Известия НАН Армении. Математика, том 48, н. 6, 2013, стр. 112-122.

ON A NON-DISSIPATIVE KIRCHHOFF VISCOELASTIC PROBLEM

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Abstract. In this paper we consider a Kirchhoff type viscoelastic problem, and prove uniform stability of the system. We do not rely on the dissipativity of the system or the boundedness of the energy as in the previous treatments. There appears a quadratic term which we cannot estimate by the initial energy as our system is not clearly dissipative in advance.

MSC2010 numbers: 35L20, 35B40, 45K05

Keywords: Exponential decay; Kirchhoff equation; memory term; relaxation function; viscoelasticity.

1. Introduction

In this paper we consider the following wave equation with a viscoelastic damping term:

$$(1.1)\quad \begin{cases} u_{tt} = \left(1 + a(t)\,\|\nabla u\|_2^2\right)\Delta u - \int_0^t h(t-s)\Delta u(s)ds + f(t,x), \ in\ \Omega\times\mathbf{R}_+ \\ u = 0,\ on\ \Gamma\times\mathbf{R}_+ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\Gamma = \partial\Omega$; the functions $u_0(x)$ and $u_1(x)$ are given initial data; the (nonnegative) relaxation function $h(t)$, the (non-negative) function $a(t)$ and $f(t,x)$ will be specified later, and $\|\cdot\|_2$ stands for the L^2-norm.

The equation in (1.1) describes the motion of a viscoelastic body according to the Kirchhoff model (see [8,18]). The integral term in (1.1) represents the memory term or the dependence on the history and the kernel involved is the relaxation function.

Kirchhoff type problems (with different dissipations) and viscoelastic problems have been investigated independently by several authors during the last decades (see, e.g., [2-19]). A number of results on well-posedness and asymptotic behavior of the solutions have been established. Among them only few papers deal with problems of the type (1.1). It should be noticed that the study of the problem of interest leads to some considerable new complications.

In the above cited papers were extensively used the boundedness of energy to estimate the quantities of interest. In our case, due to the presence of the forcing term $f(t,x)$ in the Kirchhoff problem, the derivative of the energy is not necessarily negative, and hence we cannot use the boundedness of the energy to estimate some terms like the one involving the coefficient of diffusion $a(t)\|\nabla u\|_2^2 \Delta u$. Moreover, even without this forcing term, since the relaxation function $h(t)$ generally is not non-increasing (see [17]), again we cannot use the boundedness of the energy to estimate the corresponding terms. In this paper we resolve this problem with the help of an inequality due to Airapetyan et al. [1]. Note that the well-posedness of the problem can be proved using the Faedo Galerkin method (see, [1,8,18]).

Theorem. *Let $(u_0,u_1)\in H^2(\Omega)\cap H_0^1(\Omega)\times L^2(\Omega)$ and $h(t)$ be a nonnegative summable kernel. Then there exists a unique solution u of the problem (1.1) satisfying $u\in C\left([0,T];H^2(\Omega)\cap H_0^1(\Omega)\right)$ and*

$$u_t \in C\left([0,T];L^2(\Omega)\right)\cap L^2([0,T];H_0^1(\Omega))$$

for some $T>0$.

The paper is organized as follows: in Section 2 we prepare some material needed to prove the main results of the paper (equivalence of the classical and modified energy functionals and some lemmas). Section 3 is devoted to the statement and proof of our decay result. In Section 4 we present some simple examples illustrating our findings.

2. Preliminaries

In this section we introduce different functionals we will work with, prove the equivalence of the classical and modified energy functionals, and state a useful identity and a lemma which constitutes the key tool in our contribution. We define the (classical) energy by

$$E(t)=\frac{1}{2}\left(\|u_t\|_2^2+\|\nabla u\|_2^2\right)+\frac{a(t)}{4}\|\nabla u\|_2^4,\ t\geq 0$$

where $\|.\|_2$ denotes the norm in $L^2(\Omega)$. It follows from the equation $(1)_1$ that if $a(t)$ is a differentiable function, then

$$E'(t)=\int_\Omega \nabla u_t \int_0^t h(t-s)\nabla u(s)dsdx+\frac{a'(t)}{4}\|\nabla u\|_2^4+\int_\Omega u_t f(t,x)dx,\ t\geq 0.$$

Observe that

$$2\int_\Omega \nabla u_t\int_0^t h(t-s)\nabla u(s)dsdx=\int_\Omega (h'\Box\nabla u)dx-h(t)\|\nabla u\|_2^2-$$

$$-\frac{d}{dt}\left\{\int_\Omega (h\Box\nabla u)dx - \left(\int_0^t h(s)ds\right)\|\nabla u\|_2^2\right\},$$

where

$$(h\Box v)(t) = \int_0^t h(t-s)\,|v(t)-v(s)|^2\,ds,\ t\geq 0.$$

Therefore, modifying $E(t)$ to

$$\mathcal{E}(t) = \frac{1}{2}\left\{\|u_t\|_2^2 + \left(1-\int_0^t h(s)ds\right)\|\nabla u\|_2^2 + \frac{a(t)}{2}\|\nabla u\|_2^4 + \int_\Omega (h\Box\nabla u)dx\right\}$$

we obtain, for $t\geq 0$

$$\begin{aligned}\mathcal{E}'(t) &= \tfrac{1}{2}\int_\Omega (h'\Box\nabla u)dx - \tfrac{h(t)}{2}\|\nabla u\|_2^2 + \tfrac{a'(t)}{4}\|\nabla u\|_2^4 + \int_\Omega u_t f(t,x)dx \\ &\leq \tfrac{1}{2}\int_\Omega (h'\Box\nabla u)dx - \tfrac{h(t)}{2}\|\nabla u\|_2^2 + \tfrac{a'(t)}{4}\|\nabla u\|_2^4 + \delta\|u_t\|_2^2 + \tfrac{1}{4\delta}\|f\|_2^2,\ \delta>0.\end{aligned} \tag{2.1}$$

These steps will be justified later. Clearly $\mathcal{E}'(t)$ is not necessarily negative at this stage, and this already eliminates several possible methods and techniques. We assume that the kernel is such that

$$1-\int_0^{+\infty} h(s)ds =: 1-\kappa > 0. \tag{2.2}$$

Next, we define the standard functionals: $\Phi_1(t) = \int_\Omega u_t u dx$,

$$\Phi_2(t) = -\int_\Omega u_t \int_0^t h(t-s)\left(u(t)-u(s)\right)dsdx.$$

The next functional was introduced in [17]:

$$\Phi_3(t) = \int_\Omega \int_0^t H_\gamma(t-s)\,|\nabla u(s)|^2\,dsdx,$$

where

$$H_\gamma(t) := \gamma(t)^{-1}\int_t^\infty h(s)\gamma(s)ds, \tag{2.3}$$

and $\gamma(t)$ will be determined later (see **(H3)** below). The modified energy we will work with is given by

$$L(t) := \mathcal{E}(t) + \sum_{i=1}^3 \lambda_i\Phi_i(t) \tag{2.4}$$

for some $\lambda_i>0$, $i=1,2,3$, to be determined.

The next result states that $L(t)$ and $\mathcal{E}(t)+\Phi_3(t)$ are equivalent.

Proposition 2.1. *There exist constants $\rho_i>0$, $i=1,2$, such that*

$$\rho_1[\mathcal{E}(t)+\Phi_3(t)] \leq L(t) \leq \rho_2[\mathcal{E}(t)+\Phi_3(t)]$$

for all $t\geq 0$ and small enough λ_i, $i=1,2$.

Proof. By Poincaré inequality we have

$$\Phi_1(t) = \int_\Omega u_t u dx \leq \frac{1}{2}\|u_t\|_2^2 + \frac{C_p}{2}\|\nabla u\|_2^2,$$

where C_p is the Poincaré constant, and also

$$\Phi_2(t) \leq \frac{1}{2}\|u_t\|_2^2 + \frac{C_p}{2}\kappa \int_\Omega (h\Box\nabla u)dx$$

where we have used the inequalities

$$\begin{aligned} &\int_0^t h(t-s)\,(u(t)-u(s))\,ds \\ &\leq \int_0^t \sqrt{h(t-s)}\sqrt{h(t-s)}\,(u(t)-u(s))\,ds \\ &\leq \left(\int_0^t h(t-s)ds\right)^{1/2}\left(\int_0^t h(t-s)\,(u(t)-u(s))^2\,ds\right)^{1/2}. \end{aligned}$$

The last two estimates imply

$$\begin{aligned} L(t) \leq& \tfrac{1}{2}(1+\lambda_1+\lambda_2)\|u_t\|_2^2 + \tfrac{1}{2}\left(1-\int_0^t h(s)ds + \lambda_1 C_p\right)\|\nabla u\|_2^2 \\ &+\tfrac{1}{2}(1+\lambda_2 C_p\kappa)\int_\Omega (h\Box\nabla u)dx + \lambda_3\Phi_3(t). \end{aligned}$$

On the other hand we have

$$\begin{aligned} 2L(t) \geq& (1-\lambda_1-\lambda_2)\|u_t\|_2^2 + (1-\lambda_2 C_p\kappa)\int_\Omega (h\Box\nabla u)dx \\ &+[1-\kappa-\lambda_1 C_p]\|\nabla u\|_2^2 + 2\lambda_3\Phi_3(t). \end{aligned}$$

Therefore, $\rho_1[\mathcal{E}(t)+\Phi_3(t)] \leq L(t) \leq \rho_2[\mathcal{E}(t)+\Phi_3(t)]$ for some constants $\rho_i > 0$, $i=1,2$, and small enough λ_i, $i=1,2$, such that $\lambda_1 < \min\{1,(1-\kappa)/C_p\}$ and $\lambda_2 < \min\left\{\frac{1}{C_p\kappa}, 1-\lambda_1\right\}$. Proposition 2.1 is proved.

We will need the following identity: for continuous functions h and v defined on $(0,\infty)$ and $t \geq 0$

(2.5)

$$v(t)\int_0^t h(t-s)v(s)ds = \frac{1}{2}\left(\int_0^t h(s)ds\right)v^2(t) + \frac{1}{2}\int_0^t h(t-s)v^2(s)ds - \frac{1}{2}(h\Box v)(t).$$

The proof is straightforward.

The following lemma, which was proved in [1], plays a key role in the proofs of our results.

Lemma 2.1. *Let $\chi(t)$, $\sigma(t)$, $\beta(t) \in C[0,\infty)$. If there exists a positive function $\mu(t) \in C^1[0,\infty)$ such that*

$$0 \leq \sigma(t) \leq \frac{\mu(t)}{2}\left(\chi(t) - \frac{\mu'(t)}{\mu(t)}\right),\ \beta(t) \leq \frac{1}{2\mu(t)}\left(\chi(t)-\frac{\mu'(t)}{\mu(t)}\right),$$

then a nonnegative solution $v(t)$ of the following inequality

$$v'(t) \leq -\chi(t)v(t) + \sigma(t)v^2(t) + \beta(t)$$

such that $\mu(0)v(0) < 1$, satisfies the inequality $v(t) < \frac{1}{\mu(t)}$.

3. Asymptotic behavior

In this section we state and prove the main result of this paper. We first introduce some notation (see [11]). Let h and κ be as in (1.1) and (2.2), respectively. We set $\omega = \frac{1-\kappa}{\kappa}$, and for a measurable set $\mathcal{A} \subset \mathbf{R}^+$, we define the probability measure $\hat{h}$ by

$$\hat{h}(\mathcal{A}) := \frac{1}{\kappa}\int_{\mathcal{A}} h(s)ds. \tag{3.1}$$

The flatness set and the flatness rate of h are defined by

$$\Omega_h := \left\{s \in \mathbf{R}^+ : h(s) > 0 \; and \; h'(s) = 0\right\} \tag{3.2}$$

and

$$\mathcal{R}_h := \hat{h}(\Omega_h), \tag{3.3}$$

respectively. Also, we define

$$\tilde{\Omega}_{ht} := \left\{s \in \mathbf{R}^+ : 0 \le s \le t, \; h(t-s) > 0 \; and \; h'(t-s) = 0\right\},$$

and let $t_* > 0$ be a number such that $\int_0^{t_*} h(s)ds = h_* > 0$.
We impose the following assumptions on the kernel $h(t)$.

- **(H1)** $h(t) \ge 0$ for all $t \ge 0$ and $0 < \kappa = \int_0^{+\infty} h(s)ds < 1$.
- **(H2)** $h(t)$ is an absolutely continuous function such that $h'(t) \le 0$ for almost all $t > 0$.
- **(H3)** There exists a non-decreasing function $\gamma(t) > 0$ such that $\eta(t) := \gamma'(t)/\gamma(t)$ is a decreasing function and $\int_0^{+\infty} h(s)\gamma(s)ds < +\infty$.
- **(H4)** The function $a(t)$ is a continuously differentiable, and $f \in L^2(\Omega)$ is a continuous function in t.

Remark 3.1. Note that the assumption **(H3)** is satisfied for a broad class of functions including polynomials and exponential functions. Moreover, we are considering kernels satisfying **(H2)** and **(H3)** just for simplicity. Our approach can be applied for other more general kernels as well. In particular, for occasionally increasing kernels (see [17]).

Theorem 3.1. *Let the hypotheses (H1)-(H4) be satisfied, and let* $H_\gamma(0) < \frac{9-\kappa}{8}\kappa$ *and* $\mathcal{R}_h < 1/4$, *where* $H_\gamma(t)$ *and* $\mathcal{R}_h$ *are as in (2.3) and (3.3), respectively. If there exists a positive function* $\mu(t) \in C^1[0,\infty)$ *such that*

$$\begin{array}{l} 0 \le \frac{a'_+(t)+4\lambda_2 a(t)}{\rho_1^2(1-\kappa)^2} \le \frac{\mu(t)}{2}\left(A(t) - \frac{\mu'(t)}{\mu(t)}\right), \\ \|f\|_2^2 \le \frac{B}{\mu(t)}\left(A(t) - \frac{\mu'(t)}{\mu(t)}\right) \end{array}$$

where $a'_+(t) := \sup\{0, a'(t)\}$ *and* B *is given in (3.20) below, then* $E(t) \le C/\mu(t)$ *for* $t \ge 0$ *in the cases:*

(a) $\lim_{t\to\infty}\eta(t)=\bar{\eta}\neq 0$ *and* $A(t)\equiv A=\rho_2^{-1}\max\{C_2,\lambda_3\bar{\eta}\}$ *(C_2 is as in (3.19) and λ_3 will be chosen), or*

(b) $\lim_{t\to\infty}\eta(t)=0$ *and* $A(t)=A\eta(t)=\rho_2^{-1}\max\{1,\lambda_3\}$, $t\geq 0$ *for some positive constant C provided that* $\mu(0)L(0)<1$.

Remark 3.2. The conditions imposed on $\mathcal{R}_h$ and $H_\gamma(0)$ may be relaxed with a trade-off on γ. Moreover, the existence of such a function μ is illustrated by some examples given in Section 4.

Proof of Theorem 3.1. A differentiation of $\Phi_1(t)$ with respect to t along trajectories of (1.1) gives

$$\Phi_1'(t)=\|u_t\|_2^2-\|\nabla u\|_2^2+\int_\Omega\nabla u\int_0^t h(t-s)\nabla u(s)dsdx-a(t)\|\nabla u\|_2^4+\int_\Omega ufdx$$

and, by the identity (2.5), we obtain

(3.4)
$$\begin{aligned}\Phi_1'(t)\leq{}&\|u_t\|_2^2-(1-\tfrac{\kappa}{2})\|\nabla u\|_2^2+\tfrac{1}{2}\int_0^t h(t-s)\|\nabla u(s)\|_2^2ds-\tfrac{1}{2}\int_\Omega(h\Box\nabla u)dx\\&-a(t)\|\nabla u\|_2^4+\delta_1C_p\|\nabla u\|_2^2+\tfrac{1}{4\delta_1}\|f\|_2^2,\ \delta_1>0.\end{aligned}$$

For $\Phi_2(t)$ we have

$$\begin{aligned}\Phi_2'(t)={}&-\int_\Omega u_{tt}\int_0^t h(t-s)(u(t)-u(s))dsdx\\&-\int_\Omega u_t\left[\int_0^t h'(t-s)(u(t)-u(s))ds+u_t\int_0^t h(s)ds\right]dx\end{aligned}$$

or

$$\begin{aligned}\Phi_2'(t)={}&-\int_\Omega\left[\left(1-\int_0^t h(s)ds\right)\Delta u+\int_0^t h(t-s)(\Delta u(t)-\Delta u(s))ds\right.\\&\left.+a(t)\|\nabla u\|_2^2\Delta u+f(t,x)\right]\int_0^t h(t-s)(u(t)-u(s))dsdx-\left(\int_0^t h(s)ds\right)\|u_t\|_2^2\\&-\int_\Omega u_t\int_0^t h'(t-s)(u(t)-u(s))dsdx.\end{aligned}$$

Therefore

(3.5)
$$\begin{aligned}\Phi_2'(t)={}&\left(1-\int_0^t h(s)ds\right)\int_\Omega\nabla u\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))dsdx\\&+a(t)\|\nabla u\|_2^2\int_\Omega\nabla u\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))dsdx\\&-\int_\Omega f(t,x)\int_0^t h(t-s)(u(t)-u(s))dsdx-\left(\int_0^t h(s)ds\right)\|u_t\|_2^2\\&+\int_\Omega\left|\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))ds\right|^2dx-\int_\Omega u_t\int_0^t h'(t-s)(u(t)-u(s))dsdx.\end{aligned}$$

Now we estimate the terms on the right-hand side of expression (3.5). We start with the second term, for which clearly we have

(3.6)
$$\begin{aligned}&a(t)\|\nabla u\|_2^2\int_\Omega\nabla u.\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))dsdx\\&\leq\tfrac{a(t)}{2}\|\nabla u\|_2^2\left[\|\nabla u\|_2^2+\left(\int_0^t h(s)ds\right)\int_\Omega(h\Box\nabla u)dx\right]\\&\leq\tfrac{a(t)}{2}\|\nabla u\|_2^4+\tfrac{a(t)}{2}\|\nabla u\|_2^2\left(\int_0^t h(s)ds\right)\int_\Omega(h\Box\nabla u)dx\\&\leq\tfrac{2a(t)}{(1-\kappa)^2}\mathcal{E}^2(t)+\tfrac{2a(t)}{(1-\kappa)^2}\mathcal{E}^2(t)=\tfrac{4a(t)}{(1-\kappa)^2}\mathcal{E}^2(t).\end{aligned}$$

Next, for the third term on the right-hand side of (3.5) we have

$$\begin{aligned} &\int_\Omega f(t,x)\int_0^t h(t-s)\,(u(t)-u(s))\,dsdx \\ &\le \delta_2 C_p\left(\int_0^t h(s)ds\right)\int_\Omega (h\Box\nabla u)dx+\tfrac{1}{4\delta_2}\|f\|_2^2,\ \delta_2>0. \end{aligned} \tag{3.7}$$

Regarding the first term on the right-hand side of (3.5), for any measurable sets $\mathcal{A}$ and $\mathfrak{Q}$ such that $\mathcal{A} = \mathbf{R}^+\backslash\mathfrak{Q}$, we have

$$\begin{aligned} &\int_\Omega \nabla u\int_0^t h(t-s)\,(\nabla u(t)-\nabla u(s))\,dsdx \\ &= \int_\Omega \nabla u\int_{\mathcal{A}_t} h(t-s)\,(\nabla u(t)-\nabla u(s))\,dsdx \\ &+\int_\Omega \nabla u\int_{\mathfrak{Q}_t} h(t-s)\,(\nabla u(t)-\nabla u(s))\,dsdx \\ &\le \int_\Omega \nabla u\int_{\mathcal{A}_t} h(t-s)\,(\nabla u(t)-\nabla u(s))\,dsdx \\ &+\left(\int_{\mathfrak{Q}_t} h(t-s)ds\right)\|\nabla u\|_2^2-\int_\Omega \nabla u\int_{\mathfrak{Q}_t} h(t-s)\nabla u(s)dsdx, \end{aligned} \tag{3.8}$$

where we have adopted the notation: $\mathcal{B}_t := \mathcal{B}\cap[0,t]$. It is easy to see that for $\delta_3>0$

$$\begin{aligned} &\int_\Omega \nabla u\int_{\mathcal{A}_t} h(t-s)\,(\nabla u(t)-\nabla u(s))\,dsdx \\ &\le \delta_3\|\nabla u\|_2^2+\tfrac{\kappa}{4\delta_3}\int_\Omega\int_{\mathcal{A}_t} h(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\int_\Omega \nabla u\int_{\mathfrak{Q}_t} h(t-s)\nabla u(s)dsdx \\ &\le \tfrac{1}{2}\left(\int_{\mathfrak{Q}_t} h(t-s)ds\right)\|\nabla u\|_2^2+\tfrac{1}{2}\int_{\mathfrak{Q}_t} h(t-s)\,\|\nabla u(s)\|_2^2\,ds. \end{aligned} \tag{3.10}$$

The inequalities (3.9) and (3.10) together with (3.8) imply

$$\begin{aligned} &\int_\Omega \nabla u\int_0^t h(t-s)\,(\nabla u(t)-\nabla u(s))\,dsdx \\ &\le \left(\delta_3+\tfrac{3}{2}\int_{\mathfrak{Q}_t} h(t-s)ds\right)\|\nabla u\|_2^2+\tfrac{\kappa}{4\delta_3}\int_\Omega\int_{\mathcal{A}_t} h(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx \\ &+\tfrac{1}{2}\int_{\mathfrak{Q}_t} h(t-s)\,\|\nabla u(s)\|_2^2\,ds \end{aligned} \tag{3.11}$$

where $\hat{h}$ is defined by formula (3.1).

Thus, it remains to estimate the last two terms on the right-hand side of (3.5). For the next to the last term we have

$$\begin{aligned} &\int_\Omega\left|\int_0^t h(t-s)\,(\nabla u(t)-\nabla u(s))\,ds\right|^2 dx \\ &\le (1+\tfrac{1}{\delta_4})\kappa\int_\Omega\int_{\mathcal{A}_t} h(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx \\ &+(1+\delta_4)\left(\int_{\mathfrak{Q}_t} h(t-s)ds\right)\int_\Omega\int_{\mathfrak{Q}_t} h(t-s)\,|\nabla u(t)-\nabla u(s)|^2\,dsdx,\ \delta_4>0. \end{aligned} \tag{3.12}$$

Finally, the last term on the right-hand side of (3.5) is estimated for any $\delta_5>0$ as follows

$$\begin{aligned} &\int_\Omega u_t\int_0^t h'(t-s)\,(u(t)-u(s))\,dsdx \\ &\le \delta_5\|u_t\|_2^2+\tfrac{C_p}{4\delta_5}\left(\int_0^t |h'(s)|\,ds\right)\int_\Omega(|h'|\Box\nabla u)dx \\ &\le \delta_5\|u_t\|_2^2-\tfrac{C_p}{4\delta_5}h(0)\int_\Omega(h'\Box\nabla u)dx. \end{aligned} \tag{3.13}$$

Taking into account (3.6)-(3.13), from (3.5), we obtain

$$(3.14)\quad \begin{aligned} \Phi_2'(t) \le & (1-h_*)\left[\delta_3 + \tfrac{3}{2}\int_{\Omega_t} h(t-s)ds\right]\|\nabla u\|_2^2 + (\delta_5 - h_*)\|u_t\|_2^2 \\ & +\delta_2 C_p\left(\int_0^t h(s)ds\right)\int_\Omega (h\Box\nabla u)dx + \tfrac{1}{4\delta_2}\|f\|_2^2 + \tfrac{4a(t)}{(1-\kappa)^2}\mathcal{E}^2(t) \\ & +\kappa\left[1+\tfrac{1-h_*}{4\delta_3}+\tfrac{1}{\delta_4}\right]\int_\Omega\int_{\mathcal{A}_t} h(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx \\ & +\tfrac{1}{2}(1-h_*)\int_{\Omega_t} h(t-s)\|\nabla u(s)\|_2^2 ds \\ & -\tfrac{C_p}{4\delta_5}h(0)\int_\Omega\int_{\mathcal{A}_t} h'(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx \\ & +(1+\delta_4)\left(\int_{\Omega_t} h(t-s)ds\right)\int_\Omega\int_{\Omega_t} h(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx. \end{aligned}$$

Further, a differentiation of $\Phi_3(t)$ yields

$$(3.15)\quad \begin{aligned} \Phi_3'(t) &= H_\gamma(0)\|\nabla u\|_2^2 + \int_0^t H_\gamma'(t-s)\|\nabla u(s)\|_2^2 ds \\ &= H_\gamma(0)\|\nabla u\|_2^2 - \int_0^t \tfrac{\gamma'(t-s)}{\gamma(t-s)}H_\gamma(t-s)\|\nabla u(s)\|_2^2 ds - \int_0^t h(t-s)\|\nabla u(s)\|_2^2 ds \\ &\le H_\gamma(0)\|\nabla u\|_2^2 - \eta(t)\int_0^t H_\gamma(t-s)\|\nabla u(s)\|_2^2 ds - \int_0^t h(t-s)\|\nabla u(s)\|_2^2 ds, \end{aligned}$$

where we have used the fact that $\eta(t) := \gamma'(t)/\gamma(t)$ is a non-increasing function.

Taking into account the estimates (2.1), (3.4), (3.14) and (3.15), we can write

$$(3.16)\quad \begin{aligned} L'(t) \le & \tfrac{1}{2}\int_\Omega (h'\Box\nabla u)dx - \tfrac{\lambda_2 C_p}{4\delta_5}h(0)\int_\Omega\int_{\mathcal{A}_t} h'(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx \\ & + \left[\tfrac{a'(t)}{4} - \lambda_1 a(t)\right]\|\nabla u\|_2^4 + \left[\delta + \lambda_1 + (\delta_5 - h_*)\lambda_2\right]\|u_t\|_2^2 \\ & +\tfrac{1}{4}\left[\tfrac{1}{\delta}+\tfrac{\lambda_1}{\delta_1}+\tfrac{\lambda_2}{\delta_2}\right]\|f\|_2^2 + \tfrac{4a(t)}{(1-\kappa)^2}\lambda_2\mathcal{E}^2(t) - \lambda_3\eta(t)\Phi_3(t) \\ & + \left\{\lambda_1\delta_1 C_p + \lambda_2(1-h_*)\left[\delta_3 + \tfrac{3}{2}\int_{\Omega_t} h(t-s)ds\right] + \lambda_3 H_\gamma(0) - \lambda_1(1-\tfrac{\kappa}{2})\right\} \\ & \times\|\nabla u\|_2^2 + \left(\tfrac{\lambda_1}{2} + \tfrac{\lambda_2(1-h_*)}{2} - \lambda_3\right)\int_0^t h(t-s)\|\nabla u(s)\|_2^2 ds \\ & +\lambda_2\kappa\left[\delta_2 C_p + 1 + \tfrac{1-h_*}{4\delta_3} + \tfrac{1}{\delta_4}\right]\int_\Omega\int_{\mathcal{A}_t} h(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx \\ & + \left[\lambda_2\delta_2 C_p\kappa + (1+\delta_4)\lambda_2\int_{\Omega_t} h(t-s)ds - \tfrac{\lambda_1}{2}\right] \\ & \times\int_\Omega\int_{\Omega_t} h(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 dsdx. \end{aligned}$$

Consider the following sets (see [11])

$$\mathcal{A}_n := \left\{s\in\mathbf{R}^+ : nh'(s) + h(s) \le 0\right\},\ n\in\mathbf{N},$$

and observe that

$$\bigcup_n \mathcal{A}_n = \mathbf{R}^+ \backslash \left\{\Omega_h \cup \mathcal{N}_h\right\},$$

where $\mathcal{N}_h$ is the null set where h' is not defined and Ω_h is as in (3.2).

Furthermore, denoting $\Omega_n = \mathbf{R}^+\backslash\mathcal{A}_n$, and taking into account that $\Omega_{n+1}\subset\Omega_n$ for all n and $\bigcap_n \Omega_n = \Omega_h\cup\mathcal{N}_h$, we obtain $\lim_{n\to\infty}\hat{h}(\Omega_n) = \hat{h}(\Omega_h)$. Define the sets

$$\tilde{A}_{nt} := \left\{s\in\mathbf{R}^+ : 0\le s\le t,\ nh'(t-s) + h(t-s)\le 0\right\},\ n\in\mathbf{N}.$$

In (3.16), we take $\mathcal{A} := \bar{A}_{nt}$, $\Omega := \bar{\Omega}_{nt}$ and $\lambda_1 = (h_* - \varepsilon)\lambda_2$ for some small enough $\varepsilon > 0$, to obtain

(3.17)

$$\begin{aligned}
&L'(t) \le \tfrac{1}{4}\left[1 - \tfrac{\lambda_2 C_p}{\delta_5} h(0)\right] \int_\Omega \int_{\bar{A}_{nt}} h'(t-s) \left|\nabla u(t) - \nabla u(s)\right|^2 ds dx \\
&+ \left[\tfrac{a'(t)}{4} - \lambda_1 a(t)\right] \|\nabla u\|_2^4 + \tfrac{1}{4}\left[\tfrac{1}{\delta} + \tfrac{(h_*-\varepsilon)\lambda_2}{\delta_1} + \tfrac{\lambda_2}{\delta_2}\right] \|f\|_2^2 + \tfrac{4a(t)}{(1-\kappa)^2}\lambda_2 \mathcal{E}^2(t) \\
&-\lambda_3 \eta(t)\Phi_3(t) + \Big\{\delta_1 C_p (h_* - \varepsilon)\lambda_2 + \lambda_2 (1-h_*)\left[\delta_3 + \tfrac{3}{2}\int_{\bar{\Omega}_{nt}} h(t-s) ds\right] \\
&+\lambda_3 H_\gamma(0) - (h_* - \varepsilon)\lambda_2 (1 - \tfrac{\kappa}{2})\Big\} \|\nabla u\|_2^2 + [\delta + (\delta_5 - \varepsilon)\lambda_2] \|u_t\|_2^2 \\
&+ \left(\tfrac{(1-\varepsilon)\lambda_2}{2} - \lambda_3\right) \int_0^t h(t-s) \|\nabla u(s)\|_2^2 ds \\
&+ \left[\lambda_2 \kappa \left(1 + \delta_2 C_p + \tfrac{1-h_*}{4\delta_3} + \tfrac{1}{\delta_4}\right) - \tfrac{1}{4n}\right] \int_\Omega \int_{\bar{A}_{nt}} h(t-s) \left|\nabla u(t) - \nabla u(s)\right|^2 ds dx \\
&+\lambda_2 \left[\delta_2 C_p \kappa + (1+\delta_4)\int_{\bar{\Omega}_{nt}} h(t-s) ds - \tfrac{h_* - \varepsilon}{2}\right] \\
&\times \int_\Omega \int_{\bar{\Omega}_{nt}} h(t-s) \left|\nabla u(t) - \nabla u(s)\right|^2 ds dx.
\end{aligned}$$

Noticing that $\frac{a'(t)}{4}\|\nabla u\|_2^4 \le \frac{a'_+(t)}{(1-\kappa)^2}\mathcal{E}^2(t)$, we chose $\delta_5 = \varepsilon/2$, $\lambda_3 = \frac{(1-\varepsilon)\lambda_2}{2}$ and λ_2 satisfying $\lambda_2 < \frac{\varepsilon}{2C_p h(0)}$, and

$$\lambda_2 \kappa \left(1 + \delta_2 C_p + \frac{1-h_*}{4\delta_3} + \frac{1}{\delta_4}\right) - \frac{1}{4n} \le -C_1,\ C_1 > 0,$$

to get

(3.18)

$$\begin{aligned}
&L'(t) \le -\lambda_1 a(t) \|\nabla u\|_2^4 + \tfrac{1}{4}\left[\tfrac{1}{\delta} + \tfrac{(h_*-\varepsilon)\lambda_2}{\delta_1} + \tfrac{\lambda_2}{\delta_2}\right] \|f\|_2^2 + \tfrac{a'_+(t) + 4\lambda_2 a(t)}{(1-\kappa)^2} \mathcal{E}^2(t) \\
&-\lambda_3 \eta(t)\Phi_3(t) + \lambda_2 \Big\{\delta_1 C_p (h_* - \varepsilon) + (1-h_*)\left[\delta_3 + \tfrac{3}{2}\int_{\bar{\Omega}_{nt}} h(t-s) ds\right] \\
&+\tfrac{(1-\varepsilon)}{2} H_\gamma(0) - (h_* - \varepsilon)(1 - \tfrac{\kappa}{2})\Big\} \|\nabla u\|_2^2 + [\delta - \tfrac{\varepsilon}{2}\lambda_2] \|u_t\|_2^2 \\
&+\lambda_2 \left[\delta_2 C_p \kappa + (1+\delta_4)\int_{\bar{\Omega}_{nt}} h(t-s) ds - \tfrac{h_* - \varepsilon}{2}\right] \\
&\times \int_\Omega \int_{\bar{\Omega}_{nt}} h(t-s) \left|\nabla u(t) - \nabla u(s)\right|^2 ds dx - C_1 \int_\Omega \int_{\bar{A}_{nt}} h(t-s) \left|\nabla u(t) - \nabla u(s)\right|^2 ds dx.
\end{aligned}$$

For small enough ε and δ_4 and large enough values of n and t_*, we have $\hat{h}(\Omega_h) < 1/4$ and

$$(1+\delta_4)\kappa \hat{h}(\Omega_n) - \frac{h_* - \varepsilon}{2} < 0,$$

implying

$$(1+\delta_4)\int_{\bar{\Omega}_{nt}} h(t-s) ds - \frac{h_* - \varepsilon}{2} < 0$$

$$\frac{3}{2}\kappa(1-h_*)\int_{\bar{\Omega}_{nt}} h(t-s) ds < \rho(h_* - \varepsilon)\left(1 - \frac{\kappa}{2}\right)$$

with $\rho = \frac{1}{2}\left[1 - \frac{\kappa(1+3h_*)}{4(h_*-\varepsilon)(2-\kappa)}\right]$. Note that $\rho < 1/2$. For the remaining $1-\rho$ we require that

$$\frac{1-\varepsilon}{2} H_\gamma(0) < (1-\rho)(h_* - \varepsilon)\left(1 - \frac{\kappa}{2}\right).$$

This relation is satisfied for $H_\gamma(0) < \frac{9-\kappa}{8}\kappa$ and sufficiently large t_*. Therefore, for small enough δ_i, $i = 1, 2, 3$ and δ we obtain

(3.19)
$$L'(t) \le -C_2\mathcal{E}(t) + \frac{a'_+(t) + 4\lambda_2 a(t)}{(1-\kappa)^2}\mathcal{E}^2(t) + \frac{1}{4}\left[\frac{1}{\delta} + \frac{\kappa\lambda_2}{\delta_1} + \frac{\lambda_2}{\delta_2}\right]\|f\|_2^2 - \lambda_3\eta(t)\Phi_3(t),$$

for $t \ge t_*$ and some positive constant C_2.

(a) If $\lim_{t\to\infty}\eta(t) = \bar{\eta} \neq 0$, then $\eta(t) \ge \bar{\eta}$, and by Proposition 2.1 there exist $C_3 > 0$ such that

$$L'(t) \le -C_3 L(t) + \frac{a'_+(t) + 4\lambda_2 a(t)}{\rho_1^2(1-\kappa)^2}L^2(t) + \frac{1}{4}\left(\frac{1}{\delta} + \frac{\kappa\lambda_2}{\delta_1} + \frac{\lambda_2}{\delta_2}\right)\|f\|_2^2. \tag{3.20}$$

Applying Lemma 3.1 with

$$\chi(t) = C_3,\ \sigma(t) = \frac{a'_+(t) + 4\lambda_2 a(t)}{\rho_1^2(1-\kappa)^2},\ \beta(t) = \frac{1}{4}\left(\frac{1}{\delta} + \frac{\kappa\lambda_2}{\delta_1} + \frac{\lambda_2}{\delta_2}\right)\|f\|_2^2$$

we infer that $E(t) \le C/\mu(t)$, $t \ge 0$ for some positive constant C.

(b) If $\lim_{t\to\infty}\eta(t) = 0$, there exist $\hat{t} \ge t_*$ such that $\eta(t) \le C_2$ for all $t \ge \hat{t}$. Therefore

$$L'(t) \le -C_4\eta(t)L(t) + \frac{a'_+(t) + 4\lambda_2 a(t)}{\rho_1^2(1-\kappa)^2}L^2(t) + \frac{1}{4}\left(\frac{1}{\delta} + \frac{\kappa\lambda_2}{\delta_1} + \frac{\lambda_2}{\delta_2}\right)\|f\|_2^2 \tag{3.21}$$

for some $C_4 > 0$. Taking $\chi(t) = C_4\eta(t)$ we conclude that

$$E(t) \le C/\mu(t),\ t \ge 0.$$

This completes the proof of Theorem 3.1.

4. Examples

First, as it was mentioned above (see Remark 3.1), polynomials and exponential functions satisfy the assumption **(H3)**. Indeed, for $\gamma(t) = (1+t)^\alpha$, $\alpha > 0$, we have $\eta(t) = \gamma'(t)/\gamma(t) = \alpha(1+t)^{-1}$, and for $\gamma(t) = e^{\alpha t}$, $\alpha > 0$, we find $\eta(t) = \gamma'(t)/\gamma(t) = \alpha$.

Next, we give two examples that illustrate both possible cases in Theorem 3.1.

Example 4.1. Let $\sigma(t) = \sigma_0 e^{\nu t}$ for some positive constants σ_0 and ν, which may result when $a(t) = ae^{\nu t}$) and $\beta(t) = \beta_0 e^{-\nu t}$ for some positive constant β_0. This situation can occur, for instance, if the function $f(t,x)$ is of the form $g(x)e^{-\frac{\nu}{2}t}$. Then $\mu(t) = \mu_0 e^{\nu t}$ with μ_0 satisfying $\sigma_0 \le \frac{\mu_0}{2}(\chi_0 - \nu)$ and $\beta_0 \le \frac{1}{2\mu_0}(\chi_0 - \nu)$, where $\chi_0 = C_3$ (see formula (3.20)). This is possible when $\nu < \chi_0$ and $4\sigma_0\beta_0 < (\chi_0 - \nu)^2$.

Example 4.2. Assume that $C_4\eta(t) = \alpha(1+t)^{-1}$ (see (3.21)). This can occur if $\gamma(t)$ is of the form $(1+t)^\alpha$), $\sigma(t) \le \sigma_0(1+t)^{\nu_1}$ and $\beta(t) = \beta_0(1+t)^{\nu_2}$. For instance, this is the case when the functions $a(t)$ and $f(t,x)$ are of the form $a(1+t)^{\nu_1}$ and

$h(x)(1+t)^{\nu_2/2}$, respectively, with $\nu_1+\nu_2 \leq -2$, $\alpha > 2(\nu_1+1)$ and $16\sigma_0\beta_0 < \alpha^2$. Then, there exists a constant C such that $\mu(t) = C(1+t)^{\nu}$ with $\nu = \nu_1 + 1$.

Acknowledgment. The author is grateful for the financial support and the facilities provided by King Fahd University of Petroleum and Minerals.

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Поступила 17 июля 2012