k_n-1)}{a_{m+2}(x+n-1)}-\frac{a_{m+1}(x+k_n-1)}{a_{m+1}(x+n-1)}$$
$$+\sum_{k=2,k\neq k_n}^{n}\frac{a_{m+2}(x+k-1)}{a_{m+2}(x+n-1)}-\frac{a_{m+1}(x+k-1)}{a_{m+1}(x+n-1)}>0.$$

Therefore,

$$\limsup_{n\to\infty}\ln n(1-\varepsilon_{m+1,x}(n)) \leq \limsup_{n\to\infty}\ln n(1-\varepsilon_{m,x}(n)) \leq 1$$

and the result is proven.

(2) It follows from part 1 that, for each $0 \leq j \leq m$, there is some function $\hat{\varepsilon}_{j,x}$, defined on $\mathbf{N}$, such that $\limsup_{n\to\infty}(1-\hat{\varepsilon}_{j,x}(n))\ln n \leq 1$ and

$$\prod_{k=1}^{n} a_j(x+k-1) = a_j(x+n-1)^{n\hat{\varepsilon}_{j,x}(n)}.$$

Next, we put $\varepsilon_{m,x}(1)=1$ and for $n \geq 2$ define

$$\varepsilon_{m,x}(n)=\frac{1}{\ln b_m(x+n-1)}\sum_{j=0}^{m} a_{j+1}(x+n-1)\hat{\varepsilon}_{j,x}.$$

Clearly, we have

$$b_m(x+n-1)^{n\varepsilon_{m,x}(n)}=\prod_{k=1}^{n} b_m(x+k-1).$$

The other part of the assertion follows by noting that

$$\sum_{j=0}^{m} a_{j+1}(x+n-1)/\ln b_m(x+n-1)=1.$$

(3) Follows from a simple calculation.

(4) Follows by induction and the observation that $xy \geq x+y$ for $x, y \geq 2$.

(5) Follows by induction and the observation that $\lim_{y\to 1^-} y^{1/(y-1)}=e$. □

We now proceed to the main result. In particular, the theorem that follows will produce a construction of polynomials whose existence is guaranteed in Corollary 1 of [2], and mildly extend the result of Theorem 6 in [2].

Theorem 2.2. *Suppose (λ_n) is a sequence of distinct complex numbers such that $0<\operatorname{Re}\lambda_n<\operatorname{Re}\lambda_{n+1}$ and $|\operatorname{Im}\lambda_n| \leq \operatorname{Re}\lambda_n$ for $n \geq 0$, $\lim_{n\to\infty}\operatorname{Re}\lambda_n=\infty$, and there is some $p \geq 0$ such that $|\lambda_n| \leq b_p(n)$ for $n \geq e_p$. Then for each $\ell \geq 0$, there is a sequence $(p_n) \subset \mathbb{C}[z]$ such that (p_n) satisfies (1) and (2).*

Proof. Let $\alpha > 0$ be given, fix a nonnegative integer ℓ, and define $\theta = \max(\{e_{p+1}, |\lambda_\ell|\})$. For $n, k \geq 1$, define $z_k = b_{p+1}(\theta + k - 1)$,

$$p_n(z) = \prod_{k=1}^{n} \frac{z - z_k}{\lambda_\ell - z_k}, \qquad A_n = \prod_{k=1}^{n} \frac{-z_k}{\lambda_\ell - z_k},$$

and $j_n = \min(\{j : \operatorname{Re}\lambda_j > z_n/2\})$. Note that

$$|A_n| \leq \prod_{k=1}^{n} 1 + \frac{|\lambda_\ell|}{z_k - |\lambda_\ell|} \leq e^{\frac{|\lambda_\ell| z_1}{z_1 - |\lambda_\ell|} \sum_{k=1}^{n} \frac{1}{z_k}} \leq e^{\frac{|\lambda_\ell| z_1}{z_1 - |\lambda_\ell|} \sum_{k=1}^{n} \frac{1}{k}} \leq e^{\frac{|\lambda_\ell| z_1}{z_1 - |\lambda_\ell|}} n^{\frac{|\lambda_\ell| z_1}{z_1 - |\lambda_\ell|}}.$$

By assumption, there is some J_1 such that $|\lambda_j| \leq b_p(j)$ for $j \geq J_1$. Since

$$\lim_{x \to \infty} \frac{b_p(x a_{p+1}(x))}{b_{p+1}(x)} = 1,$$

there is some M_1 such that $b_p((x/4)a_{p+1}(x/4)) \leq 2b_{p+1}(x/4)$ for $x \geq M_1$.

Thus, if $\max(\{J_1, e_p\}) \leq j_n \leq (n/4)a_{p+1}(n/4)$ and $n \geq M_1$, then

$$2\operatorname{Re}\lambda_{j_n} \leq 2|\lambda_{j_n}| \leq 2b_p(j_n) \leq 2b_p((n/4)a_{p+1}(n/4))$$

$$\leq 4b_{p+1}(n/4) = n \prod_{k=1}^{p+1} a_k(n/4) \leq z_n < 2\operatorname{Re}\lambda_{j_n}.$$

Hence, if $j_n \geq \max(\{J_1, e_p\})$ and $n \geq M_1$, then $j_n > (n/4)a_{p+1}(n/4)$. Thus, $\lim_{n\to\infty} |A_n|^{1/j_n} = 1$ and there is some N_1 such that $|A_n|^{1/j_n} < \sqrt{1+\alpha}$ for $n \geq N_1$.

By part 2 of Lemma 2.2, there is a function $\varepsilon : \mathbb{N} \to \mathbb{R}$ such that for $n \geq 1$

$$\prod_{k=1}^{n} z_k = (b_{p+1}(\theta + n - 1)^n)^{\varepsilon(n)} = z_n^{n\varepsilon(n)}$$

and $\limsup_{n\to\infty}(1 - \varepsilon(n))\ln(n) = 1$. For $n \geq 1$ and $0 \leq k \leq p$, define

$$c_{n,k} = (a_k(\theta + n - 1)a_{p+1}(\theta + n - 1))^{\frac{1}{p+1}})^{\varepsilon(n)},$$

and note that for $n \geq 1$,

$$\prod_{k=0}^{p} c_{n,k} = \prod_{k=0}^{p} (a_k(\theta + n - 1)a_{p+1}(\theta + n - 1)^{\frac{1}{p+1}})^{\varepsilon(n)} = \prod_{k=0}^{p+1} a_k(\theta + n - 1)^{\varepsilon(n)} = z_n^{\varepsilon(n)}.$$

From part 5 of Lemma 2.2, there is some M_2 such that $a_k^{-1}(x^y) \geq a_k^{-1}(x)^y$ for $x \geq M_2$, $y \in [\frac{1}{2(p+1)}, 1)$, and $0 \leq k \leq p+1$. Since $\lim_{n\to\infty} \varepsilon(n) = 1$ and $\lim_{x\to\infty} a_k(x) = \infty$ for $0 \leq k \leq p+1$, there is some N_2 such that $\frac{1}{2} \leq \varepsilon(n)$, $a_k(\theta + n - 1)^{\frac{1}{2}} \geq 2$ for $0 \leq k \leq p$, $a_{p+1}(\theta + n - 1)^{\frac{1}{2(p+1)}} \geq 2$, and $a_{p+1}(\theta + n - 1) \geq M_2$ for $n \geq N_2$.

Thus, for $n \geq N_2$ and $0 \leq k \leq p$, we have that

$$
\begin{aligned}
&n^{\varepsilon(n)} a_{p+1}(\theta+n-1)^{\frac{\varepsilon(n)}{p+1}} \leq (\theta+n-1)^{\varepsilon(n)} a_{p+1}(\theta+n-1)^{\frac{\varepsilon(n)}{p+1}} \\
\leq&(\theta+n-1)^{\varepsilon(n)} a_{p+1-k}(\theta+n-1)^{\frac{\varepsilon(n)}{p+1}} \\
\leq&a_k^{-1}(a_k(\theta+n-1)^{\varepsilon(n)}) a_k^{-1}(a_{p+1}(\theta+n-1)^{\frac{\varepsilon(n)}{p+1}}) \\
\leq&a_k^{-1}(a_k(\theta+n-1)^{\varepsilon(n)} a_{p+1}(\theta+n-1)^{\frac{\varepsilon(n)}{p+1}}) = a_k^{-1}(c_{n,k})
\end{aligned}
$$

Therefore,

$$
\sup_x \left(\frac{b_p(x)}{z_n^{\varepsilon(n)}} \right)^{\frac{n}{x}} \leq \prod_{k=0}^{p} \sup_x \left(\frac{a_k(x)}{c_{n,k}} \right)^{\frac{n}{x}} = \prod_{k=0}^{p} e^{\frac{n}{a_k^{-1}(c_{n,k})}} \leq e^{\frac{n(p+1)}{n^{\varepsilon(n)} a_{p+1}(\theta+n-1)^{\varepsilon(n)}}}.
$$

Thus, there is some N_3 such that $\sup_x (b_p(x)/z_n^{\varepsilon(n)})^{n/x} < \sqrt{1+\alpha}$ for $n \geq N_3$.

Next, since $\lim_{j\to\infty} \operatorname{Re}\lambda_j = \infty$, there is some J_2 such that $\operatorname{Re}\lambda_j > 2\operatorname{Re}\lambda_\ell$. Define $J = \max(\{J_2, e_p, j_{N_1}, j_{N_3}, \ell+1\})$, and observe that

$$
\sum_{k=1}^{n} 1/z_k \geq \int_\theta^{\theta+n} 1/b_{p+1}(x)dx = a_{p+2}(\theta+n) - a_{p+2}(\theta).
$$

Hence, $\lim_{n\to\infty} p_n(\lambda_j) = \delta_{j,\ell}$ for $j \geq \ell$. Choose N_0 such that $|p_n(\lambda_j)| \leq 1$ for $\ell < j \leq J$ and $n \geq N_0$, and define $N = \max(\{N_0, N_1, N_3\})$. If $n \geq N$, $j > \ell$, and $|p_n(\lambda_j)| > 1$, then $j > J$, $\operatorname{Re}\lambda_j > z_{N_1}/2$ and there are two possibilities.

First, there is some m with $1 \leq m \leq n-1$ such that $z_m/2 \leq \operatorname{Re}\lambda_j < z_{m+1}/2$.

Second, $z_n/2 \leq \operatorname{Re}\lambda_j$. In the first case, if $m < N_1$, then $m+1 \leq N_1$ and

$$
\frac{z_{N_1}}{2} < \operatorname{Re}\lambda_{j_{N_1}} \leq \operatorname{Re}\lambda_J \leq \operatorname{Re}\lambda_j < \frac{z_{m+1}}{2} \leq \frac{z_{N_1}}{2}.
$$

Thus, in the first case, $m \geq N_1$. Similarly, $m \geq N_3$ in the first case, and hence we have

$$
\begin{aligned}
|p_n(\lambda_j)|^{\frac{1}{j}} &= \left| A_m \prod_{k=1}^{m} \frac{\lambda_j - z_k}{-z_k} \prod_{k=m+1}^{n} \frac{\lambda_j - z_k}{\lambda_\ell - z_k} \right|^{\frac{1}{j}} \leq |A_m|^{\frac{1}{j}} \left(\prod_{k=1}^{m} \frac{|\lambda_j|}{z_k} \right)^{\frac{1}{j}} \\
&\leq |A_m|^{\frac{1}{jm}} \left(\frac{b_p(j)}{z_m^{\varepsilon(m)}} \right)^{\frac{m}{j}} < \sqrt{1+\alpha}\sqrt{1+\alpha} = 1+\alpha.
\end{aligned}
$$

Similarly, in the second case, we have $|p_n(\lambda_j)|^{1/j} < 1+\alpha$, implying that for $n \geq N$

$$
\sup_{j>\ell} |p_n(\lambda_j)|^{\frac{1}{j}} < 1+\alpha.
$$

Since α is arbitrary, we have

$$
\limsup_{n\to\infty} \sup_{j>\ell} |p_n(\lambda_j)|^{\frac{1}{j}} \leq 1
$$

and $\lim_{n\to\infty} p_n(\lambda_j) = \delta_{j,\ell}$ for $j > \ell$.

If $\ell = 0$, then we are done. If $\ell > 0$, then define $p(z) = \prod_{k=0}^{\ell-1} \frac{z-\lambda_k}{\lambda_\ell - \lambda_k}$ and $q_n = p \cdot p_n$, and observe that $\lim_{n\to\infty} q_n(\lambda_j) = \delta_{j,\ell}$ for $j \geq 0$ and

$$\limsup_{n\to\infty} \sup_{j>\ell} |q_n(\lambda_j)|^{\frac{1}{j}} \leq 1.$$

This completes the proof of Theorem 2.2. □

Corollary 2.2. *Suppose (λ_n) is a sequence of distinct complex numbers such that $0 < \operatorname{Re}\lambda_n < \operatorname{Re}\lambda_{n+1}$ and $|\operatorname{Im}\lambda_n| \leq \operatorname{Re}\lambda_n$ for $n \geq 0$, $\lim_{n\to\infty} \operatorname{Re}\lambda_n = \infty$, and there is some $p \geq 0$ such that $|\lambda_n| \leq b_p(n)$ for $n \geq e_p$. Then the diagonal operator with associated sequence (λ_n) is synthetic.*

3. Discussion

One may wonder how far one may push the technique used in Theorem 2.2 to constructively produce examples of synthetic diagonal operators on H_1. To help answer this question we turn to some results from [1]. The two theorems from [1] of the greatest importance to this paper are stated below.

Theorem 3.1 ([1] Theorem 3.1). *For any $p > 2$, writing $\lambda_n = n^p$ $(n \geq 0)$, there exists a complex sequence $\{c_n\}$ satisfying*

(3.1) $$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \delta_p = e^{-\pi \cot \frac{\pi}{p}}$$

such that

$$f(z) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n z}$$

(which converges for $\operatorname{Re} z \geq 0$, and extends as a C^∞ function to the closed right half-plane) has an infinite-order zero at $z = 0$. In other terms,

$$\sum_{n=0}^{\infty} c_n n^{pk} = 0, \ k = 0, 1, 2, \ldots.$$

Moreover, for positive x

$$|f(x)| \leq Ce^{-cx^{-1/p}},$$

where C, c are positive constants.

For integral p, the constant on the right-hand side of (3.1) is sharp, in the sense that no such sequence $\{c_n\}$ exists with $0 < \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} < \delta_p$.

Theorem 3.2 ([1] Theorem 2.1). *Let $0 < \lambda_1 < \lambda_2 < \dots$, and*

$$\limsup_{n\to\infty} \frac{(\ln n)^2}{\lambda_n} = 0.$$

Suppose, for some $\varepsilon > 0$, $|c_n| \le e^{-\varepsilon\sqrt{\lambda_n}}$. If

$$\sum_{n=1}^{\infty} c_n \lambda_n^k = 0,\ k = 0, 1, 2, \dots,$$

then all c_n vanish.

The use of the two above theorems becomes evident when compared with the following result (stated in abbreviated form) from [3].

Theorem 3.3 (Theorem 3 in [3]). *Let D be a cyclic diagonal operator on $\mathcal{H}_R$ having distinct eigenvalues $\{\lambda_n\}$. Then the following are equivalent:*

(1) *D admits spectral synthesis.*
(2) *There does not exist a sequence $\{w_n\}$ of complex numbers, not identically zero, for which $\limsup |w_n|^{1/n} < 1$ and $0 \equiv \sum_{n=0}^{\infty} w_n \lambda_n^k$ for all $k \ge 0$.*

A combination of Theorems 3.1, 3.2, and 3.3 yields the following corollary.

Corollary 3.1. *Let $D : H_1 \to H_1$ be the diagonal operator with associated sequence (n^p). Then the following hold:*

(1) *If $p > 2$, then D is not synthetic.*
(2) *If $1 < p \le 2$, then D is synthetic.*

Consider the diagonal operator $D : H_1 \to H_1$ with associated sequence (n^p).

If $p > 2$, then by Corollary 3.1 and Theorem 1.3 it would be fruitless to try to construct polynomials which separate points and satisfy the minimal growth condition.

However, if $1 < p \le 2$ then Corollary 3.1 and Theorem 1.3 guarantee the existence of polynomials which separate points and satisfy the minimal growth condition.

How shall such polynomials be constructed? Observe that since $\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty$, the ideas used to prove Theorem 2.2 may not apply. To make this precise, let $(p_n) \subseteq \mathbb{C}[z]$ be such that (p_n) satisfies conditions (2.1) and (2.2) for $j = 0$. How would such polynomials look like? Once again, as it was mentioned above, the "obvious"polynomials $o_n(z) = \prod_{k=1}^{n} (z - k^p)/(-k^p)$ fail. To see this, note that by the Mean Value Theorem $(n+1)^p - k^p \ge (n+1-k)pk^{p-1}$. Hence

$$|o_n(\lambda_{n+1})| = \prod_{k=1}^{n} \frac{(n+1)^p - k^p}{k^p} \ge \prod_{k=1}^{n} \frac{p(n+1-k)}{k} = p^n,$$

implying that $\sup_{k>0} |o_n(\lambda_k)|^{1/k} \geq p^{n/(n+1)}$.

Assume, without loss of generality, that no p_n is constant and $p_n(0) = 1$ for all $n \geq 1$. For each $n \geq 1$ write $p_n(z) = \prod_{k=1}^{d_n}(z - z_{n,k})/(-z_{n,k})$ for some $z_{n,1}, \ldots, z_{n,d_n} \in$ C. Define $q_n(z) = \prod_{k=1}^{d_n}(z - |z_{n,k}|)/(-|z_{n,k}|)$ and note that the sequence (q_n) also satisfies (2.1) and (2.2) for $j = 0$. Hence, we may assume, without loss of generality, that for $n \geq 1$, p_n is not constant, $p_n(0) = 1$, and p_n has real positive zeroes.

Suppose momentarily that $p_n(z) = \prod_{k=1}^{n}(z - z_k)/(-z_k)$ for some sequence of positive numbers (z_n) and reason heuristically rather than precisely. One possibility is that $\{\lambda_n : n \geq 1\} \subseteq \{z_n : n \geq 1\}$. However, we have already seen that the polynomials $\prod_{k=1}^{n}(z - \lambda_k)/(-\lambda_k)$ do not satisfy (2). If this is the case, then the sequence (p_n) would seem to fail for the same reason that the sequence (o_n) failed. Thus, there is some $\lambda_{n_0} \notin \{z_n : n \geq 1\}$. Then by Lemma 2.1 we have $\sum_{n=1}^{\infty} 1/z_n = \infty$. This implies that (z_n) grows slower than (λ_n), and hence that p_n grows faster than o_n. Thus, it would seem that (p_n) does not satisfy (2).

Therefore, it appears that there is not some sequence of positive numbers (z_n) for which $p_n(z) = \prod_{k=1}^{n}(z - z_k)/(-z_k)$ satisfies conditions (1) and (2), and greater creativity would be required to construct polynomials (p_n).

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