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ON THE PARTIAL SUMS OF VILENKIN-FOURIER SERIES

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Abstract. The main aim of this paper is to investigate weighted maximal operators of partial sums of Vilenkin-Fourier series. Also, the obtained results we use to prove approximation and strong convergence theorems on the martingale Hardy spaces H_p, when $0 < p \leq 1$.

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1. Introduction

It is well-known that the Vilenkin system does not form a basis in the space $L_1(G_m)$. Moreover, there is a function f in the martingale Hardy space $H_1(G_m)$, such that the partial sums of f are not bounded in $L_1(G_m)$-norm, but the partial sums S_n of the Vilenkin-Fourier series of any function $f \in L_1(G_m)$ convergence in measure (see [12]).

Uniform convergence and some approximation properties of the partial sums in $L_1(G_m)$ norms were studied by Goginava [8] (see also [9]). Fine [3] has obtained sufficient conditions for the uniform convergence, which are in complete analogy with the Dini-Lipschits conditions. Guliev [13] has estimated the rate of uniform convergence of a Walsh-Fourier series using Lebesgue constants and modulus of continuity. Uniform convergence of subsequences of partial sums was studied also in [7]. The same problem for Vilenkin group G_m has been considered by Fridli [4], Blahota [2] and Gát [6].

It is also known that a subsequence S_{n_k} is bounded from $L_1(G_m)$ to $L_1(G_m)$ if and only if n_k has uniformly bounded variation and the subsequence of partial sums S_{M_n} is bounded from the martingale Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$ for all $p > 0$.

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In this paper we prove the following rather surprising fact: there exists a martingale $f \in H_p(G_m)$ $(0 < p < 1)$, such that

$$\sup_{n\in\mathbb{N}} \|S_{M_n+1}f\|_{L_{p,\infty}} = \infty.$$

The reason of divergence of $S_{M_n+1}f$ is that for $0 < p < 1$ the Fourier coefficients of $f \in H_p(G_m)$ are not bounded (see [17]).

In [5], Gát has obtained the following strong convergence result: for all $f \in H_1(G_m)$

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0,$$

where $S_k f$ denotes the k-th partial sum of the Vilenkin-Fourier series of f.
For the trigonometric analogue of this result we refer to Smith [16], for the Walsh system see Simon [14]. For the Vilenkin system Simon [15] has proved that there is an absolute constant c_p, depending only on p, such that for all $f \in H_p(G_m)$

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \le c_p \|f\|_{H_p}^p, \quad 0 < p < 1. \tag{1.1}$$

In [18] the author proved that for any nondecreasing function $\Phi : \mathbb{N} \to [1, \infty)$ satisfying the condition $\lim_{n\to\infty} \Phi(n) = +\infty$, there exists a martingale $f \in H_p(G_m)$, such that

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_{L_{p,\infty}}^p \Phi(k)}{k^{2-p}} = \infty \text{ for } 0 < p < 1. \tag{1.2}$$

Strong convergence theorems for two-dimensional partial sums were proved by Weisz [23], Goginava [10], Gogoladze [11] and Tephnadze [19].

The main aim of this paper is to investigate weighted maximal operators of partial sums of Vilenkin-Fourier series. Also, the obtained results we use to prove approximation and strong convergence theorems on the martingale Hardy spaces H_p, when $0 < p \le 1$.

2. Definitions and notation

Let $\mathbb{N}_+$ denote the set of positive integers, and let $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. By $m := (m_0, m_1, ...)$ we denote a sequence of positive integers m_k with $m_k \ge 2$. Denote by $Z_{m_k} := \{0, 1, ..., m_k - 1\}$ the additive group of integers modulo m_k, and define the group G_m as the complete direct product of the group Z_{m_j} with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k, \quad j \in Z_{m_k}$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence $m := (m_0, m_1, ...)$ is bounded, then G_m is called a bounded Vilenkin group, otherwise – unbounded.

The elements of G_m can be represented by sequences

$$x := (x_0, x_1, ..., x_j, ...), \quad x_k \in Z_{m_k}.$$

Observe that the sequence $\{I_n(x), n \in \mathbb{N}\}$, where $I_0(x) := G_m$ and

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, ..., y_{n-1} = x_{n-1}\} \ (x \in G_m, \ n \in \mathbb{N}).$$

forms a base for the neighborhood of G_m.

Denoting $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$, we clearly have

$$\overline{I_N} = \bigcup_{s=0}^{N-1} I_s \backslash I_{s+1}. \tag{2.1}$$

If we define the so-called generalized number system based on m as follows

$$M_0 := 1, \quad M_{k+1} := m_k M_k, \quad k \in \mathbb{N},$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N})$, and only a finite number of n_j's differ from zero.

Also, we denote $|n| := \max \{j \in \mathbb{N}, n_j \neq 0\}$, and $L_1(G_m)$ will stand for the usual (one dimensional) Lebesgue space.

Next, on G_m we introduce an orthonormal system, called the Vilenkin system as follows.

We first define the complex valued functions $r_k(x) : G_m \to \mathbb{C}$ to be the generalized Rademacher functions:

$$r_k(x) := \exp(2\pi i x_k/m_k) \quad \left(\iota^2 = -1, \ x \in G_m, \ k \in \mathbb{N}\right).$$

The Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m is then defined as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad n \in \mathbb{N}.$$

In the special case where $m \equiv 2$, the Vilenkin system will be referred as a Walsh-Paley system. It is known that the Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see, e.g., [1, 20]).

Similar to the classical Fourier analysis case, for $f \in L_1(G_m)$ we can define the Fourier coefficients, the partial sums of the Fourier series and the Dirichlet kernel with respect to the Vilenkin system ψ as follows:

$$\widehat{f}(k) \; : = \int_{G_m} f\overline{\psi}_k d\mu, \quad k \in \mathbb{N},$$

$$S_n f \; : = \sum_{k=0}^{n-1} \widehat{f}(k)\, \psi_k, \quad n \in \mathbb{N}_+, \; S_0 f := 0,$$

$$D_n \; : = \sum_{k=0}^{n-1} \psi_k, \quad n \in \mathbb{N}_+.$$

Recall that (see [1])

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases} \tag{2.2}$$

and

$$D_n(x) = \psi_n(x) \left(\sum_{j=0}^{\infty} D_{M_j}(x) \sum_{u=m_j-n_j}^{m_j-1} r_j^u(x) \right). \tag{2.3}$$

The norm (or quasinorm) in the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p \, d\mu \right)^{1/p} \quad (0 < p < \infty).$$

Notice that the space $L_{p,\infty}(G_m)$ consists of all measurable functions f for which

$$\|f\|_{L_{p,\infty}} := \sup_{\lambda > 0} \lambda \mu (f > \lambda)^{1/p} < +\infty.$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n $(n \in \mathbb{N})$. Denote by $f = (f_n, n \in \mathbb{N})$ a martingale with respect to F_n, $n \in \mathbb{N}$ (for details we refer to [21]). The maximal function of a martingale f is defend by

$$f^* = \sup_{n \in \mathbb{N}} \left| f^{(n)} \right|.$$

In the case where $f \in L_1(G_m)$ the maximal function can also be defined by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|$$

For $0 < p < \infty$ the Hardy martingale space $H_p(G_m)$ consists of all martingales f satisfying

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

For $0 < p \leq 1$ the dyadic Hardy martingale spaces H_p (G_m) possess an atomic characterization. Namely, the following theorem is true (see [24]).

Theorem W. *A martingale $f=(f_n,\, n\in\mathbb{N})$ is in $H_p(G_m)$ $(0<p\leq 1)$ if and only if there exist a sequence of p-atoms $(a_k,\ k\in\mathbb{N})$ and a sequence of real numbers $(\mu_k,\ k\in\mathbb{N})$ such that for every $n\in\mathbb{N}$*

$$\sum_{k=0}^{\infty}\mu_k S_{M_n}a_k=f_n \quad \text{and} \quad \sum_{k=0}^{\infty}|\mu_k|^p<\infty. \tag{2.4}$$

Moreover, $\|f\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}|\mu_k|^p\right)^{1/p}$, where the infimum is taken over all decompositions of f of the form (2.4).

Recall, that a bounded measurable function a is a p-atom, if there exist a dyadic interval I, such that

$$\int_I a d\mu=0,\quad \|a\|_\infty\leq\mu(I)^{-1/p},\quad \operatorname{supp}(a)\subset I.$$

Let $X=X(G_m)$ denote either the space $L_1(G_m)$ or the space of continuous functions $C(G_m)$. The corresponding norm is denoted by $\|\cdot\|_X$, and the corresponding moduli of continuity are defined by

$$\omega(1/M_n,f)_X=\sup_{h\in I_n}\|f(\cdot+h)-f(\cdot)\|_X\,.$$

The modulus of continuity in $H_p(G_m)$ $(0<p\leq 1)$ can be defined as follows:

$$\omega(1/M_n,f)_{H_p(G_m)}:=\|f-S_{M_n}f\|_{H_p(G_m)}\,.$$

It is easy to show that for $f\in L_1(G_m)$ the sequence $(S_{M_n}(f):n\in\mathbb{N})$ is a martingale.

If $f=(f_n,n\in\mathbb{N})$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner, namely:

$$\widehat{f}(i):=\lim_{k\to\infty}\int_{G_m}f_k(x)\,\overline{\Psi}_i(x)\,d\mu(x)\,.$$

The Vilenkin-Fourier coefficients of a function $f\in L_1(G_m)$ are the same as the martingale $(S_{M_n}(f):n\in\mathbb{N})$ obtained from f.

For a martingale f we consider the maximal operators:

$$S^*f\ :\ =\sup_{n\in\mathbb{N}}|S_nf|,$$

$$\widetilde{S}_p^*f\ :\ =\sup_{n\in\mathbb{N}}\frac{|S_nf|}{(n+1)^{1/p-1}\log^{[p]}(n+1)},\ 0<p\leq 1,$$

where $[p]$ denotes the integer part of p.

3. Main results

In this section we state the main results of this paper.

Theorem 3.1. *The following assertions hold:*

a) *Let $0 < p \leq 1$. Then the maximal operator $\widetilde{S}_p^*$ is bounded from the Hardy space $H_p(G_m)$ to the space $L_p(G_m)$.*

b) *Let $0 < p \leq 1$ and $\varphi : \mathbb{N}_+ \to [1, \infty)$ be a nondecreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}} \frac{(n+1)^{1/p-1} \log^{[p]}(n+1)}{\varphi(n)} = +\infty. \tag{3.1}$$

Then

$$\sup_{n\in\mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_{L_{p,\infty}(G_m)} = \infty \quad \text{for} \quad 0 < p < 1$$

and

$$\sup_{n\in\mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_1 = \infty.$$

We easily infer the following result, which first was established by P. Simon [15].

Corollary 3.1. *Let $0 < p < 1$ and $f \in H_p(G_m)$. Then there is a constant c_p, depending only on p, such that*

$$\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p.$$

Theorem 3.2. *Let $0 < p \leq 1$, $f \in H_p(G_m)$ and $M_k < n \leq M_{k+1}$. Then there is a constant c_p, depending only on p, such that*

$$\|S_n(f) - f\|_{H_p(G_m)} \leq c_p n^{1/p-1} \lg^{[p]} n \omega \left(\frac{1}{M_k}, f \right)_{H_p(G_m)}.$$

Theorem 3.3. *The following assertions hold:*

a) *Let $0 < p < 1$, $f \in H_p(G_m)$ and*

$$\omega \left(\frac{1}{M_n}, f \right)_{H_p(G_m)} = o \left(\frac{1}{M_n^{1/p-1}} \right) \text{ as } n \to \infty.$$

Then

$$\|S_k(f) - f\|_{L_{p,\infty}(G_m)} \to 0 \text{ as } k \to \infty.$$

b) *For every $p \in (0, 1)$ there exists a martingale $f \in H_p(G_m)$ for which*

$$\omega \left(\frac{1}{M_{2n}}, f \right)_{H_p(G_m)} = O \left(\frac{1}{M_{2n}^{1/p-1}} \right) \text{ as } n \to \infty$$

and

$$\|S_k(f) - f\|_{L_{p,\infty}(G_m)} \nrightarrow 0 \text{ as } k \to \infty.$$

Theorem 3.4. *The following assertions hold:*

a) *Let $f \in H_1(G_m)$ and*

$$\omega\left(\frac{1}{M_n}, f\right)_{H_1(G_m)} = o\left(\frac{1}{n}\right) \quad as\ n \to \infty.$$

Then

$$\|S_k(f) - f\|_1 \to 0 \quad as\ k \to \infty.$$

b) *There exists a martingale $f \in H_1(G_m)$ for which*

$$\omega\left(\frac{1}{M_{2M_n}}, f\right)_{H_1(G_m)} = O\left(\frac{1}{M_n}\right) \quad as\ n \to \infty$$

and

$$\|S_k(f) - f\|_1 \nrightarrow 0 \quad as\ \ k \to \infty.$$

4. Auxiliary propositions

In this section we state two known lemmas that will be used in the proofs of our main results.

Lemma 4.1. ([22]) *Let T be a sublinear operator such that for some $0 < p \leq 1$ and for every p-atom a*

$$\int_{\bar{I}} |Ta|^p \, d\mu \leq c_p < \infty,$$

where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_p \leq c_p \|f\|_{H_p(G_m)}.$$

Lemma 4.2. [17] *Let $n \in \mathbb{N}$ and $x \in I_s \backslash I_{s+1}$ for $0 \leq s \leq N-1$. Then*

$$\int_{I_N} |D_n(x-t)| \, d\mu(t) \leq \frac{cM_s}{M_N}.$$

5. Proof of the theorems

Proof of Theorem 3.1. We first prove assertion a). To this end, observe that since $\widetilde{S}_p^*$ is bounded from $L_\infty(G_m)$ to $L_\infty(G_m)$, in view of Lemma 1, it is enough to show that for every p-atom a

$$\int_{\bar{I}_N} \left|\widetilde{S}_p^* a(x)\right|^p d\mu(x) \leq c < \infty \quad \text{for} \quad 0 < p \leq 1,$$

where I denotes the support of the atom.

Let a be an arbitrary p-atom with support$I$ and $\mu(I) = M_N$. We may assume that $I = I_N$. It is easy to see that $S_n(a) = 0$ when $n \leq M_N$. Therefore we can assume that $n > M_N$.

Since $\|a\|_\infty \le M_N^{1/p}$ we can write

$$\begin{aligned}|S_n(a)| &\le \int_{I_N} |a(t)|\,|D_n(x-t)|\,d\mu(t)\\ &\le \|a\|_\infty \int_{I_N} |D_n(x-t)|\,d\mu(t) \le M_N^{1/p} \int_{I_N} |D_n(x-t)|\,d\mu(t).\end{aligned}$$

For $0<p<1$ and $x \in I_s \backslash I_{s+1}$, from Lemma 2 we get

$$\frac{|S_n a(x)|}{\log^{[p]}(n+1)(n+1)^{1/p-1}} \le \frac{cM_N^{1/p-1}M_s}{\log^{[p]}(n+1)(n+1)^{1/p-1}}. \tag{5.1}$$

Combining (2.1) and (5.1) we obtain

$$\begin{aligned}\int_{\overline{I_N}} \left|\widetilde{S}_p^* a(x)\right|^p d\mu(x) &= \sum_{s=0}^{N-1} \int_{I_s \backslash I_{s+1}} \left|\widetilde{S}_p^* a(x)\right|^p d\mu(x)\\ &\le \frac{cM_N^{1-p}}{\log^{[p]p}(n+1)(n+1)^{1-p}} \sum_{s=0}^{N-1} \frac{M_s^p}{M_s} \le \frac{cM_N^{1-p}N^{[p]}}{\log^{p[p]}(n+1)(n+1)^{1-p}} < c_p < \infty.\end{aligned} \tag{5.2}$$

This completes the proof of assertion a).

To prove part b) of the theorem, we set

$$f_{n_k}(x) = D_{M_{2n_k+1}}(x) - D_{M_{2n_k}}(x)$$

and observe that

$$\widehat{f}_{n_k}(i) = \begin{cases} 1, & \text{if } i = M_{2n_k}, \dots, M_{2n_k+1}-1 \\ 0, & \text{otherwise.}\end{cases}$$

Hence we can write

$$S_i f_{n_k}(x) = \begin{cases} D_i(x) - D_{M_{2n_k}}(x), & \text{if } i = M_{2n_k}+1, \dots, M_{2n_k+1}-1, \\ f_{n_k}(x), & \text{if } i \ge M_{2n_k+1}, \\ 0, & \text{otherwise.}\end{cases} \tag{5.3}$$

From (2.2) we get

$$\|f_{n_k}\|_{H_p(G_m)} = \left\|\sup_{n\in\mathbb{N}} S_{M_n}(f_{n_k})\right\|_p = \left\|D_{M_{2n_k+1}} - D_{M_{2n_k}}\right\|_p \le c_p M_{2n_k}^{1-1/p}. \tag{5.4}$$

Let $0<p<1$, then under the condition (3.1) there exist positive integers n_k such that

$$\lim_{k\to\infty} \frac{(M_{2n_k}+2)^{1/p-1}}{\varphi(M_{2n_k}+2)} = \infty, \quad 0<p<1.$$

Applying (2.2), (2.3) and (5.3) we can write

$$\frac{\left|S_{M_{2n_k}+1} f_{n_k}\right|}{\varphi(M_{2n_k}+2)} = \frac{\left|D_{M_{2n_k}+1} - D_{M_{2n_k}}\right|}{\varphi(M_{2n_k}+2)} = \frac{\left|w_{M_{2n_k}}\right|}{\varphi(M_{2n_k}+2)} = \frac{1}{\varphi(M_{2n_k}+2)}.$$

This implies

$$\mu\left\{x\in G_m:\frac{\left|S_{M_{2n_k}+1}f_{n_k}(x)\right|}{\varphi(M_{2n_k}+2)}\geq\frac{1}{\varphi(M_{2n_k}+2)}\right\}=1. \tag{5.5}$$

Combining (5.4) and (5.5) we obtain

$$\frac{\frac{1}{\varphi(M_{2n_k}+2)}\left(\mu\left\{x\in G_m:\frac{\left|S_{M_{2n_k}+1}f_{n_k}(x)\right|}{\varphi(M_{2n_k}+2)}\geq\frac{1}{\varphi(M_{2n_k}+2)}\right\}\right)^{1/p}}{\|f_{n_k}(x)\|_{H_p}}$$

$$\geq \frac{1}{\varphi(M_{2n_k}+2)M_{2n_k}^{1-1/p}}=\frac{\left(M_{2n_k}+2\right)^{1/p-1}}{\varphi(M_{2n_k}+2)}\to\infty \text{ as } k\to\infty.$$

Now consider the case $p=1$. Under the condition (3.1) there exists a sequence $\{n_k:k\geq1\}$, such that

$$\lim_{k\to\infty}\frac{\log q_{n_k}}{\varphi(q_{n_k})}=\infty.$$

Let $q_{n_k}=M_{2n_k}+M_{2n_k-2}+M_2+M_0$ and $x\in I_{2s}\backslash I_{2s+1}$, $s=0,...,n_k$. Combining

(2.2) and (2.3) we obtain

$$\left|D_{q_{n_k}}(x)\right|\geq\left|D_{M_{2s}}(x)\right|-\left|\sum_{l=0}^{s-2}r_{2l}^{m_{2l}-1}(x)D_{M_{2l}}(x)\right|$$

$$\geq M_{2s}-\sum_{l=0}^{s-2}M_{2l}\geq M_{2s}-M_{2s-1}\geq\frac{M_{2s}}{2}.$$

Hence

$$\int_{G_m}\left|D_{q_{n_k}}(x)\right|d\mu(x)\geq\frac{1}{2}\sum_{s=0}^{n_k}\int_{I_{2s}\backslash I_{2s+1}}M_{2s}d\mu(x)\geq c\sum_{s=0}^{n_k}1\geq cn_k. \tag{5.6}$$

Finally, by (5.3), (5.4) and (5.6) we have

$$\frac{1}{\|f_{n_k}(x)\|_{H_1(G_m)}}\int_{G_m}\frac{\left|S_{q_{n_k}}f_{n_k}(x)\right|}{\varphi(q_{n_k})}d\mu(x)$$

$$\geq \frac{1}{\|f_{n_k}(x)\|_{H_1(G_m)}}\left(\int_{G_m}\frac{\left|D_{q_{n_k}}(x)\right|}{\varphi(q_{n_k})}d\mu(x)-\int_{G_m}\frac{\left|D_{M_{2n_k}}(x)\right|}{\varphi(q_{n_k})}d\mu(x)\right)$$

$$\geq \frac{c}{\varphi(q_{n_k})}(\log q_{n_k}-1)\geq\frac{c\log q_{n_k}}{\varphi(q_{n_k})}\to\infty \quad \text{as } k\to\infty,$$

and the result follows. This completes the proof of Theorem 3.1.

Proof of Corollary 3.1. Let $0<p<1$. Applying (5.1), (5.2) and Theorem W we have

$$\sum_{k=M_N}^{\infty}\frac{\|S_k a\|_p^p}{k^{2-p}}\leq\sum_{k=M_N}^{\infty}\frac{1}{k}\int_{\overline{I_N}}\left|\frac{S_k a(x)}{k^{1/p-1}}\right|^p d\mu(x)$$
$$+\sum_{k=M_N}^{\infty}\frac{M_N}{k^{2-p}}\int_{I_N}\left(\int_{I_N}|D_k(x-t)|\,d\mu(t)\right)^p d\mu(x)$$
$$\leq\ c_pM_N^{1-p}\sum_{k=M_N}^{\infty}\frac{1}{k^{2-p}}+c_pM_N^{1-p}\sum_{k=M_N}^{\infty}\frac{\log^p k}{k^{2-p}}\leq c_p<\infty.$$

This completes the proof of Corollary 3.1.

Proof of Theorem 3.2. Let $0<p\leq 1$ and $M_k<n\leq M_{k+1}$. Using Theorem 3.1 we get

$$\|S_nf\|_p\leq c_pn^{1/p-1}\log^{[p]}n\,\|f\|_{H_p(G_m)}.$$

Therefore

$$\|S_nf-f\|_p^p\leq\|S_nf-S_{M_k}f\|_p^p+\|S_{M_k}f-f\|_p^p=\|S_n(S_{M_k}f-f)\|_p^p$$
$$+\|S_{M_k}f-f\|_p^p\leq c_p\left(n^{1-p}+1\right)\log^{p[p]}n\omega^p\left(\frac{1}{M_k},f\right)_{H_p(G_m)}$$

and

$$\|S_nf-f\|_p\leq c_pn^{1/p-1}\log^{[p]}n\omega\left(\frac{1}{M_k},f\right)_{H_p(G_m)}. \tag{5.7}$$

Theorem 3.2 is proved.

Proof of Theorem 3.3. Let $0<p<1$, $f\in H_p(G_m)$ and

$$\omega\left(\frac{1}{M_{2n}},f\right)_{H_p(G_m)}=o\left(\frac{1}{M_{2n}^{1/p-1}}\right)\quad\text{as } n\to\infty.$$

It follows from (5.7) that

$$\|S_nf-f\|_p\to\infty\text{ as }n\to\infty,$$

implying assertion a) of the theorem.

To prove part b), we set

$$a_k(x)=\frac{M_{2k}^{1/p-1}}{\lambda}\left(D_{M_{2k+1}}(x)-D_{M_{2k}}(x)\right),$$

where $\lambda=\sup_{n\in\mathbb{N}}m_n$, and

$$f_A(x)=\sum_{i=0}^{A}\frac{\lambda}{M_{2i}^{1/p-1}}a_i(x).$$

Taking into account that

$$S_{M_A}a_k(x)=\begin{cases}a_k(x), & 2k\leq A,\\ 0, & 2k>A,\end{cases}$$

and

$$\text{supp}(a_k) = I_{2k}, \quad \int_{I_{2k}} a_k d\mu = 0, \quad \|a_k\|_\infty \leq M_{2k}^{1/p-1} \cdot M_{2k} = M_{2k}^{1/p} = (\text{supp } a_k)^{-1/p},$$

we can apply Theorem W to conclude that $f \in H_p$.

Next, it is easy to show that

$$\begin{aligned} (5.8) \qquad & f - S_{M_n} f \\ = & \left(f^{(1)} - S_{M_n} f^{(1)}, ..., f^{(n)} - S_{M_n} f^{(n)}, ..., f^{(n+k)} - S_{M_n} f^{(n+k)}\right) \\ = & \left(0, ..., 0, f^{(n+1)} - f^{(n)}, ..., f^{(n+k)} - f^{(n)}, ...\right) \\ = & \left(0, ..., 0, \sum_{i=n}^{k} \frac{a_i(x)}{M_i^{1/p-1}}, ...\right), \quad k \in \mathbf{N}_+ \end{aligned}$$

is a martingale. Hence, using (5.8) we get

$$\omega\left(\frac{1}{M_n}, f\right)_{H_p} \leq \sum_{i=[n/2]+1}^{\infty} \frac{1}{M_{2i}^{1/p-1}} = O\left(\frac{1}{M_n^{1/p-1}}\right),$$

where $[n/2]$ denotes the integer part of $n/2$.

Next, it is easy to show that

$$(5.9) \qquad \widehat{f}(j) = \begin{cases} 1, & \text{if } j \in \{M_{2i}, ..., M_{2i+1} - 1\}, \ i = 0, 1, ... \\ 0, & \text{if } j \notin \bigcup_{i=0}^{\infty} \{M_{2i}, ..., M_{2i+1} - 1\}. \end{cases}$$

Hence, using (5.9) we can write

$$\begin{aligned} & \limsup_{k\to\infty} \|f - S_{M_{2k+1}-1}(f)\|_{L_{p,\infty}(G_m)} \\ \geq & \limsup_{k\to\infty} \left(\|w_{M_{2k+1}-1}\|_{L_{p,\infty}(G_m)} - \|\sum_{i=k+1}^{\infty} \left(D_{M_{2i+1}} - D_{M_{2i}}\right) \|_{L_{p,\infty}(G_m)} \right) \\ \geq & \limsup_{k\to\infty} \left(1 - c/M_{2k}^{1/p-1}\right) c > 0. \end{aligned}$$

This completes the proof of Theorem 3.3.

Proof of Theorem 3.4. The proof of assertion a) is similar to that of part a) of Theorem 3, and is omitted. So, we prove only part b). To this end we set

$$a_i(x) = D_{M_{2M_i+1}}(x) - D_{M_{2M_i}}(x)$$

and

$$f_A(x) = \sum_{i=1}^{A} \frac{a_i(x)}{M_i}.$$

Taking into account that

$$S_{M_A} a_k(x) = \begin{cases} a_k(x), & 2M_k \leq A, \\ 0, & 2M_k > A, \end{cases}$$

and

$$\operatorname{supp}(a_k) = I_{2M_k}, \quad \int_{I_{2M_k}} a_k d\mu = 0, \quad \|a_k\|_\infty \le M_{2M_k} = \mu(\operatorname{supp} a_k),$$

we can apply Theorem W to conclude that $f \in H_1$.

Next, it is easy to show that

$$\omega(\frac{1}{M_n}, f)_{H_1(G_m)} \le \sum_{i=[\lg n/2]}^{\infty} \frac{1}{M_i} = O\left(\frac{1}{n}\right),$$

where $[\lg n/2]$ denotes the integer part of $\lg n/2$. By simple calculation we get

$$\widehat{f}(j) = \begin{cases} \frac{1}{M_{2i}}, & \text{if } j \in \{M_{2M_i}, ..., M_{2M_i+1} - 1\}, \ i = 0, 1, ... \\ 0, & \text{if } j \notin \bigcup_{i=0}^{\infty} \{M_{2M_i}, ..., M_{2M_i+1} - 1\}. \end{cases} \tag{5.10}$$

Finally, combining (5.6) and (5.10) we obtain

$$\begin{aligned} &\limsup_{k\to\infty} \|f - S_{q_{M_k}}(f)\|_1 \\ \ge\ &\limsup_{k\to\infty} \left(\frac{1}{M_{2k}} \|D_{q_{M_k}}\|_1 - \frac{1}{M_{2k}} \|D_{M_{2M_k+1}}\|_1 - \| \sum_{i=k+1}^{\infty} \frac{D_{M_{2M_i+1}} - D_{M_{2M_i}}}{M_{2i}} \|_1 \right) \\ \ge\ &\limsup_{k\to\infty} \left(c - \sum_{i=k+1}^{\infty} \frac{1}{M_{2i}} - \frac{1}{M_{2k}} \right) \ge c > 0, \end{aligned}$$

and the result follows. Theorem 3.4 is proved.

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