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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we consider a class of a second-order impulsive differential inclusions. Using a variational method based on the non-smooth critical point theory, we prove the existence and multiplicity of anti-periodic solutions.

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1. Introduction

The aim of this paper is to establish the existence and multiplicity of solutions for the following second-order impulsive differential inclusion subject to anti-periodic boundary conditions

$$(1.1)\quad \begin{cases} -(\Phi_p(u'(x)))' + M\Phi_p(u(x)) \in \partial F(x, u(x)), & \text{in } [0,T]\backslash\{x_1, x_2, \dots, x_m\}, \\ -\Delta\Phi_p(u'(x_k)) = I_k(u(x_k)), & k = 1, 2, \dots, m, \\ u(0) = -u(T), \quad u'(0) = -u'(T), \end{cases}$$

where $p > 1$, $T > 0$, $M \geq 0$, $\Phi_p(x) := |x|^{p-2}x$, $0 = x_0 < x_1 < \ldots < x_m < x_{m+1} = T$, $\Delta\Phi_p(u'(x_k)) = \Phi_p(u'(x_k^+)) - \Phi_p(u'(x_k^-))$, where $u'(x_k^+)$ and $u'(x_k^-)$ stand for the right and left limits of $u'(x)$ at $x = x_k$, respectively; $I_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \dots, m$, $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for all $t \in [0,T]$, $F(t, \cdot)$ is locally Lipschitz and $\partial F(t, \cdot)$ denotes the generalized subdifferential in the sense of Clarke [1].

In recent years much attention was paid to the question of existence of solutions for impulsive boundary value problems, which have a number of applications in chemotherapy, population dynamics, optimal control, ecology, industrial robotics and physics.

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For the background, theory and applications of impulsive differential equations, we refer to [2]–[12] and references therein.

In [13], Tian and Henderson studied the following equations with impulsive effects:

$$\begin{cases} -(\Phi_p(u'(x)))' + M\Phi_p(u(x)) \in \partial F(u(x)) + \mu G(x, u(x)), & \text{in } [0,T]\backslash\{x_1, x_2, \ldots, x_m\}, \\ -\Delta\Phi_p(u'(x_k)) = I_k(u(x_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = -u(T), \quad u'(0) = -u'(T). \end{cases}$$

In [13] it was proved that under some conditions imposed on F, G and I_j the problem (1) has at least three positive solutions via the three critical point theorem of Ricceri (see [14]).

In this paper, motivated by the above work, we study the question of existence of solutions of the problem (1.1), and using a variational method based on the non-smooth critical point theory, we prove the existence and multiplicity of anti-periodic solutions of problem (1).

The paper is organized as follows. In Section 2 we give some preliminary results and establish a variational principle for the problem (1.1), which are needed in the proofs of the main results. Section 3 is devoted to our main results.

2. Preliminaries

In this section we present some preliminaries, basic notion and results from the theory of non-smooth analysis – the calculus for locally Lipschitz functionals, developed by Clarke [1], which will be used in the proofs of the main results of the paper.

Let $(X, \|\cdot\|_X)$ be a Banach space, $(X^*, \|\cdot\|_{X^*})$ be its topological dual, and let $\varphi : X \to \mathbb{R}$ be a functional. Recall that a functional φ is locally Lipschitz if for all $u \in X$ there exist a neighborhood U of u and a real number $L_U > 0$ such that for all $x, y \in U$

$$|\varphi(x) - \varphi(y)| \leq L_U \|x - y\|_X.$$

If f is locally Lipschitz and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is defined by

$$\varphi^\circ(u; v) = \limsup_{w \to u, t \downarrow 0} \frac{\varphi(w + tv) - \varphi(w)}{t}.$$

The generalized gradient of φ at u is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v\rangle \leq \varphi^\circ(u; v) \text{ for all } v \in X\}.$$

So $\varphi : X \to 2^{X^*}$ is a multifunction. Observe that the function $(u, v) \mapsto \varphi^0(u; v)$ is upper semicontinuous and satisfies

$$\varphi^\circ(u; v) = \max\{\langle \xi, v\rangle : \xi \in \partial\varphi(u)\} \quad \text{for all } v \in X.$$

We say that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*. Also, an element $u \in X$ is said to be a critical point of a locally Lipschitz functional φ if $0 \in \partial\varphi(u)$.

In the proofs of our main results, we will use some facts from the non-smooth critical point theory. To state these results, we first give some definitions.

Definition 2.1. *We say that an operator $A : X \to X^*$ is of type $(S)_+$ if for any sequence $\{u_n\}$ from X the conditions $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$.*

Definition 2.2. *We say that a locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the non-smooth Palais-Smale condition (non-smooth (PS)-condition for short) if any sequence $\{u_n\}_{n\geq1} \subseteq X$ such that $\{J(u_n)\}_{n\geq1}$ is bounded and*

$$\rho(u_n) := \min\{||u^*||_{X^*} : u^* \in \partial\varphi(u_n)\} \to 0 \quad as\ n \to +\infty,$$

has a strongly convergent subsequence.

Definition 2.3. *We say that a locally Lipschitz function $J : X \to \mathbb{R}$ satisfies the non-smooth Cerami condition (non-smooth (C)-condition for short) if any sequence $\{u_n\}_{n\geq1} \subseteq X$ such that $\{J(u_n)\}_{n\geq1}$ is bounded and*

$$(1 + ||u_n||_X)\rho(u_n) \to 0 \quad as\ n \to +\infty,$$

has a strongly convergent subsequence.

For the next result we refer to [15], Proposition 1.1.

Lemma 2.1. *Let $\varphi \in C^1(X)$ be a functional. Then φ is locally Lipschitz and*

$$\varphi^\circ(u; v) = \langle\varphi'(u), v\rangle \quad \text{for all } u, v \in X,$$
$$\partial\varphi(u) = \{\varphi'(u)\} \quad \text{for all } u \in X.$$

Consider the space $X = \{u \in W^{1,p}([0,T]) : \ u(0) = -u(T)\}$ endowed with the norm

$$||u||_X = \left(\int_0^T (|u'(x)|^p + M|u(x)|^p)dx\right)^{\frac{1}{p}}, \quad u \in X.$$

Observe that the norm $||u||_X$ is equivalent to the usual norm: $\left(\int_0^T (|u'(x)|^p + |u(x)|^p)dx\right)^{\frac{1}{p}}$ (see [13], Lemma 3.1). The next lemma, which will be used in the proofs of our main results, was proved in [13] (see [13], Lemma 3.3).

Lemma 2.2. *Let $u \in X$. Then $||u||_{C^0} \leq \frac{1}{2}T^{\frac{1}{q}}||u||_X$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

It is known (see [13], Lemma 3.2) that the space X is reflexive and separable Banach space.

The functional $J : X \to \mathbb{R}$ corresponding to the problem (1.1) is defined by

$$J(u) = \frac{1}{p}\|u\|_X^p - \sum_{i=1}^{m}\int_0^{u(x_i)} I_i(s)ds - \int_0^T F(x,u(x))dx.$$

Now, we are going to establish a variational principle for the problem (1.1). To this end, we impose the following conditions on the non-smooth potential $F(x,u)$ and the real continuous function I_j:

(H1) For all $u \in \mathbb{R}$, the function $x \to F(x,u)$ is measurable.

(H2) For all $x \in [0,T]$, the function $u \to F(x,u)$ is locally Lipschitz and $F(x,0) = 0$.

(H3) There exist $a, b \in L^1([0,T];\mathbb{R})$ and $1 \le r < +\infty$ such that $|u^*| \le a(x) + b(x)|u|^{r-1}$ for all $x \in [0,T]$, $x \in \mathbb{R}$ and $u^* \in \partial F(x,u)$.

(I1) There exist constants $a_j, b_j > 0$ and $\gamma_j \in [0, p-1)$, $j = 1,2,\dots,m$ such that $|I_j(x)| \le a_j + b_j|u|^{\gamma_j}$ for all $x \in \mathbb{R}$ and $j = 1,2,\dots,m$.

Definition 2.4. *We say that $u \in X$ is a weak solution to the problem (1.1) if*

$$-\int_0^T (\Phi_p(u'(x))v'(x) + M\Phi_p(u(x))v(x) - u^*(x)v(x))dx - \sum_{i=1}^{m} I_i(u(x_i))v(x_i) = 0,$$

for all $u^* \in \partial F(x,u(x))$, $v \in X$ and for a.e. $x \in [0,T]$.

Proposition 2.1. *Assume that the potential $F(x,u)$ satisfies the conditions (H1)-(H3). Then the functional $J : X \to \mathbb{R}$ is well defined and is locally Lipschitz on X. Moreover, every critical point $u \in X$ of J is a solution to the problem (1.1).*

Proof. The proof is similar to that of Lemmas 3.5 and 4.4 from [13], and so is omitted.

Observe that, according to the Proposition 2.1, in order to find solutions of the problem (1.1), it suffices to obtain the critical points of the functional J.

3. Main results

In this section we state and prove our main results. We first establish some existence and multiplicity results for the problem (1.1), by using results from critical point theory. Our first result is as follows.

Theorem 3.1. *Assume that the conditions (H1)-(H3) and (I1) are fulfilled, and also the potential $F(x,u)$ satisfies the following conditions:*

(H4) *There exist $\mu \in (0, \frac{1}{p})$, $c_0 > 0$ and $M > 0$ such that $c_0 < F(x,u) \le -\mu F^\circ(x,u;-u)$ for all $u \in \mathbb{R}$ with $|u| \ge M$ and $x \in [0,T]$.*

(H5) $\lim_{|u|\to 0} \frac{u^*}{|u|^p} = 0$ *uniformly for all $x \in [0,T]$ and all $u^* \in \partial F(x,u)$.*

Then, the problem (1.1) has at least one nonzero solution on X.

Proof. First, we claim that J satisfies the non-smooth (PS)-condition.

By the conditions (II2), (II3) and the Lebourg's mean value theorem, for all $|u| \le M$ and $x \in [0,T]$ we have

$$|F(x,u)| = |F(x,u) - F(x,0)| = |(u^*, u)| \le a(x)|u| + b(x)|u|^r \le Ma(x) + M^r b(x).$$

Next, by the property of generalized directional derivative of a locally Lipschitz function, for all $|u| \le M$ and $x \in [0,T]$ we get

$$|F^\circ(x,u;-u)| = |\max\{(u^*, -u) : u^* \in \partial F(x,u)\}| \le a(x)|u| + b(x)|u|^r \le Ma(x) + M^r b(x).$$

Thus, for all $u \in \mathbb{R}$ with $|u| \le M$ and $x \in [0,T]$ we have

$$F(x,u) + \mu F^\circ(x,u;-u) \le a_1(x), \tag{3.1}$$

where $a_1(x) \in L^1([0,T],\mathbb{R})$.

Suppose $\{u_n\} \subset X$ satisfies

$$|J(u_n)| \le C \quad \text{and} \quad \rho(u_n) \to 0. \tag{3.2}$$

Since $\partial J(u_n) \subset X^*$ is a weak* compact set and the norm function in a Banach space is weakly semi-continuous, by the Weierstrass theorem we can find $u_n^* \in \partial J(u_n)$ to satisfy

$$\rho(u_n) = \|u_n^*\|_{X^*} \quad \text{and} \quad u_n^* = Au_n - v_n \quad \text{for every } n \ge 1, \tag{3.3}$$

with $v_n \in L^{q'}([0,T],\mathbb{R})$, $\frac{1}{q} + \frac{1}{q'} = 1$, and $v_n \in \partial F(x, u_n(x))$ for all $x \in [0,T]$. Here $A : X \to X^*$ is an operator defined by

$$\langle Au_n, v\rangle = \int_0^T [\Phi_p(u_n'(x))v'(x) + M\Phi_p(u_n(x))v(x)]dx - \sum_{i=1}^m I_i(u_n(x_i))v(x_i), \quad \forall\, v \in X.$$

By the condition (I1), when $u(x_j) < 0$, we get

$$a_j u(x_j) + \frac{b_j(-1)^{\gamma_j}}{\gamma_j + 1} u^{\gamma_j+1}(x_j) \le \int_{u(x_j)}^0 I_j(s)ds \le -a_j u(x_j) - \frac{b_j(-1)^{\gamma_j}}{\gamma_j+1} u^{\gamma_j+1}(x_j),$$
$$I_j(u(x_j))u(x_j) \ge (a_j + b_j|u(x_j)|^{\gamma_j})u(x_j) = -a_j(-u(x_j)) - b_j(-u(x_j))^{\gamma_j+1}.$$

When $u_j(t) \ge 0$, we obtain

$$-a_j u(x_j) - \frac{b_j}{\gamma_j+1} u^{\gamma_j+1}(x_j) \le \int_0^{u(x_j)} I_j(s)ds \le a_j u(x_j) + \frac{b_j}{\gamma_j+1} u^{\gamma_j+1}(x_j),$$
$$I_j(u(x_j))u(x_j) \ge (-a_j - b_j|u(x_j)|^{\gamma_j})u(x_j) = -a_j u(x_j) - b_j(u(x_j))^{\gamma_j+1}.$$

Thus, we have

$$\left|\int_0^{u(x_j)} I_j(s)ds\right| \le a_j|u(x_j)| + \frac{b_j}{\gamma_j+1}|u(x_j)|^{\gamma_j+1}, \tag{3.4}$$
$$I_j(u(x_j))u(x_j) \ge -a_j|u(x_j)| - b_j|u(x_j)|^{\gamma_j+1}.$$

Therefore, by Lemma 2.2 we get

$$\left|\sum_{j=1}^{m}\int_{0}^{u(x_j)} I_j(s)ds\right| \leq \sum_{j=1}^{m}\left(a_j\frac{1}{2}T^{\frac{1}{q}}\|u\|_X + \frac{b_j}{\gamma_j+1}\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_j+1}\|u\|_X^{\gamma_j+1}\right), \tag{3.5}$$

$$I_j(u(x_j))u(x_j) \geq -a_j\frac{1}{2}T^{\frac{1}{q}}\|u\|_X - b_j\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_j+1}\|u\|_X^{\gamma_j+1}. \tag{3.6}$$

Thus, in view of condition (H4) and the formulas (3.1), (3.2), (3.5) and (3.6), we can write

$$\begin{aligned}
C+ \quad & \mu\|u_n\|_X \geq J(u_n) - \mu\langle u_n^*, u_n\rangle \\
= \quad & \left(\frac{1}{p}-\mu\right)\|u\|_X - \int_0^T F(x,u_n(x))dx - \mu\langle v_n, -u_n\rangle \\
& -\sum_{i=1}^{m}\int_0^{u_n(x_i)} I_i(s)ds + \mu\sum_{i=1}^{m} I_i(u_n(x_i))u_n(x_i) \\
\geq \quad & \left(\frac{1}{p}-\mu\right)\|u\|_X - \int_{\{|u_n|\leq M\}}(F(x,u_n(x)) + \mu F^{\circ}(x,u_n(x);-u_n(x)))dx \\
& -\int_{\{|u_n|\geq M\}}(F(x,u_n(x)) + \mu F^{\circ}(x,u_n(x);-u_n(x)))dx \\
& -\sum_{i=1}^{m}\left(a_i\frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + \frac{b_j}{\gamma_i+1}\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right) \\
& -\mu\sum_{i=1}^{m}\left(a_i\frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + b_i\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right) \\
= \quad & \left(\frac{1}{p}-\mu\right)\|u\|_X - C_1 - \sum_{i=1}^{m}\left(a_i\frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + \frac{b_j}{\gamma_i+1}\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right) \\
& -\mu\sum_{i=1}^{m}\left(a_i\frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + b_i\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right),
\end{aligned}$$

where C_1 is a constant.

Thus, the sequence $\{u_n\}$ in X is bounded, and hence by passing to a subsequence if necessary and using Sobolev's embedding theorem, we can assume that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } X, \\ u_n \to u & \text{a.e. in } C^1([0,T]), \\ u_n \to u & \text{a.e. in } L^p([0,T]). \end{cases} \tag{3.7}$$

Now, we prove the following fact

$$\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0.$$

To this end, first observe that by (3.2) and (3.3), we have with $\epsilon_n \downarrow 0$

$$\epsilon_n\|u_n - u\| \geq \langle u_n^*, u_n - u\rangle = \langle Au_n, u_n - u\rangle - \int_0^T v_n(x)(u_n(x) - u(x))dx$$

Next, by (3.7) and the Hölder inequality we get

$$\int_0^T v_n(x)(u_n(x) - u(x))dx \to 0 \quad \text{as } n \to +\infty.$$

So, $\limsup_{n\to+\infty}\langle Au_n, u_n - u\rangle \leq 0$, and thus (3.8) is fulfilled. Since A is of type $(S)_+$ (see [13], Lemma 4.2), we obtain $u_n \to u$ in X.

Further, by conditions (H2), (H3), (H5) and the Lebourg's mean value theorem, for all $x \in [0,T]$ and $u \in \mathbb{R}$, we obtain

$$|F(x,u)| \leq \epsilon|u|^p + a_2(x)|u|^\xi, \tag{3.8}$$

where $\xi > p$, $\epsilon > 0$ is an arbitrary real number and $a_2 \in L^1([0,T], \mathbb{R}^+)$.

Therefore, by Lemma 2.2 and formulas (3.5), (3.8), we can write

$$\begin{aligned}
J(u) &= \frac{1}{p}\|u\|_X^p - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds - \int_0^T F(x,u(x))dx \\
&\geq \frac{1}{p}\|u\|_X^p - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds - \epsilon\int_0^T |u(x)|^p dx + \int_0^T a_2(x)|u(x)|^\xi dx \\
&\geq \frac{1}{p}\|u\|_X^p - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds - \epsilon\|u\|_\infty^p + \|u\|_\infty^\xi \int_0^T a_2(x)dx \\
&\geq \frac{1}{p}\|u\|_X^p - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds - \epsilon\Big(\frac{1}{2}T^{\frac{1}{q}}\Big)^p\|u\|_X^p + \Big(\frac{1}{2}T^{\frac{1}{q}}\Big)^\xi \int_0^T a_2(x)dx\|u\|_X^\xi \\
&\geq \left(\frac{1}{p} - \Big(\frac{1}{2}T^{\frac{1}{q}}\Big)^p\right)\|u\|_X^p
\end{aligned}$$

$$-\sum_{i=1}^m\left(a_i\frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + \frac{b_j}{\gamma_i+1}\Big(\frac{1}{2}T^{\frac{1}{q}}\Big)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right) + \Big(\frac{1}{2}T^{\frac{1}{q}}\Big)^\xi \int_0^T a_2(x)dx\|u\|_X^\xi.$$

Hence, we can find $R > 0$ and $\delta > 0$ such that

$$J(u) \geq \delta \quad \text{for all } u \in X \text{ with } \|u\|_X = R. \tag{3.9}$$

Now we claim that

$$J(tu) \to -\infty \quad \text{as} \quad t \to +\infty. \tag{3.10}$$

To prove this, let $\mathcal{N}$ be a Lebesgue-null set outside of which the hypotheses (H3) and (H4) hold, and let $x \in [0,T]\backslash\mathcal{N}$, $u \in \mathbb{R}$ with $|u| \geq M$. We set $\mathcal{J}(x,\lambda) = F(x,\lambda u)$, $\lambda \in \mathbb{R}$, and observe that $\mathcal{J}(x,\cdot)$ is locally Lipschitz. By Rademacher's theorem, we find that for every $x \in [0,T]$ the function $\lambda \to \mathcal{J}(x,\lambda)$ is differentiable a.e. on $\mathbb{R}$, and at a point of differentiability $\lambda \in \mathbb{R}$ we have $\frac{d}{d\lambda}\mathcal{J}(x,\lambda) \in \partial\mathcal{J}(x,\lambda)$. Moreover, by the chain rule we have $\partial\mathfrak{J}(x,\lambda) \subset (\partial_u F(x,\lambda u), u)_{\mathbb{R}}$, implying $\partial\mathfrak{J}(x,\lambda) \subset (\partial_u F(x,\lambda u), \lambda u)_{\mathbb{R}}$. Next, from (H4) we infer

$$\lambda\frac{d}{d\lambda}\mathfrak{J}(x,\lambda) \geq \frac{1}{\mu}\mathfrak{J}(x,\lambda) \Longrightarrow \frac{\frac{d}{d\lambda}\mathfrak{J}(x,\lambda)}{\mathfrak{J}(x,\lambda)} \geq \frac{1}{\lambda\mu}.$$

Integrating the above inequality from 1 to λ_0 we get $\ln\frac{\mathfrak{J}(x,\lambda_0)}{\mathfrak{J}(x,1)} \geq \ln\lambda_0^{\frac{1}{\mu}}$. So, we have proved that $\lambda_0^{-\frac{1}{\mu}}F(x,\lambda u) \geq \lambda_0^{\frac{1}{\mu}}F(x,u)$ for $x \in [0,T]\backslash\mathcal{N}$, $|u| \geq M$ and $\lambda \geq 1$.

Let $z(x) = \min\{F(x,u) : |u| = M\}$. Clearly we have $z \in L^p([0,T],\mathbb{R}^+)$ and $z(x) \geq c_0$ for every $x \in [0,T]$. Therefore, for every $x \in [0,T]\backslash\mathcal{N}$ and $|u| \geq M$ we have

$$\text{(3.11)} \quad F(x,u) = F(x,|u|M^{-1}Mu|u|^{-1}) \geq \Big(\frac{|u|}{M}\Big)^{\frac{1}{\mu}}F\Big(x,\frac{u}{|u|}M\Big) \geq z(x)\Big(\frac{|u|}{M}\Big)^{\frac{1}{\mu}}.$$

On the other hand, in view of the equivalence between two norms in the finite-dimensional spaces, for any finite-dimensional subspace $U \subset X$ and any $u \in U$, there exists a constant $C > 0$ such that

$$||u||_\delta = \left(\int_a^b |u(x)|^\delta dx\right)^{\frac{1}{\delta}} \geq C||u||_X, \quad \delta \geq 1.$$

Then, by (3.5) and (3.11), there exists a positive constant C_1 such that

$$J(u) \leq \frac{1}{p}||u||_X^p + \sum_{i=1}^m\left(a_i\frac{1}{2}T^{\frac{1}{q}}||u||_X + \frac{b_j}{\gamma_i+1}\Big(\frac{1}{2}T^{\frac{1}{q}}\Big)^{\gamma_i+1}||u||_X^{\gamma_i+1}\right) - \int_0^T z(x)\Big(\frac{|u(x)|}{M}\Big)^{\frac{1}{\mu}}dx$$

$$\leq \frac{1}{p}||u||_X^p + \sum_{i=1}^m\left(a_i\frac{1}{2}T^{\frac{1}{q}}||u||_X + \frac{b_j}{\gamma_i+1}\Big(\frac{1}{2}T^{\frac{1}{q}}\Big)^{\gamma_i+1}||u||_X^{\gamma_i+1}\right) - c_0\Big(\frac{1}{M}\Big)^{\frac{1}{\mu}}||u||_{\frac{1}{\mu}}^{\frac{1}{\mu}}$$

$$\leq \frac{1}{p}||u||_X^p + \sum_{i=1}^m\left(a_i\frac{1}{2}T^{\frac{1}{q}}||u||_X + \frac{b_j}{\gamma_i+1}\Big(\frac{1}{2}T^{\frac{1}{q}}\Big)^{\gamma_i+1}||u||_X^{\gamma_i+1}\right) - c_0C_1\Big(\frac{1}{M}\Big)^{\frac{1}{\mu}}||u||_X^{\frac{1}{\mu}}.$$

Since $\mu < \frac{1}{p}$ and $\gamma_j < p-1$, for any $u \in X\backslash\{0\}$ we have $J(tu) \to -\infty$ as $t \to +\infty$, implying the desired claim (3.10).

Finally, for large $t_0 > 0$ we have $J(t_0u) < 0$ with fixed $u \in X\backslash\{0\}$. Hence, observing that $J(0) = 0$, in view of formula (3.9) and the non-smooth mountain pass theorem (see [16, 17]), we obtain $u \in X$, $u \neq 0$ such that $0 \in \partial J(u)$. An application of Proposition 2.1 completes the proof. Theorem 1 is proved.

In the following result we replace the condition (H4) by conditions (H6)-(H8).

Theorem 3.2. *Assume that the conditions (H1)-(H3) and (I1) are fulfilled, and there exist two positive constants β, γ with $\gamma > p$ and $\beta > \gamma - p$, such that $F(x,u)$ and I_i $(i = 1,2,\ldots,m)$ satisfy the following conditions:*

(H6) $\lim_{|u|\to+\infty}\frac{F(x,u}{|u|^p} = +\infty$ *uniformly for all* $x \in [0,T]$.

(H7) $\limsup_{|u|\to+\infty}\frac{F(x,u)}{|u|^\gamma} \leq M < +\infty$ *uniformly for some* $M > 0$ *and all* $x \in [0,T]$.

(H8) $\liminf_{|u|\to+\infty} \frac{pF(x,u+F^{\circ}(x,u;-u)}{|u|^{\beta}} > 0$ *uniformly for all* $x \in [0,T]$.

(I2) I_i $(i = 1, 2, \ldots, m)$ *are odd and nondecreasing.*

Then, the problem (1.1) has at least one nonzero solution on X.

Proof. We first prove that J satisfies the non-smooth (C)-condition (see Definition 2.3). Let $\{u_n\}_{n\geq1} \subseteq X$ be such that $\{J(u_n)\}_{n\geq1}$ is bounded and $(1+\|u_n\|_X)\rho(u_n) \to 0$ as $n \to +\infty$. Then, there exists $C > 0$ to satisfy

$$|J(u_n)| \leq C \quad \text{and} \quad (1+\|u_n\|_X)\rho(u_n) \leq C \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

By condition (H7), there exist $\varrho_1 > 0$ and $\delta_1 > 0$ such that $F(x,u) \leq \varrho_1|u|^{\gamma}$ for all $|u| \geq \delta_1$ and $x \in [0,T]$. It follows from the above inequality, the conditions (H1), (H2), and the Lebourg's mean value theorem that $|F(x,u)| \leq \overline{a}_2(x)$ for all $|u| \leq \delta_1$ and $x \in [0,T]$, where $\overline{a}_2(x) \in L^1([0,T],\mathbb{R}^+)$. Therefore

$$|F(x,u)| \leq \varrho_1|u|^{\gamma} + \overline{a}_2(x) \quad \text{for all } u \in \mathbb{R},\ x \in [0,T]. \tag{3.13}$$

Also, by condition (H8) there exist $\varrho_2 > 0$ and $\delta_2 > 0$ such that $pF(x,u)+F^{\circ}(x,u;-u) \geq \varrho_2|u|^{\beta}$ for all $|u| \geq \delta_2$ and $x \in [0,T]$. By arguments similar to those used in the derivation of (3.1), we obtain $|pF(x,u) + F^{\circ}(x,u;-u)| \geq \overline{a}_3(x)$ for all $|u| \leq \delta_2$ and $x \in [0,T]$. Thus, for all $u \in \mathbb{R}$ and $x \in [0,T]$ we can write

$$pF(x,u) + F^{\circ}(x,u;-u) \geq \varrho_2|u|^{\beta} - \varrho_1\delta_2^{\beta} - \overline{a}_3(x), \tag{3.14}$$

where $\overline{a}_3(x) \in L^1([0,T],\mathbb{R}^+)$, Therefore, by (3.4), (3.12) and (3.13) we get

$$\begin{aligned} C \geq |J(u_n)| \ &\geq\ \frac{1}{p}\|u_n\|_X^p - \sum_{i=1}^{m}\left(a_i\|u_n\|_\infty + \frac{b_j}{\gamma_i+1}\|u_n\|_\infty^{\gamma_i+1}\right) \\ &\quad -\varrho_1\int_0^T |u_n(x)|^{\gamma}dx - \int_0^T \overline{a}_2(x)dx. \end{aligned}$$

Thus, we have

$$\begin{aligned} \frac{1}{p}\|u_n\|_X^p \ &\leq\ \sum_{i=1}^{m}\left(a_i\|u_n\|_\infty + \frac{b_j}{\gamma_i+1}\|u_n\|_\infty^{\gamma_i+1}\right) \\ &\quad +\varrho_1\int_0^T |u_n(x)|^{\gamma}dx + \int_0^T \overline{a}_2(x)dx + C. \end{aligned} \tag{3.15}$$

From (3.12) and (3.14) we obtain

$$\begin{aligned}(p+1)C &\geq pJ(u_n) - \langle u_n^*, u_n\rangle \\ &\geq \varrho_2\int_0^T |u_n(x)|^\beta dx - \int_0^T \overline{a}_3(x)dx - \varrho_1\delta_2^\beta \\ &\quad -p\sum_{i=1}^m\int_0^{u_n(x_i)} I_i(s)ds + \sum_{i=1}^m I_i(u_n(x_i))u_n(x_i) \\ &\geq \varrho_2\int_0^T |u_n(x)|^\beta dx - \int_0^T \overline{a}_3(x)dx - \varrho_1\delta_2^\beta \\ &\quad -p\sum_{i=1}^m\left(a_i||u_n||_\infty + \frac{b_j}{\gamma_i+1}||u_n||_\infty^{\gamma_i+1}\right) \\ &\quad -\sum_{i=1}^m\left(a_i||u_n||_\infty + b_i||u_n||_\infty^{\gamma_i+1}\right),\end{aligned}$$

where $u_n^* \in \partial J(u_n)$ and $v_n \in \partial F(x, u_n)$. Therefore the sequence $\{u_n\}$ is bounded both in $L^\beta([0,T],\mathbb{R})$ and $L^\infty([0,T],\mathbb{R})$.

Since $\gamma > p$ and $\beta > \gamma - p$, then assuming $\gamma \leq \beta$ and using HölderŸs inequality we have $\int_0^T |u_n(x)|^\gamma dx \leq \left(\int_0^T |u_n(x)|^\beta dx\right)^{\frac{\gamma}{\beta}}$, which together with (3.15) implies that $\{u_n\}$ is bounded in X. If $\beta < \gamma$, then by Lemma 2.2 we get

$$\int_0^T |u_n(x)|^\gamma dx \leq ||u_n||_\infty^{\gamma-\beta}\int_0^T |u_n(x)|^\beta dx \leq \left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma-\beta}||u_n||_X^{\gamma-\beta}\int_0^T |u_n(x)|^\beta dx.$$

Hence, taking into account (3.15), we conclude that $\{u_n\}$ is bounded in X. By arguments similar to those used in the proof of Theorem 3.1 we infer that $\{u_n\}$ strongly converges in X.

Also, by conditions (H3) and (H5) we can find $R > 0$ and $\delta > 0$ to satisfy

$$J(u) \geq \delta \quad \text{for all } u \in X \text{ with } ||u|| = R. \tag{3.16}$$

Next, we prove that there exists $u_0 \in X$ such that $J(u_0) < 0$. To this end, observe first that by condition (H6), for

$$\varrho_3 = \frac{p+1}{p}\cdot\frac{T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}}{\left(\frac{T}{2}\right)^{p+1}} > 0$$

there exists $\delta_3 > 0$ such that

$$F(x,u) \geq \varrho_3|u|^p \quad \text{for all } |u| \geq \delta_3,\ x \in [0,T].$$

It follows from (H2), (H3) and the Lebourg's mean value theorem that

$$F(x,u) \geq \varrho_3|u|^p - \varrho_3\delta_3^p - \overline{a}_4(x), \qquad \text{for all } u \in \mathbb{R},\ x \in [0,T], \tag{3.17}$$

where $\overline{a}_4(x) \in L^1([0,T],\mathbb{R}^+)$. Therefore, by (3.17), (H3) and Lemma 2.2, we choose $u_0(x) = \frac{T}{2} - x$ and observe that $u_0 \in X$.

Since by (I2) the functions I_i $(i = 1, 2, \ldots, m)$ are odd and nondecreasing, the functions $\int_0^\zeta I_i(s)ds$ are even and satisfy $\int_0^\zeta I_i(s)ds \geq 0$ for any $\zeta \geq 0$. Hence

$$\sum_{i=1}^{m} \int_0^{u_0(x_i)} I_i(s)ds \geq 0.$$

So, in view of (3.17) we can write

$$\begin{aligned} J(su_0) &= \frac{s^p}{p}\int_0^T \left(1 + M\left|\frac{T}{2} - x\right|^p\right) dx - \sum_{i=1}^{m}\int_0^{u_0(x_i)} I_i(s)ds - \int_0^T F(x, su_0(x))dx \\ &\leq \frac{s^p}{p}\int_0^T \left(1 + M\left|\frac{T}{2} - x\right|^p\right) dx - \int_0^T F(x, su_0(x))dx \\ &\leq \frac{s^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - s^p\varrho_3\int_0^T |u_0(x)|^p dx + C_1 \\ &\leq s^p\left[\frac{1}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \varrho_3\frac{2}{p+1}\left(\frac{T}{2}\right)^{p+1}\right], \end{aligned}$$

where $C_1 := \varrho_3\delta_3^pT + \int_0^T \overline{a}_4(x)dx$ is a positive constant. Taking into account that

$$\frac{1}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \varrho_3\frac{2}{p+1}\left(\frac{T}{2}\right)^{p+1} = -\frac{1}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) < 0,$$

we conclude that there exists a large enough $s_0 > 0$ such that $J(s_0u_0) < 0$.

Finally, observing that $J(0) = 0$, we use the formula (3.16), the non-smooth mountain pass theorem (under the non-smooth (C)-condition (see [17])) and Proposition 2.1, to complete the proof. Theorem 2 is proved. □

Theorem 3.3. *Assume that the conditions (H1)-(H5), (I1) and (I2) are fulfilled, and the potential $F(x, u)$ satisfies the following condition:*
(H9) $F(x, u) = F(x, -u)$ for all $x \in [0, T]$, $u \in \mathbb{R}$.
Then the problem (1.1) has an unbounded sequence of solutions $\{u_n\} \subset X$ such that $||u_n||_\infty \to \infty$ as $n \to \infty$.

Proof. It follows from the conditions (I2) and (H9) that J is even. Hence using the arguments of the proof of Theorem 3.1 and the non-smooth symmetric mountain pass theorem [16], we conclude that J possesses an unbounded sequence of critical values $\{c_n\}$ satisfying $J(u_n) = c_n$, where $0 \in \partial J(u_n)$ for $n = 1, 2, \ldots$.

Since $0 \in \partial J(u_n)$, by (3.3) we get

$$||u_n||_x^p - \sum_{i=1}^{m} I_i(u_n(x_i))u_n(x_i) - \int_0^T (v_n(x), u_n(x))dx = 0, \tag{3.18}$$

where $v_n \in \partial F(x, u_n)$.

Next, in view of (3.5), (3.6), (3.11), (3.18), (H5) and (H9), we can write

$$\frac{1}{p}||u_n||_X^p = \frac{p+1}{p}||u_n||_X^p - \sum_{i=1}^{m} I_i(u_n(x_i))u_n(x_i) - \int_0^T (v_n(x), u_n(x))dx \geq$$

$$\geq (p+1)c_n + \int_0^T ((p+1)F(x,u_n(x)) - F^\circ(x,u_n(x);u_n(x)))dx +$$

$$+(p+1)\sum_{i=1}^m \int_0^{u_n(x_i)} I_i(s)ds - \sum_{i=1}^m I_i(u_n(x_i))u_n(x_i) \geq$$

$$\geq (p+1)c_n + \left((p+1) + \frac{1}{\mu}\right)\int_0^1 (F(x,u_n(x)) + F(x,-u_n(x))dx -$$

$$-(p+1)\sum_{i=1}^m \left(a_i \frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + \frac{b_j}{\gamma_i+1}\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right) -$$

$$-\sum_{i=1}^m \left(a_i \frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + b_i\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right) \geq$$

$$\geq (p+1)c_n + \left((p+1) + \frac{1}{\mu}\right)\int_{\{|u_n(x)|\leq M\}} F(x,u_n(x)dx$$

$$-(p+1)\sum_{i=1}^m \left(a_i \frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + \frac{b_j}{\gamma_i+1}\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right)$$

$$-\sum_{i=1}^m \left(a_i \frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + b_i\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right).$$

Thus, we have

$$\frac{1}{p}\|u_n\|_X^p + (p+1)\sum_{i=1}^m \left(a_i \frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + \frac{b_j}{\gamma_i+1}\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right)$$

$$+\sum_{i=1}^m \left(a_i \frac{1}{2}T^{\frac{1}{q}}\|u_n\|_X + b_i\left(\frac{1}{2}T^{\frac{1}{q}}\right)^{\gamma_i+1}\|u_n\|_X^{\gamma_i+1}\right)$$

$$\geq (p+1)c_n + \left((p+1) + \frac{1}{\mu}\right)\int_{\{|u_n(x)|\leq M\}} F(x,u_n(x)dx.$$

Since $c_n \to +\infty$ as $n \to +\infty$, it follows from the above inequality that $\|u_n\|_X \to +\infty$ as $n \to +\infty$. Also, with some constant C'', we have

$$(p+1)c_n \leq \frac{1}{p}\|u_n\|_X^p + C''. \tag{3.19}$$

Hence, by the condition (H3) and formulas (3.4), (3.19), we can write

$$pc_n \leq \frac{1}{p}\|u_n\|_X^p - c_n + C'' = \sum_{i=1}^m \int_0^{u_n(x_i)} I_i(s)ds + \int_0^T F(x,u_n(x)\,dx + C''$$

$$= \sum_{i=1}^m \int_0^{u_n(x_i)} I_i(s)ds + \int_0^T (u_n^*(x), u_n(x)dx + C''$$

$$\leq \sum_{i=1}^m \left(a_i\|u_n\|_\infty + \frac{b_j}{\gamma_i+1}\|u_n\|_\infty^{\gamma_i+1}\right) + \|u_n\|_\infty \int_0^1 a(x)dx + \|u_n\|_\infty^q \int_0^T b(x)dx + C'',$$

where $u_n^* \in \partial F(x, su_n)$ with $s \in (0,1)$. Thus, we have $||u_n||_\infty \to +\infty$ as $n \to +\infty$, and the result follows. Theorem 3 is proved.

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REFERENCES

[1] F. Clarke, Optimization and Nonsmooth Analysis, John Wiley and Sons, New York (1983).
[2] R. P. Agarwal, D. Franco, D. O'Regan, Singular Boundary Value Problems for First and Second Order Impulsive Differential Equations, World Scientific, Singapore (1989).
[3] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, Theory of Impulsive Differential Equations, Aequationes Math., **69**, 83 – 96 (2005).
[4] J. J. Nieto, R. Rodriguez-Lopez, "Boundary value problems for a class of impulsive functional equations", Comput. Math. Appl., **55**, 2715 – 2731 (2008).
[5] J. Li, J.J. Nieto, J. Shen, "Impulsive periodic boundary value problems of first-order differential equations", J. Math. Anal. Appl., **325**, 226 – 299 (2007).
[6] J. J. Nieto, R. Rodriguez-Lopez, "New comparison results for impulsive integro-differential equations and applications", J. Math. Anal. Appl., **328**, 1343 – 1368 (2007).
[7] A. M. Samoilenko, N. A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore (1995).
[8] H. Zhang, L. Chen, J. J. Nieto, "A delayed epidemic model with stage structure and pulses for management strategy", Nonlinear Anal. RWA, **9**, 1714 – 1726 (2008).
[9] N. Zhang, B. Dai, X. Qian, "Periodic solutions for a class of higher-dimension functional differential equations with impulses", Nonlinear Anal. TMA 68, 629 -Џ 638 (2008).
[10] M. Benchohra, J. Henderson, S.K. Ntouyas, Impulsive Differential Equations and Inclusions, vol. 2, Hindawi Publishing Corporation, New York (2006).
[11] G. Zeng, F. Wang, J.J. Nieto, "Complexity of a delayed predator-prey model with impulsive harvest and Holling-type II functional response", Adv. Complex Syst., **11**, 77 – 97 (2008).
[12] X. Meng, Z. Li, J.J. Nieto, "Dynamic analysis of Michaelis-Menten chemostat-type competition models with time delay and pulse in a polluted environment", J. Math. Chem., **47**, 123 – 144 (2009).
[13] Y. Tian, J. Henderson, "Three anti-periodic solutions for second-order impulsive differential inclusions via nonsmooth critical point theory", Nonlinear Anal., **75**, 6496 – 6505 (2012).
[14] B. Ricceri, "Existence of three solutions for a class of elliptic eigenvalue problems", Math. Comput. Modelling, **32**, 1485 – 1494 (2000).
[15] D. Motreanu, P.D. Panagiotopoulos, Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities, Kluwer Academic Publishers, Dordrecht (1999).
[16] K. C. Chang, "Variational methods for non-differential functions and their applications to partial differential equations", J. Math. Anal. Appl., **80**, 102 – 129 (1981).
[17] L. Gasinki, N. Papageorgiou, Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems, Chapman Hall/CRC, Boca Raton (2005).

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