Известия НАН Армении. Математика, том 49, н. 2, 2014, стр. 25-36.
[1]

CONVERGENCE IN MEASURE OF LOGARITHMIC MEANS OF MULTIPLE FOURIER SERIES

U. GOGINAVA AND L. GOGOLADZE

Ivane Javakhishvili Tbilisi State University, Tbilisi, Georgia

E-mails: *zazagoginava@gmail.com; lgogoladze1@hotmail.com*

Abstract. The maximal Orlicz space such that the mixed logarithmic means of rectangular partial sums of multiple Fourier series for the functions from this space converge in measure is found.

MSC2010 numbers: 42A24

Keywords: Multiple Fourier series, Orlicz space, convergence in measure.

1. Introduction and main results

Let $\mathbb{T}^d := [-\pi, \pi)^d$ denote a cube in the d-dimensional Euclidean space $\mathbb{R}^d$. The elements of $\mathbb{R}^d$ are denoted by $\mathbf{x} := (x_1, ..., x_d)$.
Let $D = \{1, 2, ..., d\}$, $B = \{l_1, l_2, ..., l_r\}$, $1 \le r \le d$, $B \subset D$, $l_k < l_{k+1}$, $k = 1, 2, ..., r-1$, $B' = D \backslash B$. For any $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$ and any $B \subset D$, denote $\mathbf{x}_B = (x_{l_1}, x_{l_2}, ..., x_{l_r}) \in \mathbb{R}^r$. The number of elements of a set B we denote by $|B|$. If $B \neq \varnothing$, then for any natural number n we suppose that $n_B := (n, n, ..., n) \in \mathbb{R}^{|B|}$. The notation $a \lesssim b$ stands for $a \le cb$, where c is a constant depending on the dimension d. Below we will identify the symbols

$$\sum_{i_B=0_B}^{n_B} \text{ and } \sum_{i_{l_1}=0}^{n_{l_1}} \cdots \sum_{i_{l_r}=0}^{n_{l_r}}; \quad d\mathbf{t}_B \text{ and } dt_{l_1} \cdots dt_{l_r}, \text{ respectively.}$$

We denote by $L_0(\mathbb{T}^d)$ the Lebesgue space of functions that are measurable and finite almost everywhere on $\mathbb{T}^d$. The Lebesgue measure of a set $A \subset \mathbb{T}^d$ we denote by $\mathrm{mes}(A)$. Also, we denote by $L_p(\mathbb{T}^d)$ the class of all measurable functions f that are

[1]The research of U. Goginava was supported by Shota Rustaveli National Science Foundation grant DI/9/5-100/13 (Function spaces, weighted inequalities for integral operators and problems of summability of Fourier series)

2π-periodic with respect to all variables and satisfy

$$\|f\|_p := \left(\int_{\mathbb{T}^d} |f|^p\right)^{1/p} < \infty.$$

The $weak-L_1(\mathbb{T}^d)$ space consists of all measurable, 2π-periodic with respect to each variable functions f, satisfying

$$\|f\|_{weak-L_1(\mathbb{T}^d)} := \sup_{\lambda} \lambda \operatorname{mes}\left\{\mathbf{x} \in \mathbb{T}^d : |f(\mathbf{x})| > \lambda\right\} < \infty.$$

The Fourier series of a function $f \in L_1(\mathbb{T}^d)$ with respect to the trigonometric system is the series

$$S[f] := \sum_{n_1,\ldots,n_d=-\infty}^{+\infty} \widehat{f}(n_1,\ldots,n_d)\, e^{i(n_1x_1+\cdots+n_dx_d)},$$

where

$$\widehat{f}(n_1,\ldots,n_d) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(x_1,\ldots,x_d) e^{-i(n_1x_1+\cdots+n_dx_d)}dx_1\cdots dx_d$$

are the Fourier coefficients of f. The rectangular partial sums are defined as follows:

$$S_{N_D}(f;\mathbf{x}) := \sum_{n_D=-N_D}^{N_D} \widehat{f}(n_1,\ldots,n_d)\, e^{i(n_1x_1+\cdots+n_dx_d)}.$$

In the literature, it is known the notion of Riesz logarithmic means of a Fourier series. Given a natural n, the n-th Riesz logarithmic mean of the Fourier series of an integrable function f is defined by

$$\frac{1}{l_n}\sum_{k=0}^{n}\frac{S_k(f)}{k+1}, \qquad l_n := \sum_{k=0}^{n}\frac{1}{k+1},$$

where $S_k(f)$ is the partial sum of the Fourier series of f. The Riesz logarithmic means of Fourier series with respect to trigonometric system has been studied by a number of authors. Here we mention the papers by Szasz [13] and Yabuta [16], where additional references can be found. The Riesz logarithmic means of Fourier series with respect to Walsh and Vilenkin systems were discussed by Gát [2] and Simon [12].

Let $\{q_k : k \geq 0\}$ be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of f are defined by

$$\frac{1}{\sum_{k=0}^{n} q_k}\sum_{k=0}^{n} q_k S_{n-k}(f).$$

In the special case where $q_k = \frac{1}{k+1}$, we have the Nųrlund logarithmic means:

$$L_n(f;x) := \frac{1}{l_n} \sum_{k=0}^{n} \frac{S_{n-k}(f)}{k+1}, \tag{1.1}$$

which represent kind of "reverse"Riesz logarithmic means. In [6] we have proved some convergence and divergence properties for the logarithmic means of Walsh-Fourier series of functions in the class of continuous functions and in the Lebesgue space L.

The Reisz and Nörlund logarithmic means of multiple Fourier series are defined by the following formulas:

$$R_{n_D}(f;\mathbf{x}) := \frac{1}{\prod\limits_{i\in D} l_i} \sum_{i_D=0_D}^{n_D} \frac{S_{i_D}(f;\mathbf{x})}{\prod\limits_{j\in D}(i_j+1)},$$

$$L_{n_D}(f;\mathbf{x}) := \frac{1}{\prod\limits_{i\in D} l_i} \sum_{i_D=0_D}^{n_D} \frac{S_{n_D-i_D}(f;\mathbf{x})}{\prod\limits_{j\in D}(i_j+1)}.$$

It is easy to see that

$$L_{n_D}(f;\mathbf{x}) = \frac{1}{\pi^d} \int\limits_{\mathbb{T}^d} f(\mathbf{t}) F_{n_D}(\mathbf{x}-\mathbf{t})\, d\mathbf{t}$$

and

$$R_{n_D}(f;\mathbf{x}) = \frac{1}{\pi^d} \int\limits_{\mathbb{T}^d} f(\mathbf{t}) G_{n_D}(\mathbf{x}-\mathbf{t})\, d\mathbf{t},$$

where

$$F_{n_D}(\mathbf{x}) := \prod_{j\in D} F_{n_j}(x_j), \qquad G_{n_D}(\mathbf{x}) := \prod_{j\in D} G_{n_j}(x_j),$$

$$F_n(u) := \frac{1}{l_n} \sum_{i=0}^{n} \frac{D_{n-i}(u)}{i+1}, \qquad G_n(u) := \frac{1}{l_n} \sum_{i=0}^{n} \frac{D_i(u)}{i+1}.$$

Let $B \subset D$. Then the mixed logarithmic means of multiple Fourier series are defined by

$$(L_{n_B} \circ R_{n_{B'}})(f;\mathbf{x}) := \frac{1}{\prod\limits_{i\in D} l_i} \sum_{i_D=0_D}^{n_D} \frac{S_{n_B-i_B,i_{B'}}(f;\mathbf{x})}{\prod\limits_{j\in D}(i_j+1)}.$$

It is easy to show that

$$(L_{n_B} \circ R_{n_{B'}})(f;\mathbf{x}) = \frac{1}{\pi^d} \int\limits_{\mathbb{T}^d} f(\mathbf{t}) F_{n_B}(\mathbf{x}_B - \mathbf{t}_B) G_{n_{B'}}(\mathbf{x}_{B'} - \mathbf{t}_{B'})\, d\mathbf{t}.$$

Let $L_Q = L_Q(\mathbb{T}^d)$ be the Orlicz space generated by Young function Q, that is, Q is a convex continuous even function such that $Q(0) = 0$ and (see [10], Ch. 2)

$$\lim_{u\to+\infty} \frac{Q(u)}{u} = +\infty, \quad \lim_{u\to 0} \frac{Q(u)}{u} = 0.$$

This space is endowed with the norm

$$\|f\|_{L_Q(\mathbb{T}^d)} = \inf\{k > 0 : \int_{\mathbb{T}^d} Q(|f|/k) \le 1\}.$$

In particular, if $Q(u) = u\log^{\beta}(1+u)$ $(u, \beta > 0)$, then the corresponding space will be denoted by $L\log^{\beta} L(\mathbb{T}^d)$.

The rectangular partial sums of double Fourier series $S_{n,m}(f;x,y)$ of a function $f \in L_p(\mathbb{T}^2)$, $1 < p < \infty$, converge in L_p norm to the function f as $n \to \infty$ (see [17]). In the space $L_1(\mathbb{T}^2)$ this result does not hold. But for $f \in L_1(\mathbb{T})$, the operator $S_n(f;x)$ is of weak type $(1,1)$ (see [18]). This fact implies convergence of $S_n(f;x)$ in measure on $\mathbb{T}$ to the function $f \in L_1(\mathbb{T})$. However, for double Fourier series this result does not hold (see [8, 11]). Moreover, it is proved that quadratic partial sums $S_{n,n}(f;x,y)$ of double Fourier series do not converge in two-dimensional measure on $\mathbb{T}^2$ even for functions from Orlicz spaces wider than the Orlicz space $L\log L(\mathbb{T}^2)$. On the other hand, it is well-known that the rectangular partial sums $S_{n,m}(f;x,y)$ of a function $f \in L\log L(\mathbb{T}^2)$ converge in measure on $\mathbb{T}^2$.

Notice that the classical regular summation methods often improve the convergence of Fourier series. For instance, the Fejãr means of the double Fourier series of a function $f \in L_1(\mathbb{T}^2)$ converge in $L_1(\mathbb{T}^2)$ norm to the function f (see [17]). These means represent the particular case of the Nцrlund means.

It is well known that the method of Nцrlund logarithmic means of double Fourier series is weaker than the Cesáro method of any positive order. In [14] Tkebuchava proved that these means of double Fourier series in general do not converge in two-dimensional measure on $\mathbb{T}^d$ even for functions from Orlicz spaces wider than the Orlicz space $L\log^{d-1} L(\mathbb{T}^d)$. Thus, not all classical regular summation methods can improve the convergence in measure of double Fourier series.

For the results on summability of logarithmic means of Walsh-Fourier series we refer the papers [4]–[6], [13, 16].

In this paper we consider the mixed logarithmic means $(L_{n_B} \circ R_{n_{B'}})(f)$ of rectangular partial sums of multiple Fourier series and prove that these means are acting from the space $L\log^{|B|-1} L(\mathbb{T}^d)$ into the space $weak - L_1(\mathbb{T}^d)$ (Theorem 1.1). This fact implies the convergence in measure of mixed logarithmic means of rectangular partial sums of multiple Fourier series (Theorem 1.2). We also prove the sharpness of this result (Theorem 1.3).

Theorem 1.1. *Let $B \subset D$ and $f \in L\log^{|B|-1} L\left(\mathbb{T}^d\right)$. Then*

$$\left\|\left(L_{n_B} \circ R_{n_{B'}}\right)(f)\right\|_{weak-L_1(\mathbb{T}^d)} \lesssim 1 + \left\||f|\log^{|B|-1}|f|\right\|_{L_1(\mathbb{T}^d)}.$$

Theorem 1.2. *Let $B \subset D$ and $f \in L\log^{|B|-1} L\left(\mathbb{T}^d\right)$. Then*

$$\left(L_{n_B} \circ R_{n_{B'}}\right)(f) \to f \text{ in measure on } \mathbb{T}^d \quad \text{as } n_i \to \infty, i \in D.$$

Theorem 1.3. *Let $B \subset D$, $|B| > 1$ and $L_Q\left(\mathbb{T}^d\right)$ be an Orlicz space, such that*

$$L_Q\left(\mathbb{T}^d\right) \nsubseteq L\log^{|B|-1} L\left(\mathbb{T}^d\right).$$

Then the set of functions from the Orlicz space $L_Q\left(\mathbb{T}^d\right)$ with logarithmic means $\left(L_{n_B} \circ R_{n_{B'}}\right)(f)$ of rectangular partial sums of multiple Fourier series convergent in measure on $\mathbb{T}^d$ is of first Baire category in $L_Q\left(\mathbb{T}^d\right)$.

Corollary 1.1. *Let $B \subset D$, $|B| > 1$ and $\varphi : [0,\infty[\to [0,\infty[$ be a nondecreasing function satisfying the condition*

$$\varphi(x) = o(x\log^{|B|-1} x) \quad \text{as} \quad x \to +\infty.$$

Then there exists a function $f \in L_1(\mathbb{T}^d)$ such that

a)

$$\int_{\mathbb{T}^d} \varphi(|f|) < \infty;$$

b) the logarithmic means $\left(L_{n_B} \circ R_{n_{B'}}\right)(f)$ of rectangular partial sums of multiple Fourier series of f diverge in measure on $\mathbb{T}^d$.

2. Auxiliary results

In this section we state some auxiliary results that will be used in the proofs of our main results. For the next result we refer to ([1], Ch. 1).

Theorem 2.1. *Let $H : L_1(\mathbb{T}^d) \to L_0(\mathbb{T}^d)$ be a linear continuous operator, which commutes with a family of translations $\mathcal{E}$, that is, $HEf = EHf$ for all $E \in \mathcal{E}$ and all $f \in L_1(\mathbb{T}^d)$. Let $\|f\|_{L_1(\mathbb{T}^d)} = 1$ and $\lambda > 1$. Then for any $1 \le r \in \mathbb{N}$ under the condition $mes\{\mathbf{x} \in \mathbb{T}^d : |Hf| > \lambda\} \ge \frac{1}{r}$ there exist $E_1, ..., E_r \in \mathcal{E}$, $j = 1, 2, ..., d$, and $\varepsilon_i = \pm 1$, $i = 1, ..., r$, such that*

$$\text{mes}\left\{\mathbf{x} \in \mathbb{T}^d : \left|H\left(\sum_{i=1}^{r} \varepsilon_i f(E_i \mathbf{x})\right)\right| > \lambda\right\} \ge \frac{1}{8}.$$

The proof of the next result can be found in [3].

Lemma 2.1. *Let $\{H_m\}_{m=1}^{\infty}$ be a sequence of linear continuous operators, acting from Orlicz space $L_Q(\mathbb{T}^d)$ into the space $L_0(\mathbb{T}^d)$. Suppose that there exists a sequence of functions $\{\xi_k\}_{k=1}^{\infty}$ from the unit ball $S_Q(0,1)$ of the space $L_Q(\mathbb{T}^d)$, sequences of integers $\{m_k\}_{k=1}^{\infty}$ and $\{\nu_k\}_{k=1}^{\infty}$ increasing to infinity such that*

$$\varepsilon_0 = \inf_k \operatorname{mes}\{\mathbf{x} \in \mathbb{T}^d : |H_{m_k}\xi_k(x,y)| > \nu_k\} > 0.$$

Then the set K of functions f from the space $L_Q(\mathbb{T}^d)$, for which the sequence $\{H_m f\}$ converges in measure to an a. e. finite function, is of first Baire category in the space $L_Q(\mathbb{T}^d)$.

For the next lemma we refer to [4].

Lemma 2.2. *Let $L_\Phi(\mathbb{T}^d)$ be an Orlicz space and let $\varphi : [0,\infty) \to [0,\infty)$ be a measurable function satisfying the condition $\varphi(x) = o(\Phi(x))$ as $x \to \infty$. Then there exists an Orlicz space $L_\omega(\mathbb{T}^d)$, such that $\omega(x) = o(\Phi(x))$ as $x \to \infty$, and $\omega(x) \geq \varphi(x)$ for $x \geq c \geq 0$.*

To state the next lemma the proof of which can be found in [7], we first introduce the following notation:

$$\alpha_{mn} := \frac{\pi(12m+1)}{6(2^{2n}+1/2)},\ \beta_{mn} := \frac{\pi(12m+5)}{6(2^{2n}+1/2)},\ \gamma_n := \frac{\pi}{6(2^{2n}+1/2)},$$

and set

$$J_n := \bigcup_{m=1}^{2^n-1} [\alpha_{mn}+\gamma_n, \beta_{mn}-\gamma_n].$$

Lemma 2.3. *Let $0 \leq z \leq \gamma_n$ and $x \in J_n$. Then*

$$F_{2^{2n}}(x-z) \gtrsim \frac{1}{x}.$$

3. Proof of main results

Proof of Theorem 1.1. First, we prove that the one dimensional operator $L_n(f)$ (see (1.1)), has weak type $(1,1)$, that is, for $f \in L_1(\mathbb{T}^1)$ we have

$$\|L_n(f)\|_{weak-L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)}. \tag{3.1}$$

Setting $\alpha_n(t) := \sin((n+1)t)$ and $\beta_n(t) := \cos((n+1)t)$, we can write

$$(3.2)\quad S_{n-k}(f;x) = \frac{1}{\pi}\int_{\mathbb{T}} f(t)\frac{\sin((n-k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$= \frac{1}{\pi}\int_{\mathbb{T}} f(t)\sin((n+1)(x-t))\frac{\cos((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$-\frac{1}{\pi}\int_{\mathbb{T}} f(t)\cos((n+1)(x-t))\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$= \frac{1}{\pi}\int_{\mathbb{T}} f(t)\sin((n+1)(x-t))\left(\frac{\cos((k+1/2)(x-t))}{2\sin((x-t)/2)} - \frac{\cos((x-t)/2)}{2\sin((x-t)/2)}\right)dt$$

$$+\frac{1}{\pi}\int_{\mathbb{T}} f(t)\frac{\sin((n+1)(x-t))}{2\tan((x-t)/2)}dt$$

$$-\frac{1}{\pi}\int_{\mathbb{T}} f(t)\cos((n+1)(x-t))\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$= -\frac{\alpha_n(x)}{\pi}\int_{\mathbb{T}} f(t)\beta_n(t)\widetilde{D}_k(x-t)\,dt + \frac{\beta_n(x)}{\pi}\int_{\mathbb{T}} f(t)\alpha_n(t)\widetilde{D}_k(x-t)\,dt$$

$$+\frac{1}{\pi}\int_{\mathbb{T}} f(t)\frac{\sin((n+1)(x-t))}{2\tan((x-t)/2)}dt$$

$$-\frac{\beta_n(x)}{\pi}\int_{\mathbb{T}} f(t)\beta_n(t)\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$-\frac{\alpha_n(x)}{\pi}\int_{\mathbb{T}} f(t)\alpha_n(t)\frac{\sin((k+1/2)(x-t))}{2\sin((x-t)/2)}dt$$

$$= -\alpha_n(x)\widetilde{S}_k(f\beta_n;x) + \beta_n(x)\widetilde{S}_k(f\alpha_n;x)$$

$$-\beta_n(x)S_k(f\beta_n;x) - \alpha_n(x)S_k(f\alpha_n;x) + S^*_{n+1}(f;x),$$

where $\widetilde{S}_n(f;x)$, $S^*_n(f;x)$ and $\widetilde{D}_k(x)$ stand for the conjugate partial sums, the modified partial sums and the conjugate kernel, respectively. Taking into account that for $f \in L_1(\mathbb{T}^1)$ (see [18], Ch. 7):

$$\|S^*_n(f)\|_{weak-L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)},$$

$$\left\|\sup\frac{1}{n+1}\left|\sum_{k=0}^{n} S_k(f)\right|\right\|_{weak-L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)}$$

and

$$\left\| \sup \frac{1}{n+1} \left| \sum_{k=0}^{n} \widetilde{S}_k(f) \right| \right\|_{weak-L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)},$$

we can apply Abel's transform to infer (3.1) from (3.2).

We apply the following special case of the Marcinkiewicz interpolation theorem ([9], p. 173). Let $T : L_1(\mathbb{T}^1) \to L_0(\mathbb{T}^1)$ be a quasilinear operator of weak type $(1,1)$ and of type (α, α) for some $1 < \alpha < \infty$, that is, T satisfies the following conditions:

a) for all $y > 0$

$$\text{mes}\left\{x \in \mathbb{T}^1 : |T(f,x)| > y\right\} \lesssim \frac{1}{y} \int_{\mathbb{T}^1} |f(x)|\, dx; \quad \forall f \in L^1(\mathbb{T}^1); \tag{3.3}$$

b) for all $f \in L_\alpha(\mathbb{T}^1)$

$$\|Tf\|_{L_\alpha(\mathbb{T}^1)} \lesssim \|f\|_{L_\alpha(\mathbb{T}^1)}. \tag{3.4}$$

Then for all $\beta \geq 0$

$$\int_{\mathbb{T}^1} |T(f,x)| \ln^\beta |T(f,x)|\, dx \lesssim \int_{\mathbb{T}^1} |f(x)| \ln^{\beta+1} |f(x)|\, dx + 1. \tag{3.5}$$

On the other hand, it is easy to show that the operator $f * G_n$ has type (1,1). Indeed, applying Abel's transform we get

$$\begin{aligned} l_n G_n(x) &= \sum_{i=0}^{n-1} \left(\frac{1}{i+1} - \frac{1}{i+2} \right) \sum_{j=0}^{i} D_j(x) + \frac{1}{n+1} \sum_{j=0}^{n} D_j(x) \\ &= \sum_{i=0}^{n-1} \frac{K_i(x)}{i+2} + K_n(x), \end{aligned}$$

where

$$K_i(x) := \frac{1}{i+1} \sum_{j=0}^{i} D_j(x).$$

Hence

$$f * G_n = \frac{1}{l_n} \sum_{i=0}^{n-1} \frac{f * K_i}{i+2} + \frac{f * K_n}{l_n}. \tag{3.6}$$

Since $\|f * K_n\|_1 \lesssim \|f\|_1$ from (3.6) we conclude that

$$\|f * G_n\|_{L_1(\mathbb{T}^1)} \lesssim \|f\|_{L_1(\mathbb{T}^1)}. \tag{3.7}$$

Next, setting

$$\Omega := \left\{ \mathbf{x} \in \mathbb{T}^d : \left| \left(L_{n_B} \circ R_{n_{B'}} \right)(f, \mathbf{x}) \right| > \lambda \right\}.$$

where $B' = \{s_1, s_2, ..., s_{r'}\}$, in view of (3.1)-(3.7) we can write

$$\begin{aligned}
&\lambda \text{mes}\left\{\mathbf{x} \in \mathbb{T}^d : \left|\left(L_{n_B} \circ R_{n_{B'}}\right)(f, \mathbf{x})\right| > \lambda\right\} \\
=\ & \lambda \int_{\mathbb{T}^d} 1_\Omega(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{T}^{d-1}} \left(\int_{\mathbb{T}} 1_\Omega(\mathbf{x})\, dx_{l_1}\right) d\mathbf{x}_{D\setminus\{l_1\}} \\
\lesssim\ & \left\|\left(L_{n_{B\setminus\{l_1\}}} \circ R_{B'}\right)(f)\right\|_{L_1(\mathbb{T}^d)} \\
=\ & \left\|\left(R_{s_1} \circ \cdots \circ R_{s_{r'}} \circ L_{n_{l_2}} \circ \cdots \circ L_{n_{l_r}}\right)(f)\right\|_{L_1(\mathbb{T}^d)} \\
\lesssim\ & \cdots \lesssim \left\|\left(L_{n_{l_2}} \circ \cdots \circ L_{n_{l_r}}\right)(f)\right\|_{L_1(\mathbb{T}^d)} \\
\lesssim\ & 1 + \left\|\left|L_{n_{l_3}} \circ \cdots \circ L_{n_{l_r}}(f)\right| \log \left|L_{n_3} \circ \cdots \circ L_{n_{l_r}}(f)\right|\right\|_{L_1(\mathbb{T}^d)} \\
\lesssim\ & \cdots \lesssim 1 + \left\|\left|L_{n_{l_r}}(f)\right| \log^{r-2} \left|L_{n_{l_r}}(f)\right|\right\|_{L_1(\mathbb{T}^d)} \\
\lesssim\ & 1 + \left\||f| \log^{r-1} |f|\right\|_{L_1(\mathbb{T}^d)},
\end{aligned}$$

and the result follows. Theorem 1.1 is proved. □

Proof of Theorem 1.2. By virtue of standard arguments (see, e.g., [18]), the result can easily be deduced from Theorem 1.1. So we omit the details. □

Proof of Theorem 1.3. By Lemma 2.1 the proof will be completed if we show that there exist sequences of integers $\{n_k : k \geq 1\}$ and $\{\nu_k : k \geq 1\}$ increasing to infinity, and a sequence of functions $\{\xi_k : k \geq 1\}$ from the unit ball $S_Q(0,1)$ of Orlicz space $L_Q(\mathbb{T}^d)$, such that for all k

$$\text{mes}\{\mathbf{x} \in \mathbb{T}^d : \left|L_{2^{2n_k}(B)} \circ R_{2^{2n_k}(B')}(\xi_k; \mathbf{x})\right| > \nu_k\} \geq \frac{1}{8}. \tag{3.8}$$

First, we prove that

$$\begin{aligned}
&\text{mes}\left\{\mathbf{x} \in \mathbb{T}^d : \left|L_{2^{2n}(B)} \circ R_{2^{2n}(B')}\left(\frac{1_{[0,\gamma_n]^{|B|}}}{\gamma_n^{|B|}}; \mathbf{x}\right)\right| \gtrsim 2^{n(2|B|-1)}\right\} \\
\gtrsim\ & \frac{n^{|B|-1}}{2^{n(2|B|-1)}}, |B| > 1.
\end{aligned} \tag{3.9}$$

By Lemma 2.3 we have

$$\begin{aligned}
&L_{2^{2n}(B)} \circ R_{2^{2n}(B')}\left(\frac{1_{[0,\gamma_n]^{|B|}}}{\gamma_n^{|B|}}; \mathbf{x}\right) = \frac{1}{\gamma_n^{|B|}} \frac{1}{\pi^d} \int_{[0,\gamma_n]^{|B|}} \prod_{j \in B} F_{2^{2n}}(x_j - z_j)\, d\mathbf{z}_B \\
&\times \int_{\mathbb{T}^{|B'|}} \prod_{i \in B'} G_{2^{2n}}(x_i - z_i)\, d\mathbf{z}_{B'} = \frac{1}{\gamma_n^{|B|}} \frac{1}{\pi^{|B|}} \int_{[0,\gamma_n]^{|B|}} \prod_{j \in B} F_{2^{2n}}(x_j - z_j)\, d\mathbf{z}_B \\
\gtrsim\ & \prod_{j \in B} \frac{1}{x_j}, x_j \in J_n, j \in B.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d:\left|L_{2^{2n}(B)}\circ R_{2^{2n}(B')}\left(\frac{\mathbf{1}_{[0,\gamma_n]^{|B|}}}{\gamma_n^{|B|}};\mathbf{x}\right)\right|\gtrsim 2^{n(2|B|-1)}\right\}\\
\geq\ &\operatorname{mes}\left\{\mathbf{x}\in J_n^{|B|}\times\mathbb{T}^{|B'|}:\prod_{j\in B}\frac{1}{x_j}\gtrsim 2^{n(2|B|-1)}\right\}\\
=\ &(2\pi)^{|B'|}\operatorname{mes}\left\{\mathbf{x}_B\in J_n^{|B|}:x_{l_1}\lesssim\frac{1}{2^{n(2|B|-1)}\prod\limits_{j\in B\setminus\{l_1\}}x_j}\right\}.
\end{aligned}$$

Setting

$$r_{n,m_{l_2},\dots,m_{l_r}}:=\max\left\{l:\beta_{ln}\leq\frac{1}{2^{n(2|B|-1)}\prod\limits_{j\in B\setminus\{l_1\}}(\beta_{m_jn}-\gamma_n)}+\gamma_n\right\},$$

after some algebra we obtain $r_{n,m_{l_2},\dots,m_{l_r}}\sim\frac{2^n}{\prod_{j\in B\setminus\{l_1\}}m_j}$. Then we have

$$\begin{aligned}
&\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d:\left|L_{2^{2n}(B)}\circ R_{2^{2n}(B')}\left(\frac{\mathbf{1}_{[0,\gamma_n]^{|B|}}}{\gamma_n^{|B|}};\mathbf{x}\right)\right|\gtrsim 2^{n(2|B|-1)}\right\}\\
\gtrsim\ &\frac{1}{2^{2n|B|}}\sum_{m_{B\setminus\{l_1\}}=1_{B\setminus\{l_1\}}}^{2^n(B\setminus\{l_1\})}\ \sum_{l=1}^{r_{n,m_{l_2},\dots,m_{l_r}}}1\\
\gtrsim\ &\frac{1}{2^{2n|B|}}\sum_{m_{B\setminus\{l_1\}}=1_{B\setminus\{l_1\}}}^{2^n(B\setminus\{l_1\})}\frac{2^n}{\prod\limits_{j\in B\setminus\{l_1\}}m_j}\gtrsim\frac{n^{|B|-1}}{2^{n(2|B|-1)}},|B|>1,
\end{aligned}$$

yielding (3.9).

Next, under the conditions of the theorem we have

$$\liminf_{u\to\infty}\frac{Q(u)}{u\log^{|B|-1}u}=0.$$

Therefore there exists a sequence of integers $\{n_k:k\geq1\}$ increasing to infinity, such that for all k

$$\lim_{k\to\infty}\frac{Q(2^{2n_k|B|})}{2^{2n_k|B|}n_k^{|B|-1}}=0\quad\text{and}\quad\frac{Q(2^{2n_k|B|})}{2^{2n_k|B|+4|B|}}\geq1. \tag{3.10}$$

From (3.9) we have

$$\begin{aligned}
&\operatorname{mes}\left\{\mathbf{x}\in\mathbb{T}^d:\left|L_{2^{2n_k}(B)}\circ R_{2^{2n_k}(B')}\left(\frac{\mathbf{1}_{[0,\gamma_{n_k}]^{|B|}}}{\gamma_{n_k}^{|B|}};\mathbf{x}\right)\right|\gtrsim 2^{n_k(2|B|-1)}\right\}\\
\gtrsim\ &\frac{n_k^{|B|-1}}{2^{n_k(2|B|-1)}}.
\end{aligned}$$

Then, by Theorem 2.1, there exist $E_1, ..., E_{r_k} \in \mathcal{E}$ and $\varepsilon_1, ..., \varepsilon_{r_k}$ $(\varepsilon_i = \pm 1)$ such that

$$\text{(3.11)} \qquad \begin{aligned} \operatorname{mes}\{\mathbf{x} \in \mathbb{T}^d : & \left| \sum_{i=1}^{r_k} \varepsilon_i L_{2^{2n_k}(B)} \circ R_{2^{2n_k}(B')} \left(\frac{\mathbf{1}_{[0,\gamma_{n_k}]^{|B|}}}{\gamma_{n_k}^{|B|}}; E_i \mathbf{x} \right) \right| \\ & > 2^{n_k(2|B|-1)}\} > \frac{1}{8}, \end{aligned}$$

where $r_k \sim \frac{2^{n_k(2|B|-1)}}{n_k^{|B|-1}}$.

Denoting

$$\nu_k = \frac{2^{n_k(4|B|-1)-1}}{r_k Q\left(2^{2n_k|B|}\right)}, \quad \xi_k(\mathbf{x}) = \frac{2^{2|B|n_k-1}}{Q\left(2^{2n_k|B|}\right)} M_k(\mathbf{x}),$$

where

$$M_k(\mathbf{x}) = \frac{1}{r_k} \sum_{i=1}^{r_k} \varepsilon_i \frac{\mathbf{1}_{[0,\gamma_{n_k}]^{|B|}}(E_i \mathbf{x})}{\gamma_{n_k}^{|B|}},$$

from (3.11) we obtain (3.8).

Finally, we prove that $\xi_k \in S_Q(0,1)$. To this end, observe that since

$$\|M_k\|_\infty \le 2^{2|B|(n_k+2)},$$

$$\|M_k\|_{L_1(\mathbb{T}^d)} \le 1,$$

$$\|\xi_k\|_{L_Q(\mathbb{T}^d)} \le \frac{1}{2}\left[\int_{\mathbb{T}^d} Q(2|\xi_k|) + 1\right],$$

and $\frac{Q(u)}{u} < \frac{Q(u')}{u'}$ for $0 < u < u'$, in view of (3.10) we can write

$$\begin{aligned} \|\xi_k\|_{L_Q(\mathbb{T}^d)} & \le \frac{1}{2}\left[1 + \int_{\mathbb{T}^d} Q\left(\frac{2^{2|B|n_k}|M_k(x)|}{Q(2^{2|B|n_k})}\right) d\mathbf{x}\right] \\ & \le \frac{1}{2}\left[1 + \int_{\mathbb{T}^d} \frac{Q\left(\frac{2^{2|B|n_k} 2^{2|B|(n_k+2)}}{Q(2^{2|B|n_k})}\right)}{\frac{2^{2|B|n_k} 2^{2|B|(n_k+2)}}{Q(2^{2|B|n_k})}} \frac{2^{2|B|n_k}|M_k(\mathbf{x})|}{Q(2^{2|B|n_k})} d\mathbf{x}\right] \\ & \le \frac{1}{2}\left[1 + \int_{\mathbb{T}^d} \frac{Q\left(2^{2|B|n_k}\right)}{2^{2|B|n_k}} \frac{2^{2|B|n_k}|M_k(\mathbf{x})|}{Q(2^{2|B|n_k})} d\mathbf{x}\right] \le 1, \end{aligned}$$

implying that $\xi_k \in S_Q(0,1)$. This completes the proof of Theorem 1.3. □

Proof of Corollary 1.1. The result follows from Theorem 1.3 and Lemma 2.2. □

Acknowledgment. The authors would like to thank the referee for helpful suggestions.

REFERENCES

[1] A. Garsia, *Topic in almost everywhere convergence*, Chicago, 1970.

[2] G. Gát, Investigation of certain operators with respect to the Vilenkin system, Acta Math. Hungar. **61** (1993), no. 1-2, 131–149.

[3] G. Gát, U. Goginava, G. Tkebuchava, *Convergence in measure of Logarithmic means of double Walsh-Fourier series*, Georgian Math. J. 12 (2005), no. 4, 607–618.

[4] G. Gát, U. Goginava, G. Tkebuchava, Convergence in measure of logarithmic means of quadratical partial sums of double Walsh-Fourier series. J. Math. Anal. Appl. 323 (2006), no. 1, 535–549.

[5] G. Gát, U. Goginava, K. Nagy, On the Marcinkiewicz-Fejer means of double Fourier series with respect to the Walsh-Kaczmarz system. Studia Sci. Math. Hungar. 46 (2009), no. 3, 399–421.

[6] G. Gát and U. Goginava, uniform and L-convergence of logarithmic means of Walsh-Fourier series, Acta Math. Sin. (Engl. Ser.) 22 (2006), no. 2, 497–506.

[7] U. Goginava and G. Tkebuchava, Convergence of the logarithmic means of Fourier series. J. Math. Anal. Approx. Theory 1 (2006), no. 1, 30–41.

[8] R. Getsadze, *On the divergence in measure of multiple Fourier seties*, Some problems of functions theory, **4** (1988), 84-117(in Russian).

[9] R. E. Edwards, Fourier series a modern introduction, vol. 1, Springer-Verlang, New-York, Heidelberg, Berlin 1982.

[10] M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex functions and Orlicz space*(English translation), P. Noorhoff (Groningen, 1961).

[11] S. V. Konyagin, Divergence with respect to measure of multiple Fourier series. (Russian) Mat. Zametki 44 (1988), no. 2, 196–201, 286; translation in Math. Notes 44 (1988), no. 1-2, 589–592 (1989)

[12] P. Simon, Strong convergence of certain means with respect to the Walsh-Fourier series, Acta Math. Hungar. **49** (1987), 425–431.

[13] O. Szász, On the logarithmic means of rearranged partial sums of Fourier series, Bull. Amer. Math. Soc. **48** (1942), 705–711.

[14] G. Tkebuchava, Divergence in measure of logarithmic means of multiple Fourier series. Bull. Georgian Acad. Sci. 170 (2004), no. 2, 224–225.

[15] G. Tkebuchava, Subsequences of partial sums of multiple Fourier and Fourier-Walsh series. Bull. Georgian Acad. Sci. 169 (2004), no. 2, 252–253.

[16] K. Yabuta, Quasi-Tauberian theorems, applied to the summability of Fourier series by Riesz's logarithmic means, Тфhфku Math. Journ. **22** (1970), 117–129.

[17] L. V. Zhizhiashvili, *Some problems of multidimensional harmonic analysis*, Tbilisi, TGU, 1996 (Russian).

[18] A. Zygmund, *Trigonometric Series, vol. 1*, Cambridge Univ. Press, Cambridge, 1959.

Поступила 5 декабря 2013