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BOUNDARY VALUE PROBLEMS OF SECOND ORDER NONLINEAR DIFFERENCE EQUATIONS WITH JACOBI OPERATORS

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Abstract. The paper is devoted to a question of existence and multiplicity of solutions of boundary value problems for a class of nonlinear difference equations with Jacobi second order operators. By using the critical point theory, some sufficient conditions are obtained.

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1. Introduction

Throughout the paper the following notation will be used. The letters $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural, integer and real numbers, respectively; k will stand for a positive integer. For any a, $b \in \mathbf{Z}$ we define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \leq b$. By Δ we denote the forward difference operator defined by $\Delta u_n = u_{n+1} - u_n$. Also, the symbol * will denote the transpose of a vector.

The second order forward-backward differential-difference equation

$$c^2 u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1)), \ t \in \mathbf{R} \tag{1.1}$$

has been studied extensively by many scholars. For example, Smets and Willem [26] have established the existence of solitary waves of (1.1).

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A generalization of (1.1) is the following equation

$$Su(t) = f(t, u(t+1), u(t), u(t-1)),\ t \in \mathbf{R}, \tag{1.2}$$

where S stands for the Sturm-Liouville differential expression and $f \in C(\mathbf{R}^4, \mathbf{R})$.

The present paper considers the second order difference equation

$$Lu_n = f(n, u_{n+1}, u_n, u_{n-1}), \tag{1.3}$$

with boundary value conditions

$$u_a + \alpha u_{a+1} = A,\ u_{b+2} + \beta u_{b+1} = B, \tag{1.4}$$

where L is the Jacobi operator defined by $Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n$, a_n and b_n are real-valued for each $n \in \mathbf{Z}$, $f \in C(\mathbf{R}^4, \mathbf{R})$, and α, β, A and B are constants.

Observe that the equation (1.3) can be considered as a discrete analogue of (1.2), and the operator L leads to a symmetric matrix representation. Also, notice that the boundary value conditions in equation (1.4) include the following special cases: the Dirichlet boundary value conditions, the mixed boundary value conditions and the Neumann boundary value conditions:

$$u_a = A,\ u_{b+2} = B; \tag{1.5}$$

$$u_a = A,\ \Delta u_{b+1} = B; \tag{1.6}$$

$$\Delta u_a = A,\ u_{b+2} = B;\ \textit{and} \tag{1.7}$$

$$\Delta u_a = A,\ \Delta u_{b+1} = B. \tag{1.8}$$

It is worthwhile to observe that the Jacobi operators appear in a variety of applications (see, e.g., [27], and references therein). They can be viewed as the discrete analogues of the Sturm-Liouville operators and their investigation has many similarities with that of Sturm-Liouville theory. It should be noted that there are a number of books devoted to the Sturm-Liouville operators, whereas there are only few on Jacobi operators. Moreover, there is a small number of researches available that cover some basic topics, such us positive solutions, periodic operators, boundary value problems, etc., which typically can be found in the books on Sturm-Liouville operators (see, e.g., [17]).

Without loss of generality, we can assume that $a = 0$ and $b = k-1$ for some positive number k. Then the boundary value problem (BVP) (1.3) with (1.4) becomes

$$Lu_n = f(n, u_{n+1}, u_n, u_{n-1}),\ n \in \mathbf{Z}(1, k) \tag{1.9}$$

with boundary value conditions

$$u_0 + \alpha u_1 = A,\ u_{k+1} + \beta u_k = B. \tag{1.10}$$

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields, such as computer science, economics, neural network, ecology, cybernetics, etc. Since the last decade, there has been much literature on qualitative properties of difference equations, those studies cover many of the branches of difference equations (see, e.g., [1, 8, 9,14-16, 18, 20, 22, 24, 25], and references therein).

In recent years, the boundary value problems for differential equations was studied extensively. By using various methods and techniques, such as the Schauder fixed point theorem, the cone theoretic fixed point theorem, the method of upper and lower solutions, coincidence degree theory, a series of results of nontrivial solutions for differential equations have been obtained in the literature (see, e.g., [2-6, 13, 28]. Notice that the critical point theory is also an important tool to deal with problems on differential equations (see [19, 23, 31]). Because of applications of difference equations in many areas (see [1, 8, 15, 16, 20, 25]), recently, some authors have gradually paid attention to applying critical point theory to deal with periodic and homoclinic solutions of discrete systems (see [7, 10-12, 29, 30, 32, 33]).

In particular, using the critical point theory, Chen and Fang [7] have obtained a sufficient condition for the existence of periodic and subharmonic solutions of the following second-order p-Laplacian difference equation

$$\Delta(\varphi_p(\Delta u(n-1))) + f(n, u(n+1), u(n), u(n-1)) = 0, \ n \in \mathbf{Z}.$$

We also refer the reader to [29, 30] for the discrete boundary value problems. Notice that, however, all these topics do not concern with the Jacobi operators.

As far as we know results obtained in the literature for the (BVP) (1.9) with (1.10) are very scarce. Since the function f in (1.9) depends on u_{n+1} and u_{n-1}, the traditional methods of establishing the functional, developed in [10-12, 29, 30, 32, 33], are inapplicable to our case. The present paper aims to fill this gap.

In this paper, motivated by the above arguments and results, we use the critical point theory and obtain sufficient conditions for the existence and multiplicity of the solutions of the BVP (1.9) with (1.10). The main idea is to transfer the question of existence of the solutions of the BVP (1.9) with (1.10) into that of the critical points of some functional. We also demonstrate the power of the critical point theory in the study of the existence of multiple solutions for boundary value problems for difference equations. For the basic concepts and results on variational methods, we refer the reader to [19,21,23,31].

Throughout the paper we suppose that $B = 0$ and $a_n < 0$ for $n \in \mathbf{Z}(1, k)$. The main results of the paper are the following theorems.

Theorem 1.1. *Assume that the following conditions are satisfied:*

(L_1) $b_1 - \alpha a_0 + a_1 > 0,\ b_k - \beta a_k + a_{k-1} > 0,\ b_i + a_i + a_{i-1} > 0,\ 2 \le i \le k-1;$

(F_1) *there exists a functional* $F(n, \cdot) \in C^1(\mathbf{Z} \times \mathbf{R}^2, \mathbf{R})$ *with* $F(0, \cdot) = 0$ *such that*

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3),\ \forall n \in \mathbf{Z}(1, k);$$

(F_2) *there exist constants* $c_1 > 0$, $c_2 > 0$ *and* $1 < \tau_1 < 2$ *such that for any* $n \in \mathbf{Z}(1, k)$,

$$F(n, v_1, v_2) \le c_1 \left(\sqrt{v_1^2 + v_2^2}\right)^{\tau_1} + c_2. \tag{1.11}$$

Then the BVP (1.9) with (1.10) possesses at least one solution.

Corollary 1.1. *Assume that the conditions* (F_1) *and* (L_1) *are satisfied. And, also*

(F_3) *there exists a constant* $M_0 > 0$ *such that for all* $(n, v_1, v_2) \in \mathbf{Z}(1, k) \times \mathbf{R}^2$

$$\left|\frac{\partial F(n, v_1, v_2)}{\partial v_1}\right| \le M_0,\ \left|\frac{\partial F(n, v_1, v_2)}{\partial v_2}\right| \le M_0.$$

Then the BVP (1.9) with (1.10) possesses at least one solution.

Remark 1.1. *Assumption* (F_3) *implies that there exists a constant* $M_1 > 0$ *such that*
(F_3') $|F(n, v_1, v_2)| \le M_1 + M_0(|v_1| + |v_2|),\ \forall (n, v_1, v_2) \in \mathbf{Z}(1, k) \times \mathbf{R}^2.$

Theorem 1.2. *Assume that the conditions* (F_1) *and* (F_3) *are satisfied. And, also*

(L_2) $A = 0,\ b_1 - \alpha a_0 + a_1 = 0,\ b_k - \beta a_k + a_{k-1} = 0,\ b_i + a_i + a_{i-1} = 0,\ 2 \le i \le k-1;$

(F_4) $F(n, v_1, v_2) \to +\infty$ *for* $n \in \mathbf{Z}(1, k)$ *as* $\sqrt{v_1^2 + v_2^2} \to +\infty$.

Then the BVP (1.9) with (1.10) possesses at least one solution.

Theorem 1.3. *Assume that the condition* (F_1) *is satisfied. And, also*

(F_5) *there exists a constant* $\sigma_1 > 2$ *such that for any* $n \in \mathbf{Z}(1, k)$,

$$0 < \sigma_1 F(n, v_1, v_2) \le \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2,\ \forall \sqrt{v_1^2 + v_2^2} \ge R. \tag{1.12}$$

Then the BVP (1.9) with (1.10) possesses at least one solution.

Remark 1.2. *The condition (1.12) implies that there exist constants* $c_3 > 0$ *and* $c_4 > 0$ *such that*

$$F(n, v_1, v_2) \ge c_3 \left(\sqrt{v_1^2 + v_2^2}\right)^{\sigma_1} - c_4,\ \forall n \in \mathbf{Z}(1, k). \tag{1.13}$$

The next two theorems contain sufficient conditions ensuring at least two nontrivial solutions for BVP (1.9) with boundary value conditions (1.10).

Theorem 1.4. *Assume that* $A = 0$, *and the conditions* (L_1), (F_1) *and* (F_5) *are satisfied. And, also*

(F_6) *there exists a functional* $F(n,\cdot)\in C^1(Z\times R^2,R)$ *such that*

$$\lim_{r\to 0}\frac{F(n,v_1,v_2)}{r^2}=0,\ r=\sqrt{v_1^2+v_2^2},\ \forall n\in Z(1,k).$$

Then the BVP (1.9) with (1.10) possesses at least two nontrivial solutions.

Theorem 1.5. *Assume that the conditions* (L_2) *and* (F_1) *are satisfied. And, also*
(F_7) *there exist constants* $\delta>0$, $\tau_2\in\left(0,\frac{1}{2(\beta^2+2)}\lambda_{\min}\right)$ *such that*

$$F(n,v_1,v_2)\le\tau_2\left(v_1^2+v_2^2\right),\ \textit{for } n\in Z(1,k) \textit{ and } v_1^2+v_2^2\le\delta^2;$$

(F_8) *there exist constants* $\rho>0$, $\gamma>0$, $\sigma_2\in\left(\frac{1}{2}\lambda_{\max},+\infty\right)$ *such that*

$$F(n,v_1,v_2)\ge\sigma_2\left(v_1^2+v_2^2\right)-\gamma,\ \textit{for } n\in Z(1,k) \textit{ and } v_1^2+v_2^2\ge\rho^2,$$

where $\lambda_{\min}$ *and* $\lambda_{\max}$ *are defined in formula (2.4).*

Then the BVP (1.9) with (1.10) possesses at least two nontrivial solutions.

Remark 1.3. *It follows from* (F_8) *that there exists a constant* $\gamma'>0$ *such that*

$$(F_8')\quad F(n,v_1,v_2)\ge\sigma_2\left(v_1^2+v_2^2\right)-\gamma',\ \forall(n,v_1,v_2)\in\mathbf{Z}(1,k)\times\mathbf{R}^2.$$

2. Variational structure and some lemmas

For a given $r>1$, define the norm $\|\cdot\|_r$ on $\mathbf{R}^k$ as follows: for all $u\in\mathbf{R}^k$

$$\|u\|_r=\left(\sum_{j=1}^{k}|u_j|^r\right)^{\frac{1}{r}}.$$

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants $k_1,\ k_2$ such that $k_2\ge k_1>0$, and

$$k_1\|u\|_2\le\|u\|_r\le k_2\|u\|_2,\ \forall u\in\mathbf{R}^k. \tag{2.1}$$

Clearly, $\|u\|=\|u\|_2$. When $k>2$, for the BVP (1.9) with (1.10), consider the functional J on $\mathbf{R}^k$ defined as follows:

$$J(u)=\frac{1}{2}\langle Pu,u\rangle+\langle\eta,u\rangle-\sum_{n=1}^{k}F(n,u_{n+1},u_n), \tag{2.2}$$

$\forall u=(u_1,u_2,\cdots,u_k)^*\in\mathbf{R}^k,\ u_0+\alpha u_1=A,\ u_{k+1}+\beta u_k=B$, where

$$P=\begin{pmatrix} b_1-\alpha a_0 & a_1 & 0 & \cdots & 0 & 0\\ a_1 & b_2 & a_2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & b_{k-1} & a_{k-1}\\ 0 & 0 & 0 & \cdots & a_{k-1} & b_k-\beta a_k \end{pmatrix},\ \eta=\begin{pmatrix} a_0A\\ 0\\ \cdots\\ 0\\ a_kB \end{pmatrix}. \tag{2.3}$$

Clearly, $J \in C^1(\mathbf{R}^k, \mathbf{R})$, and for any $u = \{u_n\}_{n\in\mathbf{Z}(1,k)} \in \mathbf{R}^k$, by using the equalities $u_0 + \alpha u_1 = A$ and $u_{k+1} + \beta u_k = B$, one can easily compute the partial derivative:

$$\frac{\partial J}{\partial u_n} = Lu_n - f(n, u_{n+1}, u_n, u_{n-1}),\ n \in \mathbf{Z}(1,k).$$

Threfore, u is a critical point of J on $\mathbf{R}^k$ if and only if $\{u_n\}_{n=0}^{k+1} = (u_0, u_1, u_2, \cdots, u_k, u_{k+1})$ is a solution of the BVP (1.9) with (1.10), where $u_0 = A - \alpha u_1$ and $u_{k+1} = B - \beta u_k$. Thus, the existence of solutions of the BVP (1.9) with (1.10) is reduced to that of the critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (1.9) with (1.10).

Remark 2.1. *The case $k = 1$ is trivial. For the case $k = 2$, P has a different form, namely,*

$$P = \begin{pmatrix} b_1 - \alpha a_0 & a_1 \\ a_1 & b_2 - \beta a_2 \end{pmatrix}.$$

However, in this case, it is easy to complete the proofs of Theorems 1.1 – 1.5.

Remark 2.2. *It follows from (L_2) that 0 is an eigenvalue of P and $\zeta = \frac{1}{\sqrt{k}}(1, 1, \cdots, 1)^* \in E_k$ is an eigenvector of P corresponding to 0. Let $\lambda_1, \lambda_2, \cdots, \lambda_{k-1}$ be the other eigenvalues of P. Applying matrix theory, we infer that $\lambda_j > 0$ for all $j \in \mathbf{Z}(1, k-1)$.*

Define

$$\lambda_{\min} = \min\{\lambda_j | j \in \mathbf{Z}(1, k-1)\} > 0,\ \lambda_{\max} = \max\{\lambda_j | j \in \mathbf{Z}(1, k-1)\} > 0. \quad (2.4)$$

Denote $W = \{(u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k | u_n \equiv c,\ c \in \mathbf{R},\ n \in \mathbf{Z}(1,k)\}$ and let V be the direct orthogonal complement of $\mathbf{R}^k$ to W, i.e., $\mathbf{R}^k = V \oplus W$.

Let E be a real Banach space and let $J \in C^1(E, \mathbf{R})$, that is, J is a continuously Fréchet-differentiable functional defined on E. We say that the functional J satisfis the Palais-Smale (PS) condition (see [12]), if any sequence $\{u^{(k)}\} \subset E$ for which $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \to 0$ as $k \to \infty$ possesses a convergent subsequence in E. Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1. *(Saddle Point Theorem, [23]). Let E be a real Banach space, $E = E_1 \oplus E_2$, where $E_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(E, \mathbf{R})$ satisfies the (PS) condition and also*
(J_1) there exist constants μ, $\rho > 0$ such that $J|_{\partial B_\rho \cap E_1} \leq \mu$; (J_2) there exists $e \in B_\rho \cap E_1$ and a constant $\omega \geq \mu$ such that $J|_{e+E_2} \geq \omega$.
Then J possesses a critical value $c \geq \omega$ given by

$$c = \inf_{h\in\Gamma} \max_{u\in B_\rho \cap E_1} J(h(u)),$$

where

$$\Gamma = \{h \in C(\bar{B}_\rho \cap E_1, E) \mid h|_{\partial B_\rho \cap E_1} = id\}$$

and id denotes the identity operator.

Lemma 2.2. *(Mountain Pass Lemma, [23]). Let E be a real Banach space and let $J \in C^1(E, R)$ satisfy the (PS) condition. If $J(0) = 0$ and*

(J_3) there exist constants ρ, $a > 0$ such that $J|_{\partial B_\rho} \geq a$, and

(J_4) there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.

Then J possesses a critical value $c \geq a$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} J(g(s)), \tag{2.5}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0, \; g(1) = e\}. \tag{2.6}$$

Lemma 2.3. *(Linking Theorem, [23]). Let E be a real Banach space, $E = E_1 \oplus E_2$, where E_1 is finite dimensional. Suppose that $J \in C^1(E, R)$ satisfies the (PS) condition and also*

(J_5) there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;

(J_6) there exists an $e \in \partial B_1 \cap E_2$ and a constant $R_0 \geq \rho$ such that $J|_{\partial Q} \leq 0$, where $Q = (\bar{B}_{R_0} \cap E_1) \oplus \{re | 0 < r < R_0\}$.

Then J possesses a critical value $c \geq a$ given by

$$c = \inf_{h \in \Gamma} \sup_{u \in Q} J(h(u)),$$

where $\Gamma = \{h \in C(\bar{Q}, E) \mid h|_{\partial Q} = id\}$ and id denotes the identity operator.

Lemma 2.4. *Assume that the conditions (L_2), (F_1), (F_3) and (F_4) are satisfied. Then the functional J satisfies the (PS) condition.*

Proof. Let $u^{(l)} \in \mathbf{R}^k$ and $l \in \mathbf{Z}(1)$ be such that $\left\{J\left(u^{(l)}\right)\right\}$ is bounded and $J'\left(u^{(l)}\right) \to 0$ as $l \to \infty$. Then there exists a positive constant M_2 such that $\left|J\left(u^{(l)}\right)\right| \leq M_2$.

Let $u^{(l)} = v^{(l)} + w^{(l)} \in V + W$. For l large enough, since

$$-\|u\| \leq \left\langle -J'\left(u^{(l)}\right), u\right\rangle = -\left\langle Pu^{(l)}, u\right\rangle + \sum_{n=1}^{k} f\left(n, u_{n+1}^{(l)}, u_n^{(l)}, u_{n-1}^{(l)}\right) u_n,$$

in view of (F_1) and (F_3), we can write

$$\begin{aligned}\left\langle Pu^{(l)}, v^{(l)}\right\rangle &\leq \sum_{n=1}^{k} f\left(n, u_{n+1}^{(l)}, u_n^{(l)}, u_{n-1}^{(l)}\right) v_n^{(l)} + \left\|v^{(l)}\right\| \\ &\leq 2M_0 \sum_{n=1}^{k} \left|v_n^{(l)}\right| + \left\|v^{(l)}\right\| \leq \left(2M_0\sqrt{k} + 1\right) \left\|v^{(l)}\right\|.\end{aligned}$$

On the other hand, we have $\langle Pu^{(l)}, v^{(l)}\rangle = \langle Pv^{(l)}, v^{(l)}\rangle \geq \lambda_{\min}\left\|v^{(l)}\right\|^2$. Therefore $\lambda_{\min}\left\|v^{(l)}\right\|^2 \leq (2M_0\sqrt{k}+1)\left\|v^{(l)}\right\|$, implying that $\{v^{(l)}\}$ is bounded.

Next, we show that $\{w^{(l)}\}$ is bounded. Since

$$M_2 \geq -J\left(u^{(l)}\right) = -\frac{1}{2}\left\langle Pu^{(l)}, u^{(l)}\right\rangle + \sum_{n=1}^{k} F\left(n, u_{n+1}^{(l)}, u_n^{(l)}\right),$$

$$= -\frac{1}{2}\left\langle Pu^{(l)}, u^{(l)}\right\rangle + \sum_{n=1}^{k}\left[F\left(n, u_{n+1}^{(l)}, u_n^{(l)}\right) - F\left(n, w_{n+1}^{(l)}, w_n^{(l)}\right)\right] + \sum_{n=1}^{k} F\left(n, w_{n+1}^{(l)}, w_n^{(l)}\right),$$

we get

$$\sum_{n=1}^{k} F\left(n, w_{n+1}^{(l)}, w_n^{(l)}\right)$$

$$\leq M_2 + \frac{1}{2}\left\langle Pv^{(l)}, v^{(l)}\right\rangle + \sum_{n=1}^{k}\left|F\left(n, u_{n+1}^{(l)}, u_n^{(l)}\right) - F\left(n, w_{n+1}^{(l)}, w_n^{(l)}\right)\right|$$

$$\leq M_2 + \frac{1}{2}\lambda_{\max}\left\|v^{(l)}\right\|^2$$

$$+\sum_{n=1}^{k}\left|\frac{\partial F\left(n, w_{n+1}^{(l)}+\theta v_{n+1}^{(l)}, w_n^{(l)}+\theta v_n^{(l)}\right)}{\partial v_1}\cdot v_{n+1}^{(l)} + \frac{\partial F\left(n, w_{n+1}^{(l)}+\theta v_{n+1}^{(l)}, w_n^{(k)}+\theta v_n^{(l)}\right)}{\partial v_2}\cdot v_n^{(l)}\right|$$

$$\leq M_2 + \frac{1}{2}\lambda_{\max}\left\|v^{(l)}\right\|^2 + \sqrt{4k-4\beta+\beta^2}M_0\left\|v^{(l)}\right\|,$$

where $\theta \in (0,1)$. It is easy to see that $\left\{\sum_{n=1}^{k} F\left(n, w_{n+1}^{(l)}, w_n^{(l)}\right)\right\}$ is bounded.

It follows from (F_4) that $\{w^{(l)}\}$ is bounded. If otherwise, we assume that $\left\|w^{(l)}\right\| \to +\infty$ as $l \to \infty$. Since there exist $z^{(l)} \in \mathbf{R}^k, l \in \mathbf{N}$, such that $w^{(l)} = (z^{(l)}, z^{(l)}, \cdots, z^{(l)})^* \in E_k$, then

$$\left\|w^{(l)}\right\| = \left(\sum_{n=1}^{k}\left|w_n^{(l)}\right|^2\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{k}\left|z^{(l)}\right|^2\right)^{\frac{1}{2}} = \sqrt{k}|z^{(l)}| \to +\infty \text{ as } l \to \infty.$$

Since

$$F\left(n, w_{n+1}^{(l)}, w_n^{(l)}\right) = \begin{cases} F\left(n, z^{(l)}, z^{(l)}\right), \text{ when } n \in \mathbf{Z}(1,k), \\ F\left(n, -\beta z^{(l)}, z^{(l)}\right), \text{ when } n = k \end{cases}$$

then $F\left(n, w_{n+1}^{(l)}, w_n^{(l)}\right) \to +\infty$ as $l \to \infty$.

This contradicts the fact that $\left\{\sum_{n=1}^{k} F\left(n, w_{n+1}^{(l)}, w_n^{(l)}\right)\right\}$ is bounded. Lemma 2.4 is proved. □

Lemma 2.5. *Assume that the conditions (L_1), (F_1), (F_5) and (F_6) are satisfied.*

Then the functional J satisfies the (PS) condition.

Proof. It follows from (L_1) that P is positive definite. We denote by $\lambda_1, \lambda_2, \cdots, \lambda_k$ its eigenvalues, and define

$$\tilde{\lambda}_{\min} = \min\{\lambda_j | j \in \mathbf{Z}(1,k)\} > 0, \ \tilde{\lambda}_{\max} = \max\{\lambda_j | j \in \mathbf{Z}(1,k)\} > 0. \tag{2.7}$$

Let $u^{(l)} \in \mathbf{R}^k$ and $l \in \mathbf{Z}(1)$ be such that $\{J(u^{(l)})\}$ is bounded and $J'(u^{(l)}) \to 0$ as $l \to \infty$. Then there exists a positive constant M_3 such that $-M_3 \le J(u^{(l)}) \le M_3$, $\forall l \in \mathbf{N}$. By (1.13), we have

$$\begin{aligned} -M_3 \le J\left(u^{(l)}\right) &= \frac{1}{2}\left\langle Pu^{(l)}, u^{(l)}\right\rangle - \sum_{n=1}^{k} F\left(n, u_{n+1}^{(l)}, u_n^{(l)}\right) \\ &\le \frac{1}{2}\tilde{\lambda}_{\max}\left\|u^{(l)}\right\|^2 - c_3 \sum_{n=1}^{k}\left[\sqrt{\left(u_{n+1}^{(l)}\right)^2 + \left(u_n^{(l)}\right)^2}\right]^{\sigma_1} + c_4 k \\ &\le \frac{1}{2}\tilde{\lambda}_{\max}\left\|u^{(l)}\right\|^2 - c_3 k_1^{\sigma_1}\left\|u^{(l)}\right\|^{\sigma_1} + c_4 k. \end{aligned}$$

This implies

$$c_3 k_1^{\sigma_1}\left\|u^{(l)}\right\|^{\sigma_1} - \frac{1}{2}\tilde{\lambda}_{\max}\left\|u^{(l)}\right\|^2 \le M_3 + c_4 k.$$

Since $\sigma_1 > 2$, there exists a constant $M_4 > 0$ such that

$$\left\|u^{(l)}\right\| \le M_4, \ \forall l \in \mathbf{N}.$$

Therefore, $\{u^{(l)}\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\{u^{(l)}\}$ possesses a convergent subsequence in $\mathbf{R}^k$. Thus the (PS) condition is satisfied. Lemma 2.5 is proved. □

Lemma 2.6. *Assume that the conditions (L_2), (F_1), (F_7) and (F_8) are satisfied. Then the functional J satisfies the (PS) condition.*

Proof. Let $u^{(l)} \in \mathbf{R}^k$, $l \in \mathbf{Z}(1)$ be such that $\{J(u^{(l)})\}$ is bounded and $J'(u^{(l)}) \to 0$ as $l \to \infty$. Then there exists a positive constant M_5 such that $|J(u^{(l)})| \le M_5$. By (F_8'), for any $u^{(l)} \in E_k$ and $l \in \mathbf{Z}(1)$ we have

$$\begin{aligned} -M_5 \le J\left(u^{(l)}\right) &= \frac{1}{2}\left\langle Pu^{(l)}, u^{(l)}\right\rangle - \sum_{n=1}^{k} F\left(n, u_{n+1}^{(l)}, u_n^{(l)}\right) \\ &\le \frac{1}{2}\lambda_{\max}\left\|u^{(l)}\right\|^2 - \sum_{n=1}^{k}\left\{\sigma_2\left[\left(u_{n+1}^{(l)}\right)^2 + \left(u_n^{(l)}\right)^2\right] - \gamma'\right\} \\ &\le \left(\frac{1}{2}\lambda_{\max} - \sigma_2\right)\left\|u^{(l)}\right\|^2 + k\gamma' \le k\gamma'. \end{aligned}$$

Therefore,

$$\left(\sigma_2 - \frac{1}{2}\lambda_{\max}\right)\left\|u^{(l)}\right\|^2 \le M_5 + k\gamma'.$$

Since $\sigma_2 > \frac{1}{2}\lambda_{\max}$, it is easy to see that $\{u^{(l)}\}$ is a bounded sequence in E_k. As a consequence, $\{u^{(l)}\}$ possesses a convergent subsequence in E_k, and the result follows. Lemma 2.6 is proved. □

3. Proofs of the main results

In this section, we prove our main results by using the critical point method.

Proof of Theorem 1.1. It follows from (L_1) that the matrix P is positive definite. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the eigenvalues of P. Applying matrix theory, we have $\lambda_j > 0$, $j = 1, 2, \cdots, k$. Without loss of generality, we may assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k.$$

Then for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$ we have

$$\begin{aligned} J(u) &\geq \frac{1}{2}\lambda_1\|u\|^2 - \|\eta\| \cdot \|u\| - c_1 \sum_{n=1}^{k} \left(\sqrt{u_{n+1}^2 + u_n^2}\right)^{r_1} - c_2 k \\ &\geq \frac{1}{2}\lambda_1\|u\|^2 - \|\eta\| \cdot \|u\| - c_1 \sum_{n=1}^{k} [(1+|\beta|) \cdot \|u\| + \|u\|]^{r_1} - c_2 k \\ &\geq \frac{1}{2}\lambda_1\|u\|^2 - \|\eta\| \cdot \|u\| - c_1 k (2+|\beta|)^{r_1} \|u\|^{r_1} - c_2 k \to +\infty \text{ as } \|u\| \to +\infty, \end{aligned}$$

which by (F_2) implies that J is bounded from below. From this, we conclude that a (PS) sequence must be bounded in $\mathbf{R}^k$. This means that $J(u)$ is coercive. By the continuity of $J(u)$, there exists $\bar{u} \in E_k$ such that $J(\bar{u}) = c_0$. Clearly, $\bar{u}$ is a critical point of J. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Observe first that by Lemma 2.4, J satisfies the (PS) condition. Next, we verify the conditions (J_1) and (J_2). For any $v \in V$, by (F_3') we have

$$\begin{aligned} -J(v) &= -\frac{1}{2}\langle Pv, v\rangle + \sum_{n=1}^{k} F(n, v_{n+1}, v_n) \\ &\leq -\frac{1}{2}\lambda_{\min}\|v\|^2 + kM_1 + M_0 \sum_{n=1}^{k} (|u_{n+1}| + |u_n|) \\ &\leq -\tfrac{1}{2}\lambda_{\min}\|v\|^2 + kM_1 + \sqrt{4k - 4\beta + \beta^2} M_0 \|v\| \to -\infty \text{ as } \|v\| \to +\infty, \end{aligned}$$

implying that the condition (J_1) is satisfied.

Next, for any $w \in W$, $w = (w_1, w_2, \cdots, w_k)^*$, there exists $z \in \mathbf{R}$ such that $w_n = z$ for all $n \in \mathbf{Z}(1, k)$. By (F_4), there exists a constant $R_0 > 0$ such that

$F(k,-\beta z,z)>0$ and $F(n,z,z)>0$ for $n\in\mathbf{Z}(1,k-1)$ and $|z|>R_0/\sqrt{2}$. Let $M_6=\min_{n\in\mathbf{Z}(1,k-1),|z|\le R_0/\sqrt{2}}\{F(n,z,z),F(k,-\beta z,z)\}$ and $M_7=\min\{0,M_6\}$. Then

$$F(k,-\beta z,z)\ge M_7,\ F(n,z,z)\ge M_7,\ \forall(n,z,z)\in\mathbf{Z}(1,k-1)\times\mathbf{R}^2.$$

So, we have

$$-J(w)=\sum_{n=1}^{k}F(n,w_{n+1},w_n)=\sum_{n=1}^{k-1}F(n,z,z)+F(k,-\beta z,z)\ge kM_7,\ \forall w\in W.$$

It can easily be seen that the functional $-J$ satisfies all the assumptions of Lemma 2.1, and the result follows. Theorem 1.2 is proved. □

Proof of Theorem 1.3. Since the matrix P (see (2.3)) is symmetric, its eigenvalues (denoted by $\lambda_1,\lambda_2,\cdots,\lambda_k$) are real, and without loss of generality, we may assume that $\lambda_1\le\lambda_2\le\cdots\le\lambda_k$. Therefore, for any $u=(u_1,u_2,\cdots,u_k)^*\in\mathbf{R}^k$, we have

$$\begin{aligned}J(u)&\le\frac{1}{2}\lambda_k\|u\|^2+\|\eta\|\cdot\|u\|-c_3\sum_{n=1}^{k}\left[\sqrt{u_{n+1}^2+u_n^2}\right]^{\sigma_1}+c_4k\\&\le\frac{1}{2}\lambda_k\|u\|^2+\|\eta\|\cdot\|u\|-c_3k_1^{\sigma_1}\|u\|^{\sigma_1}+c_4k\to-\infty\ \text{ as }\ \|u\|\to+\infty.\end{aligned}$$

Due to the continuity of $J(u)$, the above inequality implies that there exist upper bounds of values of functional J. Classical calculus shows that J attains its maximal value at some point, which is just the critical point of J. Theorem 1.3 is proved. □

Proof of Theorem 1.4. First observe that by (F_6), for any $\epsilon=\frac{1}{4(\beta^2+2)}\tilde{\lambda}_{\min}$, where $\tilde{\lambda}_{\min}$ is defined in (2.7), there exists $\rho>0$, such that

$$|F(n,v_1,v_2)|\le\frac{1}{4(\beta^2+2)}\tilde{\lambda}_{\min}\left(v_1^2+v_2^2\right),\forall n\in\mathbf{Z}(1,k),$$

for $\sqrt{v_1^2+v_2^2}\le\sqrt{2}\rho$.

Next, for any $u=(u_1,u_2,\cdots,u_k)^*\in\mathbf{R}^k$ and $\|u\|\le\rho$, we have $|u_n|\le\rho$, $n\in\mathbf{Z}(1,k)$, and for $k>2$ we can write

$$\begin{aligned}J(u)&\ge\frac{1}{2}\tilde{\lambda}_{\min}\|u\|^2-\frac{1}{4(\beta^2+2)}\tilde{\lambda}_{\min}\sum_{n=1}^{k}\left(u_{n+1}^2+u_n^2\right)\\&\ge\frac{1}{2}\tilde{\lambda}_{\min}\|u\|^2-\frac{1}{4}\tilde{\lambda}_{\min}\|u\|^2=\frac{1}{4}\tilde{\lambda}_{\min}\|u\|^2.\end{aligned}$$

Taking $a\triangleq\frac{1}{4}\tilde{\lambda}_{\min}\rho^2>0$, we get $J(u)\ge a>0$, $\forall u\in\partial B_\rho$. Observe also that there exist constants $a>0$ and $\rho>0$ such that $J|_{\partial B_\rho}\ge a$. This implies that J satisfies the condition (J_3) of the Mountain Pass Lemma.

Clearly we have $J(0)=0$, and in order to exploit the Mountain Pass Lemma in critical point theory, we need to verify that the other conditions of the Mountain Pass

Lemma are also satisfied. By Lemma 2.5, J satisfies the (PS) condition. So it remains to verify the condition (J_4).
From the proof of the (PS) condition it follows that $J(u) \le \frac{1}{2}\bar{\lambda}_{\max} \|u\|^2 - c_3 k_1^{\sigma_1} \|u\|^{\sigma_1} + c_4 k$. Since $\sigma_1 > 2$, we can choose $\bar{u}$ large enough to satisfy $J(\bar{u}) < 0$. By the Mountain Pass Lemma, J possesses a critical value $c \ge a > 0$, where $c = \inf_{h\in\Gamma} \sup_{s\in[0,1]} J(h(s))$ and $\Gamma = \{h \in C([0,1], \mathbf{R}^k) \mid h(0) = 0,\ h(1) = \bar{u}\}$.

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated to the critical value c of J, that is, $J(\tilde{u}) = c$. Using the arguments of the proof of (PS) condition, we infer that there exists $\hat{u} \in \mathbf{R}^k$ such that $J(\hat{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s))$.

Clearly, $\hat{u} \neq 0$. If $\tilde{u} \neq \hat{u}$, then the conclusion of Theorem 1.4 holds. Otherwise, $\tilde{u} = \hat{u}$, and $c = J(\tilde{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s))$. This implies $\sup_{u\in\mathbf{R}^k} J(u) = \inf_{h\in\Gamma} \sup_{s\in[0,1]} J(h(s))$. Therefore, $c_{\max} = \max_{s\in[0,1]} J(h(s))$ for any $h \in \Gamma$.
By the continuity of $J(h(s))$ with respect to s, $J(0) = 0$ and $J(\bar{u}) < 0$ imply that there exists $s_0 \in (0,1)$ such that $J(h(s_0)) = c_{\max}$. Choose $h_1,\ h_2 \in \Gamma$ such that $\{h_1(s) \mid s \in (0,1)\} \cap \{h_1(s) \mid s \in (0,1)\}$ is empty, then there exists $s_1,\ s_2 \in (0,1)$ such that $J(h_1(s_1)) = J(h_2(s_2)) = c_{\max}$. Thus, we get two different critical points of J on $\mathbf{R}^k$ denoted by $u^1 = h_1(s_1),\ u^2 = h_2(s_2)$. The above arguments imply that the BVP (1.9) with (1.10) possesses at least two nontrivial solutions. Theorem 1.4 is proved. □

Proof of Theorem 1.5. By Lemma 2.6, J is bounded from above on E_k. We define $c_0 = \sup_{u\in E_k} J(u)$. The arguments of the proof of Lemma 2.6 imply $\lim_{\|u\|\to+\infty} J(u) = -\infty$. This means that $-J(u)$ is coercive. By the continuity of $J(u)$, there exists $\bar{u} \in E_k$ such that $J(\bar{u}) = c_0$. Clearly, $\bar{u}$ is a critical point of J.

Next, we claim that $c_0 > 0$. Indeed, by (F_7), for any $u \in V,\ \|u\| \le \rho$, we have

$$J(u) \ge \frac{1}{2}\lambda_{\min}\|u\|^2 - \tau_2 \sum_{n=1}^{k} \left(u_{n+1}^2 + u_n^2\right)$$

$$\ge \left[\frac{1}{2}\lambda_{\min} - \left(\beta^2 + 2\right)\tau_2\right] \|u\|^2.$$

Taking $a = \left[\frac{1}{2}\lambda_{\min} - \left(\beta^2 + 2\right)\tau_2\right]\rho^2$, we have $J(u) \ge a,\ \forall u \in V \cap \partial B_\rho$. Therefore, we have proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap V} \ge a$. This implies that J satisfies the condition (J_5) of the Linking Theorem.

Noting that $Pu = 0$ for all $u \in W$, we have

$$J(u) = \frac{1}{2}\langle Pu, u\rangle - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) = -\sum_{n=1}^{k} F(n, u_{n+1}, u_n) \le 0.$$

Thus, the critical point $\bar{u}$ of J corresponding to the critical value c_0 is a nontrivial solution of the BVP (1.9) with (1.10).

It follows from Lemma 2.6 that J satisfies the (PS) condition on E_k. Now we are going to verify the condition (J_6).

We take $e \in \partial B_1 \cap V$, and for any $z \in W$ and $r \in \mathbf{R}$ we set $u = re + z$. Then we have

$$J(u) = \frac{1}{2}\langle P(re+z), re+z\rangle - \sum_{n=1}^{k} F(n, re_{n+1} + z_{n+1}, re_n + z_n)$$

$$= \frac{1}{2}\langle P(re), re\rangle - \sum_{n=1}^{k}\left\{\sigma_2\left[(re_{n+1}+z_{n+1})^2 + (re_n + z_n)^2\right] - \gamma'\right\}$$

$$\le \frac{1}{2}\lambda_{\max} r^2 - \sigma_2 \sum_{n=1}^{k}(re_n + z_n)^2 + k\gamma'$$

$$= \left(\frac{1}{2}\lambda_{\max} - \sigma_2\right) r^2 - \sigma_2\|z\|^2 + k\gamma' \le -\sigma_2\|z\|^2 + k\gamma'.$$

Thus, there exists a positive constant $R_2 > \delta$ such that for any $u \in \partial Q$, $J(u) \le 0$, where $Q = (\bar{B}_{R_2} \cap W) \oplus \{re | 0 < r < R_2\}$. By the Linking Theorem, J possesses a critical value $c \ge \rho > 0$, where $c = \inf_{h\in\Gamma}\sup_{u\in Q} J(h(u))$ and $\Gamma = \{h \in C(\bar{Q}, E_k) \mid h|_{\partial Q} = id\}$.

Let $\tilde{u} \in E_k$ be a critical point associated to the critical value c of J, that is, $J(\tilde{u}) = c$. If $\tilde{u} \ne \bar{u}$, then the conclusion of Theorem 1.5 holds. Otherwise, $\tilde{u} = \bar{u}$. Then $c_0 = J(\bar{u}) = J(\tilde{u}) = c$, that is, $\sup\limits_{u\in E_k} J(u) = \inf\limits_{h\in\Gamma}\sup\limits_{u\in Q} J(h(u))$. Choosing $h = id$, we have $\sup\limits_{u\in Q} J(u) = c_0$. Since the choice of $e \in \partial B_1 \cap V$ is arbitrary, we can take $-e \in \partial B_1 \cap V$. Similarly, there exists a positive number $R_3 > \sigma$, such that for any $u \in \partial Q_1$ we have $J(u) \le 0$, where $Q_1 = (\bar{B}_{R_3} \cap W) \oplus \{-re | 0 < r < R_3\}$.

Again applying Lemma 2.3, we conclude that J possesses a critical value $c' \ge \rho > 0$, where $c' = \inf_{h\in\Gamma_1}\sup_{u\in Q_1} J(h(u))$, and $\Gamma_1 = \{h \in C(\bar{Q}_1, E_k) \mid h|_{\partial Q_1} = id\}$.

If $c' \ne c_0$, then the proof is finished. If $c' = c_0$, then $\sup\limits_{u\in Q_1} J(u) = c_0$. Due to the inequalities $J|_{\partial Q} \le 0$ and $J|_{\partial Q_1} \le 0$, J attains its maximum at some points in the interiors of the sets Q and Q_1. However, $Q \cap Q_1 \subset W$ and $J(u) \le 0$ for any $u \in W$. Therefore, there exists a point $u' \in E_k$ such that $u' \ne \tilde{u}$ and $J(u') = c' = c_0$. This completes the proof of Theorem 1.5. □

4. Examples

In this section we give two examples that illustrate our results obtained in Theorems 1.4 and 1.5.

Example 4.1. For $n \in \mathbf{Z}(1,k)$ consider the BVP:

(4.1)
$$-u_{n+1} - u_{n-1} + 3u_n = \sigma_1 u_n \left[\varphi(n)\left(u_{n+1}^2 + u_n^2\right)^{\frac{\sigma_1}{2}-1} + \varphi(n-1)\left(u_n^2 + u_{n-1}^2\right)^{\frac{\sigma_1}{2}-1}\right]$$

with boundary value conditions

$$u_0 + \alpha u_1 = 0,\ u_{k+1} + \beta u_k = 0, \tag{4.2}$$

where $\sigma_1 > 2$, $\alpha > -2$ and $\beta > -2$, $\varphi(s)(s \in \mathbf{R})$ is continuously differentiable and $\varphi(n) > 0$ with $\varphi(0) = 0$. It is easy to check that all the conditions of Theorem 1.4 are satisfied, and hence the BVP (4.1) with (4.2) possesses at least two nontrivial solutions.

Example 4.2. For $n \in \mathbf{Z}(1,k)$ consider the BVP:

(4.3)
$$-6u_{n+1} - 6u_{n-1} + 12u_n = \mu u_n \left[n^2\left(u_{n+1}^2 + u_n^2\right)^{\frac{\mu}{2}-1} + (n-1)^2\left(u_n^2 + u_{n-1}^2\right)^{\frac{\mu}{2}-1}\right]$$

with boundary value conditions

$$u_0 + \alpha u_1 = 0,\ u_{k+1} + \beta u_k = 0, \tag{4.4}$$

where $\mu > 2$, $\alpha = \beta = -1$. It is easy to verify that all conditions of Theorem 1.5 are satisfied, and hence the BVP (4.3) with (4.4) possesses at least two nontrivial solutions.

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