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CHEEGER-GROMOLL TYPE METRICS ON THE (1,1)–TENSOR BUNDLES

E. PEYGHAN, A. TAYEBI AND L. NOURMOHAMMADIFAR

Department of Mathematics, Faculty of Science, Arak University, Arak, Iran
Department of Mathematics, Faculty of Science, University of Qom, Qom, Iran
E-mails: *epeyghan@gmail.com; akbar.tayebi@gmail.com*

Abstract. Using a Riemannian metric on a differentiable manifold, a Cheeger-Gromoll type metric is introduced on the (1,1)–tensor bundle of the manifold. Then the Levi-Civita connection, Riemannian curvature tensor, Ricci tensor, scalar curvature and sectional curvature of this metric are calculated. Also, a para-Nordenian structure on the the (1,1)–tensor bundle with this metric is constructed and the geometric properties of this structure are studied.

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1. Introduction

Geometry of the tangent bundle TM of an n-dimensional Riemannian manifold (M, g) with Sasaki metric has been extensively studied since the $60'$s. Nevertheless, the rigidity of this metric has incited some geometers to tackle the problem of construction and study of other metrics on TM (see [1, 21]). The Cheeger-Gromoll metric has appeared as a nicely fitted one to overcome this rigidity (see [6]). Then using the concept of naturality, O. Kowalski and M. Sekizawa [17] have given a complete classification of metrics which are naturally constructed from a metric g on the base M, assuming that M is oriented. Other presentations of the basic results from [17] (involving also the non-oriented case and something more) can be found in [16]. These metrics, called in [2] g-natural metrics on TM, had been extensively studied during last years. It has been proved that some subclasses of g-natural metrics such as natural diagonal lift type metrics offer very interesting geometrical features and research horizons (see [11, 12]).

Fiber bundles have important applications in geometry and modern theoretical physics. They are used in a number of physical fields, such as gauge theory, which was invented by Weyl [7]. Tensor bundles T^p_qM of type (p, q) over a differentiable manifold M are particular examples of fiber bundles, which were studied by a number of mathematicians such as Ledger, Yano, Cengiz, Gezer and Salimov (see [3] – [5], [14, 18, 19]). The tangent bundle TM and cotangent bundle T^*M are the special cases of tensor bundles (see, e.g., [8] – [13], which deal with g-natural structures on TM and T^*M).

In [20], Salimov and Gezer have introduced the Sasaki metric ${}^Sg$ on the (1,1)–tensor bundle T_1^1M of a Riemannian manifold M and have calculated the Levi-Civita connection of this metric and its Riemannian curvature tensor.

In this paper, using a similar method, applied to a tangent bundle, we define the Cheeger-Gromoll type metric ${}^{CG}g$ on T_1^1M, which is an extension of Sasaki metric. Then we calculate the Levi-Civita connection, Riemannian curvature tensor, Ricci tensor, scalar curvature and sectional curvature of this metric and establish some relationships between the geometric properties of the base manifold (M, g) and $(T_1^1M, {}^{CG}g)$. Finally, we introduce a para-Nordenian structure on $(T_1^1M, {}^{CG}g)$ and find equivalence conditions for para-Kählerian properties of this structure.

2. Preliminaries

Let M be an C^∞ manifold of finite dimension n. Then the set $T_1^1M = \coprod_{p\in M} T_1^1(p)$ is defined to be the tensor bundle of type $(1,1)$ over M, where $\coprod$ denotes the disjoint union of the tensor spaces $T_1^1(p)$ for all $p \in M$. For any point $\widetilde{p} \in T_1^1M$ the surjective correspondence $\widetilde{p} \to p$ determines the natural projection $\pi : T_1^1M \to M$. The projection π defines the natural differentiable manifold structure of T_1^1M, that is, T_1^1M is a C^∞-manifold of dimension $n + n^2$. A local coordinate neighborhood $\{(U; x^j, j = 1, \ldots, n)\}$ in M induces on T_1^1M a local coordinate neighborhood

$$\{\pi^{-1}(U); x^j, x^{\bar{j}} = t^i_j, j = 1, \ldots, n, \bar{j} = n + j(\bar{j} = n + 1, \ldots, n + n^2)\},$$

where $x^{\bar{j}} = t^i_j$ are the components of the $(1,1)$–tensor field t in each $(1,1)$–tensor space $T_1^1(p)$ $(p \in U)$ with respect to the natural base.

We denote by $\Im_1^1(M)$ the module over $F(M)$ of all C^∞ tensor fields of type $(1,1)$ on M, where $F(M)$ is the ring of real-valued C^∞ functions on M. If $\alpha \in \Im_1^1(M)$, then by contraction it is regarded as a function on T_1^1M, which we denote by $\imath\alpha$. If α has the local expression $\alpha = \alpha_i^j \frac{\partial}{\partial x^j} \otimes dx^i$ in a coordinate neighborhood $U(x^j) \subset M$, then $\imath(\alpha) = \alpha(t)$ has the local expression $\imath\alpha = \alpha_i^j t^i_j$ with respect to the coordinates $(x^j, x^{\bar{j}})$ in $\pi^{-1}(U)$. Suppose that $A \in \Im_1^1(M)$. Then there is a unique vector field ${}^VA \in \Im_0^1(T_1^1M)$ such that for $\alpha \in \Im_1^1(M)$

$$ {}^VA(\imath\alpha) = \alpha(A) \circ \pi = {}^V(\alpha(A)), \tag{2.1}$$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in F(M)$ (see [18]. We note that the vertical lift ${}^Vf = f \circ \pi$ of an arbitrary function $f \in F(M)$ is constant along each fibre $\pi^{-1}(p)$. Put ${}^VA = {}^VA^k\partial_k + {}^VA^{\bar{k}}\partial_{\bar{k}}$, where

$$\partial_k := \frac{\partial}{\partial x^k}, \ \ \partial_{\bar{k}} := \frac{\partial}{\partial x^{\bar{k}}} = \frac{\partial}{\partial t^h_k}, \ \ {}^VA^{\bar{k}} := {}^VA^h_k.$$

Then by (2.1), we obtain ${}^VA^k = 0$ and ${}^VA^{\bar{k}} = A^h_k$. Thus the vertical lift ${}^VA$ of A has the components

$$ {}^{V}A = \begin{pmatrix} {}^{V}A^{j} \\ {}^{V}A^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^{i}_{j} \end{pmatrix}, \tag{2.2} $$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in T^1_1M (see [3]). Let $\mathcal{L}_V$ be the Lie derivation with respect to $V \in \Im^1_0(M)$. The complete lift ${}^{C}V$ of V to T^1_1M is defined by

$$ {}^{C}V(\imath\alpha) = \imath(\mathcal{L}_V\alpha), \tag{2.3} $$

for $\alpha \in \Im^1_1(M)$ (see [18]). If ${}^{C}V = {}^{C}V^k\partial_k + {}^{C}V^{\bar{k}}\partial_{\bar{k}}$, then by (2.3), it follows that the complete lift ${}^{C}V$ has the following components

$$ {}^{C}V = \begin{pmatrix} {}^{C}V^{j} \\ {}^{C}V^{\bar{j}} \end{pmatrix} = \begin{pmatrix} V^{j} \\ t^{m}_{j}(\partial_m V^{i}) - t^{i}_{m}(\partial_j V^{m}) \end{pmatrix}, \tag{2.4} $$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in T^1_1M (see [3]).

The horizontal lift ${}^{H}V \in \Im^1_0(T^1_1M)$ of $V \in \Im^1_0(M)$ to T^1_1M is defined by (see [18]): ${}^{H}V(\imath\alpha) = \imath(\nabla_V\alpha)$, $\alpha \in \Im^1_1(M)$, where ∇ is a symmetric affine connection on M. It is easy to see that ${}^{H}V$ has the components

$$ {}^{H}V = \begin{pmatrix} {}^{H}V^{j} \\ {}^{H}V^{\bar{j}} \end{pmatrix} = \begin{pmatrix} V^{j} \\ V^{s}(\Gamma^{m}_{sj}t^{i}_{m} - \Gamma^{i}_{sm}t^{m}_{j}) \end{pmatrix}, \tag{2.5} $$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in T^1_1M, where Γ^k_{ij} are the local components of ∇ on M. Let $U(x^h)$ be a local chart in M. Using (2.2) and (2.5) we obtain

$$ e_j := {}^{H}\partial_j = {}^{H}(\delta^h_j\partial_h) = \delta^h_j\partial_h + (\Gamma^s_{jh}t^k_s - \Gamma^k_{js}t^s_h)\partial_{\bar{h}}, \tag{2.6} $$

$$ e_{\bar{j}} := {}^{V}(\partial_i \otimes dx^j) = {}^{V}(\delta^k_i\delta^j_h\partial_k \otimes dx^h) = \delta^k_i\delta^j_h\partial_{\bar{h}}, \tag{2.7} $$

where δ^h_j is the Kronecker symbol and $\bar{j} = n+1, \ldots, n+n^2$. These $n+n^2$ vector fields are linearly independent and generate the horizontal distribution of ∇ and the vertical distribution of T^1_1M, respectively. Indeed, we have ${}^{H}X = X^j e_j$ and ${}^{V}A = A^i_j e_{\bar{j}}$ (see [20]). The set $\{e_\beta\} = \{e_j, e_{\bar{j}}\}$ is called the frame adapted to the affine connection ∇ on $\pi^{-1}(U) \subset T^1_1M$.

3. A Cheeger-Gromoll type metric on T^1_1M

For each $p \in M$ the extension of a scalar product g, denoted by G, is defined on the tensor space $\pi^{-1}(p) = T^1_1(p)$ by $G(A,B) = g_{it}g^{jl}A^i_jB^t_l$, $\quad A, B \in \Im^1_1(p)$, where g_{ij} and g^{ij} are the local covariant and contravariant tensors, respectively, associated with the metric g on M.

We consider a Riemannian metric ${}^{CG}g$ of Cheeger-Gromoll type defined on T^1_1M as follows:

$$ {}^{CG}g({}^{V}A, {}^{V}B) = {}^{V}(aG(A,B) + bG(t,A)G(t,B)), \tag{3.1} $$

$$ {}^{CG}g({}^{V}A, {}^{H}Y) = 0, \tag{3.2} $$

$$ {}^{CG}g({}^{H}X, {}^{H}Y) = {}^{V}(g(X,Y)), \tag{3.3} $$

for all $X, Y \in \Im^1_0(M)$ and $A, B \in \Im^1_1(M)$, where a and b are smooth functions of $\tau = ||t||^2 = t^i_j t^t_l g_{it}(x) g^{jl}(x)$ defined on $T^1_1 M$ and satisfying the conditions $a > 0$ and $a + b\tau > 0$. The symmetric $(2n \times 2n)$–type matrix

$$\begin{pmatrix} g_{jl} & 0 \\ 0 & a g^{jl} g_{it} + b \bar{t}^j_i \bar{t}^l_t \end{pmatrix}, \tag{3.4}$$

associated with the metric ${}^{CG}g$ in the adapted frame $\{e_\beta\}$, has the inverse

$$\begin{pmatrix} g^{jl} & 0 \\ 0 & \frac{1}{a} g_{jl} g^{it} - \frac{b}{a(a+b\tau)} t^i_j t^t_l \end{pmatrix}, \quad \text{where} \quad \bar{t}^j_i = g^{jh} g_{ik} t^k_h. \tag{3.5}$$

Notice that in the special case where $a = 1$ and $b = 0$, we have the Sasaki metric ${}^S g$ (see [20]). Let $\varphi = \varphi^i_j \frac{\partial}{\partial x^i} \otimes dx^j$ be a tensor field defined on M. Then $\gamma\varphi = (t^m_j \varphi^i_m) \frac{\partial}{\partial x^J}$ and $\widetilde{\gamma}\varphi = (t^i_m \varphi^m_j) \frac{\partial}{\partial x^J}$ are vector fields defined on $T^1_1 M$, and the bracket operation of vertical and horizontal vector fields is given by the formulas:

$$[{}^V A, {}^V B] = 0, \quad [{}^H X, {}^V A] = {}^V (\nabla_X A), \tag{3.6}$$

$$[{}^H X, {}^H Y] = {}^H [X, Y] + (\widetilde{\gamma} - \gamma) R(X, Y), \tag{3.7}$$

where R denotes the curvature tensor field of the connection ∇ and $\widetilde{\gamma} - \gamma : \varphi \to \Im^1_0(T^1_1 M)$ is an operator defined by $(\widetilde{\gamma} - \gamma)\varphi = \begin{pmatrix} 0 \\ t^i_m \varphi^m_j - t^m_j \varphi^i_m \end{pmatrix}$ for $\varphi \in \Im^1_1(M)$.

Proposition 3.1. *The Levi-Civita connection ${}^{CG}\nabla$ associated with the Riemannian metric ${}^{CG}g$ on the tensor bundle $T^1_1 M$ has the following form:*

$$\begin{aligned}
{}^{CG}\nabla_{{}^H X} {}^H Y &= {}^H (\nabla_X Y) + \frac{1}{2}(\widetilde{\gamma} - \gamma) R(X, Y), \\
{}^{CG}\nabla_{{}^V A} {}^H Y &= \frac{a}{2} {}^H \Big(g^{bl} R(t_b, A_l) Y + g_{at}(t^a (g^{-1} \circ R(., Y) \widetilde{A}^t)) \Big), \\
{}^{CG}\nabla_{{}^H X} {}^V B &= {}^V (\nabla_X B) + \frac{a}{2} {}^H \Big(g^{bj} R(t_b, B_j) X + g_{ai}(t^a (g^{-1} \circ R(., X) \widetilde{B}^i)) \Big), \\
{}^{CG}\nabla_{{}^V A} {}^V B &= L(G(t, A)^V B + G(t, B)^V A) + M G(A, B)^V t + N G(t, A) G(t, B)^V t,
\end{aligned}$$

where $L := \frac{a'}{a}$, $M := \frac{-a' + 2b}{a + b\tau}$ *and* $N := \frac{b'a - 2a'b}{a(a + b\tau)}$.

Proof. By straightforward computation we obtain

$$^H X(\tau) = 0, \quad {}^V A(\tau) = 2 g_{ir} g^{jl} t^i_j A^r_l = 2G(t, A). \tag{3.8}$$

Next, using (3.6) - (3.8) and the Koszul formula:

$$\begin{aligned}
2\,{}^{CG}g({}^{CG}\nabla_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}) = &\widetilde{X}\,{}^{CG}g(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y}\,{}^{CG}g(\widetilde{Z}, \widetilde{X}) - \widetilde{Z}\,{}^{CG}g(\widetilde{X}, \widetilde{Y}) \\
&+ {}^{CG}g([\widetilde{X}, \widetilde{Y}], \widetilde{Z}) - {}^{CG}g([\widetilde{Y}, \widetilde{Z}], \widetilde{X}) + {}^{CG}g([\widetilde{Z}, \widetilde{X}], \widetilde{Y}),
\end{aligned}$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^1_0(T^1_1 M)$, we obtain the components of ${}^{CG}\nabla$.

Putting ${}^{CG}\nabla_{e_\alpha} e_\beta = {}^{CG}\Gamma^\gamma_{\alpha\beta} e_\gamma$, and using Proposition 1, we obtain:

$${}^{CG}\Gamma^r_{lj} = \Gamma^r_{lj}, \quad {}^{CG}\Gamma^{\bar{r}}_{lj} = \frac{1}{2}(R_{ljr}{}^s t^v_s - R_{ljs}{}^v t^s_r), \quad {}^{CG}\Gamma^r_{l\bar{j}} = 0, \quad {}^{CG}\Gamma^{\bar{r}}_{\bar{l}j} = 0,$$

$$ {}^{CG}\Gamma_{\bar{l}j}^{r} = \frac{a}{2}(g_{ta}R^{sl}{}_{j}{}^{r}t_{s}^{a} - g^{lb}R_{tsj}{}^{r}t_{b}^{s}), \quad {}^{CG}\Gamma_{l\bar{j}}^{r} = \frac{a}{2}(g_{ia}R^{sj}{}_{l}{}^{r}t_{s}^{a} - g^{jb}R_{isl}{}^{r}t_{b}^{s}), $$

$$ {}^{CG}\Gamma_{l\bar{j}}^{\bar{r}} = (\Gamma_{li}^{v}\delta_{r}^{j} - \Gamma_{lr}^{j}\delta_{i}^{v}), \quad {}^{CG}\Gamma_{\bar{l}\bar{j}}^{\bar{r}} = L(\bar{t}_{t}^{l}\delta_{r}^{j}\delta_{i}^{v} + \bar{t}_{i}^{j}\delta_{r}^{l}\delta_{t}^{v}) + Mg^{lj}g_{ti}t_{r}^{v} + N\bar{t}_{t}^{l}\bar{t}_{i}^{j}t_{r}^{v}. $$

4. The Curvature Tensor of ${}^{CG}\nabla$

It is known that the curvature tensor ${}^{CG}R$ of ${}^{CG}\nabla$ is obtained from the formula

$$ {}^{CG}R(\widetilde{X},\widetilde{Y})\widetilde{Z} = {}^{CG}\nabla_{\widetilde{X}}^{CG}\nabla_{\widetilde{Y}}\widetilde{Z} - {}^{CG}\nabla_{\widetilde{Y}}^{CG}\nabla_{\widetilde{X}}\widetilde{Z} - {}^{CG}\nabla_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}, \tag{4.1} $$

where $\widetilde{X},\widetilde{Y},\widetilde{Z} \in \Im_0^1(T_1^1M)$. Putting ${}^{CG}R(e_\alpha,e_\beta)e_\gamma = {}^{CG}R_{\alpha\beta\gamma}{}^{\lambda}e_\lambda$, we get

$$ {}^{CG}R_{\overline{m}\bar{l}\bar{j}}{}^{r} = 0, \tag{4.2} $$

$$ {}^{CG}R_{\overline{m}\bar{l}j}{}^{\bar{r}} = 0, \tag{4.3} $$

$$ {}^{CG}R_{m\bar{l}\bar{j}}{}^{\bar{r}} = 0, \tag{4.4} $$

$$ {}^{CG}R_{mlj}{}^{\bar{r}} = \frac{1}{2}\{\nabla_m R_{ljr}{}^{s}t_{s}^{v} - \nabla_l R_{mjr}{}^{s}t_{s}^{v} + \nabla_l R_{mjs}{}^{v}t_{r}^{s} - \nabla_m R_{ljs}{}^{v}t_{r}^{s}\}, \tag{4.5} $$

$$ {}^{CG}R_{m\bar{l}j}{}^{r} = \frac{a}{2}\{g_{ta}\nabla_m R^{sl}{}_{j}{}^{r}t_{s}^{a} - g^{lb}\nabla_m R_{tsj}{}^{r}t_{b}^{s}\}, \tag{4.6} $$

$$ {}^{CG}R_{ml\bar{j}}{}^{r} = \frac{a}{2}\{g_{ia}\nabla_m R^{sj}{}_{l}{}^{r}t_{s}^{a} - \nabla_l R^{sj}{}_{m}{}^{r}t_{s}^{a} + g^{jb}\nabla_l R_{ism}{}^{r}t_{b}^{s} - \nabla_m R_{isl}{}^{r}t_{b}^{s}\}, \tag{4.7} $$

$$ \begin{aligned} {}^{CG}R_{mlj}{}^{r} = R_{mlj}{}^{r} + \frac{a}{4}\{ & g_{ka}(R^{sh}{}_{m}{}^{r}R_{ljh}{}^{p} - R^{sh}{}_{l}{}^{r}R_{mjh}{}^{p} - 2R^{sh}{}_{j}{}^{r}R_{mlh}{}^{p})t_{s}^{a}t_{p}^{k} \\ & + g_{ka}(R^{sh}{}_{l}{}^{r}R_{mjp}{}^{k} - R^{sh}{}_{m}{}^{r}R_{ljp}{}^{k} + 2R^{sh}{}_{j}{}^{r}R_{mlp}{}^{k})t_{s}^{a}t_{h}^{p} \\ & + g^{hb}(R_{kpl}{}^{r}R_{mjh}{}^{s} - R_{kpm}{}^{r}R_{ljh}{}^{s} + 2R_{kpj}{}^{r}R_{mlh}{}^{s})t_{b}^{p}t_{s}^{k} \\ & + g^{hb}(R_{ksm}{}^{r}R_{ljp}{}^{k} - R_{ksl}{}^{r}R_{mjp}{}^{k} - 2R_{ksj}{}^{r}R_{mlp}{}^{k})t_{b}^{s}t_{h}^{p}\}, \end{aligned} \tag{4.8} $$

$$ \begin{aligned} {}^{CG}R_{ml\bar{j}}{}^{\bar{r}} = R_{mli}{}^{v}\delta_{r}^{j} - R_{mlr}{}^{j}\delta_{i}^{v} + \frac{a}{4}\{ & g_{ia}(R_{mhr}{}^{s}R^{pj}{}_{l}{}^{h} - R_{lhr}{}^{s}R^{pj}{}_{m}{}^{h})t_{s}^{v}t_{p}^{a} \\ & + g_{ia}(R_{lhp}{}^{v}R^{sj}{}_{m}{}^{h} - R_{mhp}{}^{v}R^{sj}{}_{l}{}^{h})t_{s}^{a}t_{r}^{p} + g^{jb}(R_{lhr}{}^{s}R_{ipm}{}^{h} \\ & - R_{mhr}{}^{s}R_{ipl}{}^{h})t_{b}^{p}t_{s}^{v} + g^{jb}(R_{mhs}{}^{v}R_{ipl}{}^{h} - R_{lhs}{}^{v}R_{ipm}{}^{h})t_{r}^{s}t_{b}^{p}\} \\ & + M(g_{ki}R^{sj}{}_{ml}t_{s}^{k} - g^{hj}R_{mlis}t_{h}^{s})t_{r}^{v} \\ & + L(R_{mls}{}^{v}t_{r}^{s} - R_{mlr}{}^{s}t_{s}^{v})\bar{t}_{i}^{j}, \end{aligned} \tag{4.9} $$

$$ \begin{aligned} {}^{CG}R_{m\bar{l}\bar{j}}{}^{r} = -a\bar{t}_{i}^{l}(g_{ia}R^{sj}{}_{m}{}^{r}t_{s}^{a} - g^{jb}R_{ism}{}^{r}t_{b}^{s}) + \frac{a^2}{4}\{ & g_{ta}R^{sl}{}_{h}{}^{r}g^{jb}R_{ipm}{}^{h}t_{s}^{a}t_{b}^{p} \\ & - g_{ta}R^{sl}{}_{h}{}^{r}g_{ib}R^{pj}{}_{m}{}^{h}t_{s}^{a}t_{p}^{b} + g^{lb}R_{tph}{}^{r}g_{ia}R^{sj}{}_{m}{}^{h}t_{b}^{p}t_{s}^{a} \\ & - g^{la}R_{tsh}{}^{r}g^{jb}R_{ipm}{}^{h}t_{a}^{s}t_{b}^{p}\} + \frac{a}{2}\{g^{jl}R_{itm}{}^{r} - g_{it}R^{lj}{}_{m}{}^{r} \\ & + L(g_{ia}R^{sj}{}_{m}{}^{r}t_{s}^{a} - g^{jb}R_{ism}{}^{r}t_{b}^{s})\bar{t}_{t}^{l} + L(g_{ta}R^{sl}{}_{m}{}^{r}t_{s}^{a} - g^{lb}R_{tsm}{}^{r}t_{b}^{s})\bar{t}_{i}^{j} \\ & + (Mg^{lj}g_{ti} + N\bar{t}_{t}^{l}\bar{t}_{i}^{j})t_{h}^{k}(g_{ka}R^{sh}{}_{m}{}^{r}t_{s}^{a} - g^{hb}R_{ksm}{}^{r}t_{b}^{s})\}, \end{aligned} \tag{4.10} $$

$$
\begin{aligned}
{}^{CG}R_{m\bar{l}j}{}^{\bar{r}} = &-\frac{1}{2}\{R_{mjr}{}^{l}\delta_t^v - R_{mjt}{}^{v}\delta_r^l + L(R_{mjr}{}^{s}t_s^v - R_{mjs}{}^{v}t_r^s)\bar{t}_t^l \\
&+M(g_{tk}R_{mj}{}^{ls}t_s^k - g^{lh}R_{mjst}t_h^s)t_r^v\} \\
&+\frac{a}{4}\{g_{ta}R^{pl}{}_j{}^{h}R_{mhr}{}^{s}t_s^v t_p^a - g^{lb}R_{tpj}{}^{h}R_{mhr}{}^{s}t_s^v t_b^p \\
&-g_{ta}R^{sl}{}_j{}^{h}R_{mhp}{}^{v}t_r^p t_s^a + g^{lb}R_{tpj}{}^{h}R_{mhs}{}^{v}t_r^s t_b^p\},
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
{}^{CG}R_{\bar{m}\bar{l}j}{}^{r} = &\, a'\bar{t}_n^m(g_{ta}R^{sl}{}_j{}^{r}t_s^a - g^{lb}R_{tsj}{}^{r}t_b^s) - a'\bar{t}_t^l(g_{na}R^{sm}{}_j{}^{r}t_s^a - g^{mb}R_{nsj}{}^{r}t_b^s) \\
&+a(g_{tn}R^{ml}{}_j{}^{r} - g^{lm}R_{tnj}{}^{r}) + \frac{a^2}{4}\{g_{na}R^{sm}{}_h{}^{r}g_{tb}R^{pl}{}_j{}^{h}t_s^a t_p^b \\
&-g_{ta}R^{sl}{}_h{}^{r}g_{nb}R^{pm}{}_j{}^{h}t_s^a t_p^b + g_{ta}R^{sl}{}_h{}^{r}g^{mb}R_{npj}{}^{h}t_s^a t_b^p \\
&-g_{na}R^{sm}{}_h{}^{r}g^{lb}R_{tpj}{}^{h}t_s^a t_b^p + g^{lb}R_{tph}{}^{r}g_{na}R^{sm}{}_j{}^{h}t_b^p t_s^a \\
&-g^{mb}R_{nph}{}^{r}g_{ta}R^{sl}{}_j{}^{h}t_b^p t_s^a + g^{ma}R_{nsh}{}^{r}g^{lb}R_{tsj}{}^{h}t_b^p t_a^s \\
&-g^{la}R_{tsh}{}^{r}g^{mb}R_{npj}{}^{h}t_b^p t_a^s\},
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
{}^{CG}R_{\bar{m}\bar{l}\bar{j}}{}^{\bar{r}} = &\, F_1(\bar{t}_n^m\bar{t}_i^j\delta_r^l\delta_t^v - \bar{t}_t^l\bar{t}_i^j\delta_r^m\delta_n^v) + F_2(g^{mj}g_{ni}\delta_r^l\delta_t^v - g^{lj}g_{ti}\delta_r^m\delta_n^v) \\
&+F_3(g^{lj}g_{ti}\bar{t}_n^m t_r^v - g^{mj}g_{ni}\bar{t}_t^l t_r^v),
\end{aligned} \tag{4.13}
$$

where $F_1 := 2L' - L^2 - N(1 - L\tau)$, $F_2 := L - M(1 + \tau L)$ and $F_3 := 2M' + M^2 - N(1 - \tau M)$.

Theorem 4.1. *Let (M, g) be a Riemannian manifold and T_1^1M be its $(1,1)$–tensor bundle with the metric ${}^{CG}g$. If T_1^1M is flat, then M is a flat manifold.*

Proof. Let T_1^1M be a flat manifold. Then we have ${}^{CG}R = 0$, or eqivalently, ${}^{CG}R_{\alpha\beta\gamma}{}^{\lambda} = 0$. Hence from (4.8) at the point $(x^i, t_i^j) = (x^i, 0) \in T_1^1M$ we get $0 = ({}^{CG}R_{mlj}{}^{r})_{(x^i,0)} = R_{mlj}{}^{r}(x^i)$. Thus we have $R = 0$, implying that (M, g) is a flat manifold.

Next, let M be a flat manifold. Then in view of (4.2) - (4.13) we conclude that all the components of ${}^{CG}R$ are zero except ${}^{CG}R_{\bar{m}\bar{l}\bar{j}}{}^{\bar{r}}$. But ${}^{CG}R_{\bar{m}\bar{l}\bar{j}}{}^{\bar{r}} = 0$ if and only if $F_1 = F_2 = F_3 = 0$. Thus we have the following result.

Theorem 4.2. *Let (M, g) be a flat Riemannian manifold and T_1^1M be its $(1,1)$–tensor bundle with the metric ${}^{CG}g$. Then T_1^1M is flat if and only if $F_1 = F_2 = F_3 = 0$.*

Taking into account that for Sasaki metric ${}^{S}g$ we have $F_1 = F_2 = F_3 = 0$, from Theorems 4.1 and 4.2 we infer the following result.

Corollary 4.1. Let (M, g) be a Riemannian manifold and T_1^1M be its $(1,1)$–tensor bundle with the Sasaki metric ${}^{S}g$. Then M is flat if and only if T_1^1M is flat.

The Ricci tensor and the scalar curvature of the metric ${}^{CG}g$ are defined by ${}^{CG}R_{\alpha\beta} = {}^{CG}R_{\sigma\alpha\beta}{}^{\sigma}$ and ${}^{CG}S = {}^{CG}g^{\alpha\beta}\,{}^{CG}R_{\alpha\beta}$, respectively. Using (4.2) - (4.13), we can write

$$
\begin{aligned}
{}^{CG}R_{\bar{l}\bar{j}} &= ((1-n^2)F_1 - F_3)\bar{t}_i^j\bar{t}_t^l + ((1-n^2)F_2 + \tau F_3)g_{ti}g^{lj} \\
&\quad + \frac{a^2}{4}\{g^{lb}R_{tph}{}^{r}g_{ia}R^{sj}{}_{r}{}^{h}t_b^p t_s^a - g_{ta}R^{sl}{}_{h}{}^{r}g_{ib}R^{pj}{}_{r}{}^{h}t_s^a t_p^b \\
&\quad - g^{la}R_{tsh}{}^{r}g^{jb}R_{ipr}{}^{h}t_a^s t_b^p + g_{ta}R^{sl}{}_{h}{}^{r}g^{jb}R_{ipr}{}^{h}t_s^a t_b^p\}, \\
{}^{CG}R_{\bar{l}j} &= \frac{a}{2}\{g_{ta}\nabla_r R^{sl}{}_{j}{}^{r}t_s^a - g^{lb}\nabla_r R_{tsj}{}^{r}t_b^s\}, \\
{}^{CG}R_{l\bar{j}} &= \frac{a}{2}\{g_{ia}\nabla_r R^{sj}{}_{l}{}^{r}t_s^a - g^{jb}\nabla_r R_{isl}{}^{r}t_b^s\}, \\
{}^{CG}R_{lj} &= R_{lj} + \frac{a}{2}\{g^{hb}R_{kpj}{}^{r}R_{rlh}{}^{s}t_b^p t_s^k - g_{ka}R^{sh}{}_{j}{}^{r}R_{rlh}{}^{p}t_s^a t_p^k \\
&\quad - g^{hb}R_{ksj}{}^{r}R_{rlp}{}^{k}t_b^s t_h^p + g_{ka}R^{sh}{}_{j}{}^{r}R_{rlp}{}^{k}t_s^a t_h^p\} \\
&\quad - \frac{a}{4}\{g_{ka}R^{sh}{}_{l}{}^{r}R_{rjh}{}^{p}t_s^a t_p^k + g_{va}R^{pr}{}_{j}{}^{h}R_{lhr}{}^{s}t_s^v t_p^a \\
&\quad + g^{hb}R_{ksl}{}^{r}R_{rjp}{}^{k}t_b^s t_h^p + g^{rb}R_{vpj}{}^{h}R_{lhs}{}^{v}t_r^s t_b^p\},
\end{aligned}
$$

with respect to the frame $\{e_\beta\}$. Also, the scalar curvature of metric ${}^{CG}g$ is given by

$$
\begin{aligned}
{}^{CG}S &= S + \left(\frac{\tau}{a} - \frac{b(trt^2)^2}{a(a+b\tau)}\right)((1-n^2)F_1 - F_3) \\
&\quad + \left(\frac{n^2}{a} - \frac{b\tau}{a(a+b\tau)}\right)((1-n^2)F_2 + \tau F_3) \\
&\quad - \frac{a}{4}g^{ab}g^{hk}g^{lj}g^{vr}R_{slhv}R_{pjkr}t_a^s t_b^p \\
&\quad - \frac{a}{4}g_{cd}g^{lj}g^{hk}g^{rv}R_{rlh}{}^{s}R_{vjk}{}^{p}t_s^c t_p^d + \frac{a}{2}R_{cpr}{}^{h}R_h{}^{rbs}t_s^c t_b^p.
\end{aligned}
$$

Thus we have the following result.

Theorem 4.3. *Let M be a Riemannian manifold with the metric g, and T_1^1M be its $(1,1)$–tensor bundle equipped with the metric ${}^{CG}g$. Let S be the scalar curvature of g, and ${}^{CG}S$ be the scalar curvature of ${}^{CG}g$. Then the following equation holds:*

$$
\begin{aligned}
{}^{CG}S = S &+ \left(\frac{\tau}{a} - \frac{b(trt^2)^2}{a(a+b\tau)}\right)((1-n^2)F_1 - F_3) + \left(\frac{n^2}{a} - \frac{b\tau}{a(a+b\tau)}\right)((1-n^2)F_2 + \tau F_3) \\
&- \frac{a}{4}g^{ab}g^{hk}g^{lj}g^{vr}(tR)_{alhv}(tR)_{bjkr} - \frac{a}{4}g_{cd}g^{lj}g^{hk}g^{rv}(R_t)^c_{rlh}(R_t)^d_{vjk} + \frac{a}{2}T,
\end{aligned}
$$

where $(tR)_{alhv} = R_{slhv}t_a^s$, $(R_t)^c_{rlh} = R_{rlh}{}^{s}t_s^c$ and $T = R_{cpr}{}^{h}R_h{}^{rbs}t_s^c t_b^p$.

Let (M, g) be a Riemannian manifold of dimension $n > 2$ and constant curvature κ:

$$
R_{kmj}{}^{s} = \kappa(\delta_k^s g_{mj} - \delta_m^s g_{kj}). \tag{4.14}
$$

Then $S = n(n-1)\kappa$ and we have the following theorem.

Theorem 4.4. *Let (M, g) be a Riemannian manifold of dimension $n > 2$ and constant curvature κ. Then the scalar curvature ${}^{CG}S$ of $(T_1^1M, {}^{CG}g)$ is given by*

$$\begin{aligned}{}^{CG}S = (1-n)[(\frac{\tau}{a} - \frac{b(trt^2)^2}{a(a+b\tau)})((1+n)F_1 - F_3) + (\frac{n^2}{a} - \frac{b\tau}{a(a+b\tau)})((1+n)F_2 \\ +\tau F_3) - \kappa(n - a\|t\|^2\kappa)] + a\kappa^2((trt)^2 - trt^2).\end{aligned} \tag{4.15}$$

Proof. Direct calculations yield:

$$\begin{aligned}{}^{CG}S = S + (\frac{\tau}{a} - \frac{b(trt^2)^2}{a(a+b\tau)})((1-n^2)F_1 - F_3) \\ +(\frac{n^2}{a} - \frac{b\tau}{a(a+b\tau)})((1-n^2)F_2 + \tau F_3) - \frac{a}{4} g^{ab} g^{hk} g^{lj} g^{vr} R_{slhv} R_{pjkr} t_a^s t_b^p \\ -\frac{a}{4} g_{cd} g^{lj} g^{hk} g^{rv} R_{rlh}{}^s R_{vjk}{}^p t_s^c t_p^d + \frac{a}{2} g^{re} g^{bz} R_{cpr}{}^h R_{hez}{}^s t_s^c t_b^p.\end{aligned} \tag{4.16}$$

Next, it follows from (4.14) that

$$g_{cd} g^{lj} g^{hk} g^{rv} R_{rlh}{}^s R_{vjk}{}^p t_s^c t_p^d = g^{ab} g^{hk} g^{lj} g^{vr} R_{slhv} R_{pjkr} t_a^s t_b^p = 2\kappa^2 \|t\|^2 (n-1), \tag{4.17}$$

$$g^{re} g^{bz} R_{cpr}{}^h R_{hez}{}^s t_s^c t_b^p = 2\kappa^2 (t_c^c t_p^p - t_p^c t_c^p). \tag{4.18}$$

>From formulas (4.16) - (4.18) and the equality $S = n(n-1)\kappa$, we obtain

$$\begin{aligned}{}^{CG}S = n(n-1)\kappa + (\frac{\tau}{a} - \frac{b(trt^2)^2}{a(a+b\tau)})((1-n^2)F_1 - F_3) \\ + (\frac{n^2}{a} - \frac{b\tau}{a(a+b\tau)})((1-n^2)F_2 + \tau F_3) - a\kappa^2 \|t\|^2 (n-1) + a\kappa^2 (t_c^c t_p^p - t_p^c t_c^p).\end{aligned}$$

Finally, using the equality $t_c^c t_p^p - t_p^c t_c^p = (trt)^2 - trt^2$, from the last equation we infer the relation (4.15), and the result follows. It is known that for a local frame a sectional curvature on $(T_1^1M, {}^{CG}g)$ is given by

$${}^{CG}K(\Delta) = -\frac{R_{kmij} U^k V^m U^i V^j}{{}^{CG}g(U,U)\,{}^{CG}g(V,V) - ({}^{CG}g(U,V))^2}, \tag{4.19}$$

where $\Delta = (U, V)$ denotes the plane spanned by (U, V).

Let $\{X_i\}_{i=1}^n$ be a local orthonormal frame, $\|A^{\bar{i}}\|_G^2 = G(A^{\bar{i}}, A^{\bar{i}}) = 1$, and $G(A^{\bar{i}}, A^{\bar{j}}) = 0$ for $\bar{i} \neq \bar{j}$ and $A^{\bar{i}} \in \Im_1^1(M)$, $\bar{i} = n+1, \ldots, n^2$. Then from the definition of ${}^{CG}g$, it is easy to see that $\{{}^HX_1, \ldots, {}^HX_n, {}^VA_1, \ldots, {}^VA^{n^2}\}$ is a local frame on T_1^1M. In view of (4.2) - (4.13) and (4.19), we obtain

$$\begin{aligned}{}^{CG}K({}^VA, {}^VB) = \frac{1}{\mathcal{A}}\{F_1(g^{km} g_{hn} \bar{t}_t^l - g^{kl} g_{ht} \bar{t}_n^m)\bar{t}_i^j \\ +F_2(g^{lj} g_{ti} g^{km} g_{hn} - g^{mj} g_{nt} g^{kl} g_{ht}) + F_3(g^{mj} g_{ni} \bar{t}_t^l - g^{lj} g_{ti} \bar{t}_n^m)\bar{t}_h^k\} A_m^n B_l^t A_j^i B_k^h,\end{aligned}$$

$$\begin{aligned}{}^{CG}K({}^HX, {}^VB) = \frac{a}{4\mathcal{B}}\{g_{vh} g_{ta} g^{ez} g^{ld} g^{kr} R_{emr}{}^s R_{zjd}{}^p t_s^v t_p^a \\ +g_{hv} g^{kr}(g^{lb} R_{mer}{}^s R_{tpj}{}^e t_s^v t_b^p + g_{ta} R_{mep}{}^v R^{sl}{}_j{}^e t_r^p t_s^a) \\ +g_{ve} g^{kb} g^{rl} R_{tsm}{}^v R_{hpj}{}^e t_r^s t_b^p\} X^m B_l^t X^j B_k^h,\end{aligned} \tag{4.20}$$

$$\begin{aligned} {}^{CG}K({}^{H}X,{}^{H}Y) = -\{R_{mljk} &+ \frac{3a}{4}(g^{zh}R_{mkz}{}^{s}R_{ljh}{}^{p}g_{ab}t_s^a t_p^b + g_{ve}R_{mks}{}^{v}R_{jlp}{}^{e}g^{ab}t_a^s t_b^p \\ &+ g_{ae}g^{zh}R_{mkz}{}^{s}R_{ljp}{}^{e}t_s^a t_h^p + g_{az}g^{hb}R_{mkp}{}^{z}R_{ljh}{}^{s}t_s^a t_b^p)\}X^m Y^l X^j Y^k, \end{aligned} \tag{4.21}$$

where $\mathcal{A} = {}^{V}(a + bG^2(t,A))\,{}^{V}(a + bG^2(t,B)) - b\,{}^{V}G(t,A)\,{}^{V}G(t,B)$, $\mathcal{B} = {}^{V}(a + bG^2(t,B))$, ${}^{CG}K({}^{H}X,{}^{H}Y)$, ${}^{CG}K({}^{H}X,{}^{V}B)$ denote the sectional curvature of the plane spanned by $({}^{H}X,{}^{H}Y)$, $({}^{H}X,{}^{V}A)$ and $({}^{V}A,{}^{V}B)$ on $(T_1^1M, {}^{CG}g)$. Hence we can state the following result.

Theorem 4.5. *Let (M,g) be a Riemannian manifold and T_1^1M be its $(1,1)$–tensor bundle with metric ${}^{CG}g$. Suppose that $(T_1^1M, {}^{CG}g)$ is a Riemannian manifold of constant sectional curvature ${}^{CG}K = \kappa$. Then the sectional curvature of (M,g) is equal to κ. Moreover, (M,g) cannot have a non-zero constant sectional curvature.*

Proof. Let $(T_1^1M, {}^{CG}g)$ be a Riemannian manifold of constant sectional curvature ${}^{CG}K = \kappa$. Then we have $K({}^{H}X,{}^{H}Y) = \kappa$. Writing (4.21) at $(x^i, 0)$, we obtain

$$\kappa = {}^{CG}K({}^{H}X,{}^{H}Y) = -R_{mljk}X^m Y^l X^j Y^k = K(X,Y),$$

where $K(X,Y)$ is the sectional curvature of (M,g). This implies that (M,g) has a constant sectional curvature equal to κ. Also, from the equation (4.20) written at $(x^i, 0)$, we infer $\kappa = {}^{CG}K({}^{H}X,{}^{V}B) = 0$.

5. Para-Kähler Structures on $(T_1^1M, {}^{CG}g)$

An *almost product structure* F on a differentiable manifold M is a $(1,1)$–tensor field F on M such that $F^2 = 1$. The pair (M,F) is called an almost product manifold. An *almost paracomplex manifold* (or almost B–manifold) is an almost product manifold (M,F) such that the two eigenbundles T^+M and T^-M associated with the two eigenvalues +1 and -1 of F, respectively, have the same rank.. An almost paracomplex structure on a 2n-dimensional manifold M may alternatively be defined as a G–structure on M with structural group $GL(n,R) \times GL(n,R)$. A *paracomplex manifold* (or B–manifold) is an almost paracomplex manifold (M,F) such that the G–structure defined by the tensor field F is integrable (see [15]). If an almost paracomplex manifold (M,F) admits a Riemannian metric g such that

$$g(FX,FY) = g(X,Y), \quad \forall X,Y \in \Im_0^1(M),$$

then (M,g,F) is called an *almost para-Nordenian manifold.* It is well known that, an almost para-Nordenian manifold is a para-Kähler manifold if and only if $\nabla F = 0$, where ∇ is the Levi-Civita connection of g (see [20]). Let now $E \in \Im_0^1(M)$ be a nowhere zero vector field on M. For any $X \in \Im_0^1(M)$ and $\widetilde{E} = g \circ E \in \Im_1^0(M)$, we define the vertical lift ${}^{V}(X \otimes \widetilde{E})$ of X with respect to E. The map $X \to {}^{V}(X \otimes \widetilde{E})$ is a monomorphism of $\Im_0^1(M) \to \Im_0^1(T_1^1M)$. Hence an n-dimensional C^∞ vertical distribution V^E is defined on T_1^1M. Let $V^\perp$ be the distribution on T_1^1M which is orthogonal to H and V^E. Then H, V^E and $V^\perp$ are mutually orthogonal distributions with respect to the metric ${}^{CG}g$. We define a tensor field F of type $(1,1)$ on T_1^1M by

$$\begin{cases} F({}^H X) = \alpha^V(X \otimes \widetilde{E}) + \beta g(X,E)^V(E \otimes \widetilde{E}), \\ F({}^V(X \otimes \widetilde{E})) = \delta^H X + \rho g(X,E)^H E, \\ F({}^V A) = {}^V A, \end{cases}$$

for any $X \in \Im_0^1(M)$ and $A \in \Im_1^1(M)$, where $\alpha, \beta, \delta, \rho$ are functions on $T_1^1 M$ to be determined. The condition $F^2 = I$ leads to the equations

$$\alpha\delta = 1, \quad \alpha\rho + \beta\delta + \beta\rho||E||^2 = 0. \tag{5.1}$$

Thus we have the following result.

Proposition 5.1. *Let (M,g) be a Riemannian manifold and $T_1^1 M$ be its (1,1)- tensor bundle. Then $(T_1^1 M, F)$ is an almost paracomplex manifold if and only if (5.1) holds.*

The condition ${}^{CG}g(F(\widetilde{X}), F(\widetilde{Y})) = {}^{CG}g(\widetilde{X}, \widetilde{Y})$ for all $\widetilde{X}, \widetilde{Y} \in \Im_0^1(T_1^1 M)$ implies

$$a\alpha^2||E||^2 = 1, \quad \delta^2 = a||E||^2, \quad 2\delta\rho + \rho^2||E||^2 = 0, \quad 2\alpha\beta + \beta^2||E||^2 = 0. \tag{5.2}$$

The solution of the system of equations (5.1) and (5.2) is

$$\alpha = \frac{1}{||E||\sqrt{a}}, \quad \beta = \frac{-2}{\sqrt{a}||E||^3}, \quad \delta = ||E||\sqrt{a}, \quad \rho = -\frac{2\sqrt{a}}{||E||}. \tag{5.3}$$

Thus we have the following result.

Theorem 5.1. *Let (M,g) be a Riemannian manifold and $T_1^1 M$ be its (1, 1)-tensor bundle equipped with metric ${}^{CG}g$. Then the triple $(T_1^1 M, {}^{CG}g, F)$ is an almost para-Nordenian manifold if and only if (5.3) holds.*

Now, we obtain the covariant derivative of F as follows:

$$\begin{aligned} &({}^{CG}\nabla_{{}^H X} F)({}^H Y) \\ &= \alpha^V(Y \otimes [g \circ \nabla_X E]) - \frac{1}{2}(\widetilde{\gamma} - \gamma)R(X,Y) + \beta^V(E \otimes \widetilde{E}) g(Y, \nabla_X E) \\ &+ \frac{a\alpha}{2}{}^H\Big(g^{bj} R(t_b, (Y \otimes \widetilde{E})_j)X + g_{ai}(t^a(g^{-1} \circ R(.,X)\widetilde{(Y \otimes \widetilde{E})}^i))\Big) \\ &+ \beta g(Y,E)[{}^V(\nabla_X E \otimes \widetilde{E} + E \otimes g \circ \nabla_X E) \\ &+ \frac{a}{2}{}^H\Big(g^{bj} R(t_b, (E \otimes \widetilde{E})_j)X + g_{ai}(t^a(g^{-1} \circ R(.,X)\widetilde{(E \otimes \widetilde{E})}^i))\Big)], \end{aligned} \tag{5.4}$$

$$\begin{aligned} &({}^{CG}\nabla_{{}^H X} F)({}^V B) \\ &= \frac{a}{2}{}^H\Big(g^{bj} R(t_b, B_j)X + g_{ai}(t^a(g^{-1} \circ R(.,X)\widetilde{B}^i))\Big) \\ &- \frac{a\alpha}{2}{}^V\Big([g^{bj} R(t_b, B_j)X + g_{ai}(t^a(g^{-1} \circ R(.,X)\widetilde{B}^i))] \otimes \widetilde{E}\Big) \\ &- \frac{a\beta}{2} g\Big(g^{bj} R(t_b, B_j)X + g_{ai}(t^a(g^{-1} \circ R(.,X)\widetilde{B}^i)), E\Big)^V(E \otimes \widetilde{E}), \end{aligned}$$

$$
\begin{aligned}
&({}^{CG}\nabla_{{}^HX}F)^V(Y\otimes\widetilde{E})\\
&=-{}^V(Y\otimes(g\circ\nabla_XE))+\frac{1}{2}(\widetilde{\gamma}-\gamma)[\delta R(X,Y)+\rho g(Y,E)R(X,E)]\\
&-\frac{a\alpha}{2}{}^V\Big(g^{bj}R(t_b,(Y\otimes\widetilde{E})_j)X+g_{ai}(t^a(g^{-1}\circ R(.,X)\widetilde{(Y\otimes\widetilde{E})}^i))\otimes\widetilde{E}\Big)\\
&-\frac{a\beta}{2}g\Big(g^{bj}R(t_b,(Y\otimes\widetilde{E})_j)X+g_{ai}(t^a(g^{-1}\circ R(.,X)\widetilde{(Y\otimes\widetilde{E})}^i)),E\Big)^V(E\otimes\widetilde{E})\\
&+\rho^HEg(Y,\nabla_XE)+\rho g(Y,E)^H(\nabla_XE),
\end{aligned}
$$

$$({}^{CG}\nabla_{{}^VA}F)({}^VB)=0,$$

$$
\begin{aligned}
&({}^{CG}\nabla_{{}^V(X\otimes\widetilde{E})}F)^V(Y\otimes\widetilde{E})\\
&=-MG(X\otimes\widetilde{E},Y\otimes\widetilde{E})^Vt+\frac{a\rho}{2}g(Y,E)^H\Big(g^{bl}R(t_b,{}^V(X\otimes\widetilde{E})_l)E\\
&\quad+g_{at}(t^a(g^{-1}\circ R(.,E)^V\widetilde{(X\otimes\widetilde{E})}^t))\Big)+\frac{a\delta}{2}{}^H\Big(g^{bl}R(t_b,{}^V(X\otimes\widetilde{E})_l)Y\\
&\quad+g_{at}(t^a(g^{-1}\circ R(.,Y)^V\widetilde{(X\otimes\widetilde{E})}^t))\Big),
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
&({}^{CG}\nabla_{{}^V(X\otimes\widetilde{E})}F)({}^HY)\\
&=-\frac{a\alpha}{2}{}^V\Big([g^{bl}R(t_b,(X\otimes\widetilde{E})_l)Y+g_{at}(t^a(g^{-1}\circ R(.,Y)\widetilde{(X\otimes\widetilde{E})}^t))]\otimes\widetilde{E}\Big)\\
&-\frac{a\beta}{2}g\Big(g^{bl}R(t_b,(X\otimes\widetilde{E})_l)Y+g_{at}(t^a(g^{-1}\circ R(.,Y)\widetilde{(X\otimes\widetilde{E})}^t)),E\Big)^V(E\otimes\widetilde{E})\\
&+\alpha MG(X\otimes\widetilde{E},Y\otimes\widetilde{E})^Vt,
\end{aligned}
$$

$$
\begin{aligned}
&({}^{CG}\nabla_{{}^V(X\otimes\widetilde{E})}F)({}^VB)\\
&=LG(B,t)^V(X\otimes\widetilde{E})-LG(B,t)\Big(\delta^HX+\rho g(X,E)^HE\Big).
\end{aligned}
\tag{5.6}
$$

Hence we have the following theorem.

Theorem 5.2. *Let (M,g) be a Riemannian manifold, T_1^1M be its tensor bundle with the Riemannian metric ${}^{CG}g$ and the almost paracomplex structure F. Then the triple $(T_1^1M,{}^{CG}g,F)$ is a para-Kähler-Norden manifold if and only if $a=constant$, $b=0$, $R=0$ and $\nabla E=0$.*

Proof. Obviously, if $a=constant$, $b=0$, $R=0$ and $\nabla E=0$, then ${}^{CG}\nabla F=0$, that is, $(T_1^1M,{}^{CG}g,F)$ is a para-Kähler-Norden manifold. Conversely, let ${}^{CG}\nabla F=0$. Then by (5.6) we get $L=0$, implying that $a=constant$. Moreover, from (5.5) we have $M=0$, implying $b=0$. Finally, (5.4) gives us $R=0$ and $\nabla E=0$.

As an immediate consequence of Theorem 7, we can state the following result for Sasaki metric ${}^Sg$.

Corollary 5.1. *Let (M, g) be a Riemannian manifold, T_1^1M be its tensor bundle with the Sasaki metric ${}^Sg$ and the paracomplex structure F. Then the triple $(T_1^1M, {}^Sg, F)$ is a para-Kähler-Norden manifold if and only if $R = 0$ and $\nabla E = 0$.*

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