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MEROMORPHIC FUNCTIONS WITH DEFICIENCIES GENERATING UNIQUE RANGE SETS

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Abstract. With the help of weighted sharing of sets we deal with the problem of unique range set for meromorphic functions with deficient values and obtain a result which improves, generalizes and extends some previous results. We provide two examples to show that the condition in one of our results is the best possible.

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1. Introduction: Definitions and Results

Throughout the paper by meromorphic functions we always mean meromorphic functions in the complex plane $\mathbb{C}$, and the letter E will denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying $S(r,h)=o(T(r,h))$, $(r\longrightarrow\infty,\ r\notin E)$.

We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$, and by $S(r)$ any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty$, $r\notin E$.

Also, we adopt the standard notation of the Nevanlinna theory of meromorphic functions as explained in [6]. For $a\in\mathbb{C}\cup\{\infty\}$ we define

$$\Theta(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, if $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, if $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z: f(z)=a\}$, where each point is counted according to its multiplicity. If we do not count the

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multiplicity the set $\bigcup_{a\in S}\{z : f(z) = a\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$ we say that f and g share the set S IM. Evidently, these definitions coincide with the usual definitions of CM (resp., IM) shared values, provided that the set S contains only one element.

Let S ($S \subset \mathbb{C}$) be a set, and let f and g be two non-constant meromorphic (resp., entire) functions. If $E_f(S) = E_g(S)$ implies $f \equiv g$, then S is called a unique range set for meromorphic (resp., entire) functions or in brief URSM (resp., URSE).

In 1926, R. Nevanlinna showed that a meromorphic function on the complex plane $\mathbb{C}$ is uniquely determined by the images (ignoring multiplicities) of 5 distinct values. A few years later he showed that when multiplicities are counted, then 4 points are sufficient (with one exceptional situation). In [4] Gross raised the problem of finding out a finite set S so that an entire function is determined by the single pre-image (counting multiplicities) of S.

In 1982 F. Gross and C. C. Yang [5] proved the following theorem:

Theorem A. *Let $S = \{z \in \mathbb{C} : e^z + z = 0\}$, and let f, g be two entire functions satisfying $E_f(S) = E_g(S)$. Then $f \equiv g$.*

Since in Theorem A, S is an infinite set, it does not provide a solution to the Gross' problem. In 1994 H.X. Yi [16] established a URSE with 15 elements, and in 1995 P. Li and C.C. Yang [14] established a URSM with 15 elements and a URSE with 7 elements. Since then to find a URSM with minimum cardinality becomes an increasing interest among the researchers.

In 1998 G. Frank and M. Reinders [2] obtained a URSM with 11 element, which is the smallest available URSM to the knowledge of the authors.

A polynomial P in $\mathbb{C}$ is called a strong uniqueness polynomial for meromorphic (resp., entire) functions if for any non-constant meromorphic (resp., entire) functions f and g, $P(f) \equiv cP(g)$ implies $f \equiv g$, where c is a suitable nonzero constant. We say P is SUPM (resp., SUPE) in brief. On the other hand, for a polynomial P in $\mathbb{C}$ if the condition $P(f) \equiv P(g)$ implies $f \equiv g$ for any non-constant meromorphic (resp., entire) function f and g, then P is called a uniqueness polynomial for meromorphic (resp., entire) functions. We say P is a UPM (resp., UPE) in brief.

Suppose that P is a polynomial of degree n in $\mathbb{C}$ having only simple zeros and S be the set of all zeros of P. If S is a URSM (resp., URSE), then from the definition it follows that P is UPM (resp., UPE). However the converse is not true, in general. For instance, $P(z) = az + b$ ($a \neq 0$) is clearly a UPM, but for $f = -\frac{b}{a}e^z$ and $g = -\frac{b}{a}e^{-z}$ we see that $E_f(S) = E_g(S)$, where $S = \{-\frac{b}{a}\}$ is the set of zeros of $P(z) = az + b$.

To find conditions under which the converse is true, H. Fujimoto [3] first invented a special property of polynomials, which he called the property (H). Fujimoto's property

(H) may be stated as follows: A polynomial P is said to satisfy the property (H) if $P(\alpha) \neq P(\beta)$ for any two distinct zeros α and β of the derivative P'. Fujimoto found a sufficient condition for a set of zeros S of a SUPM (resp., SUPE) P to be a URSM (resp., URSE). Specifically, in [3] H. Fujimoto proved the following result.

Theorem B. [3] *Let P be a polynomial of degree n in $\mathbb{C}$ having only simple zeros and satisfying the condition (H). Let P' have k distinct zeros and either $k \geq 3$ or $k = 2$ and P' has no a simple zero. Further suppose that P is a* SUPM (resp., SUPE). *If S is the set of zeros of P and $n \geq 2k+7$ (resp., $n \geq 2k+3$), then S is a* URSM (resp., URSE).

To deal with the the Gross' problem and its counterpart for meromorphic functions on $\mathbb{C}$, Yi [17] and Li and Yang [14]-[15] have investigated the zero sets of polynomials of the form $P(z) = z^n + az^{n-m} + b$, where $n > m \geq 1$ and a and b are chosen so that P has n distinct roots. Clearly $P(z)$ satisfies the property (H). In [18] it has been shown that when $m \geq 2$ the zero set S of $P(z)$ is a URSM and hence $P(z)$ is a UPM. But when $m = 1$, the situation is completely different. So, a natural question would be whether for $m = 1$, the zero set S of $P(z)$ can be a URSM or even a URSE.

In this direction, independently Yi [17] and Li-Yang [14] had already made some contributions for entire functions. In paricular, they proved the following result.

Theorem C. *Let $S = \{z : z^7 - z^6 - 1 = 0\}$. If f and g are two non-constant entire functions satisfying $E_f(S) = E_g(S)$ then $f \equiv g$.*

Clearly $z^7 - z^6 - 1$ is an UPE. To obtain a counterpart of Theorem C for meromorphic functions and for more general polynomials, in 1996 Yi proved the following result.

Theorem D. [18] *Let $S = \{z : z^n + az^{n-m} + b = 0\}$, where m, n are two positive integers such that m and n have no common factors, $n > 2m + 8$ ($m \geq 2$), and a, b are nonzero constants such that the algebraic equation $z^n + az^{n-m} + b = 0$ has no multiple roots. Then $E_f(S) = E_g(S)$ implies $f \equiv g$.*

>From Theorem D we infer that a URS of meromorphic functions of the form as given in Theorem B consists of 13 elements. In [18] Yi also explored the case $m = 1$, and obtained the following version of Theorem D in this case.

Theorem E. [18] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where n (≥ 11) is an integer, a and b are two nonzero constants such that the algebraic equation $z^n + az^{n-1} + b = 0$ has no multiple roots. If f and g are non-constant meromorphic functions satisfying $E_f(S) = E_g(S)$ then either $f \equiv g$ or $f = -\frac{ah(h^{n-1}-1)}{h^n-1}$, $g = -\frac{a(h^{n-1}-1)}{h^n-1}$, where $h = \frac{f}{g}$.*

Clearly under the assumptions of Theorem E, S can not be a URSM.

In 1998 Fang and Hua [1] have extended Theorem C to the case of meromorphic functions with some additional conditions on the ramification indices of f and g. Specifically, in [1] was proved the followin result.

Theorem F. [1] *Let S be as in* Theorem C. *If two meromorphic functions f and g are such that $\Theta(\infty;f) > \frac{11}{12}$, $\Theta(\infty;g) > \frac{11}{12}$ and $E_f(S) = E_g(S)$ then $f \equiv g$.*
We need the following definition, known as weighted sharing of sets and values, which renders a useful tool for the purpose of relaxation of the nature of sharing the sets.
Definition 1.1. [8, 9] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.
Definition 1.2. [8] *Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S,k)$ the set $E_f(S) = \bigcup_{a\in S}\{z : f(z)-a = 0\}$. Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.*
Definition 1.3. [7] *For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f| \le m)$ (resp., $N(r,a;f| \ge m)$ the counting function of those a-points of f whose multiplicities are not greater(resp., less) than m, where each a-point is counted according to its multiplicity. The functions $\overline{N}(r,a;f \mid\le m)$ and $\overline{N}(r,a;f \mid\ge m)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also, the functions $N(r,a;f \mid< m)$, $N(r,a;f \mid> m)$, $\overline{N}(r,a;f \mid< m)$ and $\overline{N}(r,a;f \mid> m)$ are defined analogously.*
We define $\delta_2(a;f) = 1 - \limsup\limits_{r\longrightarrow\infty} \frac{N_2(r,a;f)}{T(r,f)}$, where $N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\ge 2)$.
Lahiri [10] improved Theorem F in the following direction.
Theorem G. [10] *Let S be as in Theorem C. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f) + \Theta(\infty;g) > \frac{3}{2}$ and $E_f(S,2) = E_g(S,2)$ then $f \equiv g$.*

In 2004 Lahiri and Banerjee [11] further improved Theorem C in a more compact and convenient way, and obtained the following result.
Theorem H. [11] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where $n\ (\ge 9)$ is an integer, and a, b are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple roots. If $E_f(S,2) = E_g(S,2)$ and $\Theta(\infty;f) + \Theta(\infty;g) > \frac{4}{n-1}$, then $f \equiv g$.*

The following example shows that the set S in Theorems G-H cannot be replaced by an arbitrary set containing six distinct elements.
Example 1.1. *Let $f(z) = \sqrt{\alpha\beta\gamma}e^{z}$ and $g(z) = \sqrt{\alpha\beta\gamma}e^{-z}$, and let $S = \{\alpha\sqrt{\beta}, \alpha\sqrt{\gamma}, \beta\sqrt{\alpha}, \beta\sqrt{\gamma}, \gamma\sqrt{\alpha}, \gamma\sqrt{\beta}\}$, where α, β and γ are nonzero distinct complex numbers. Then it is easy to see that $E_f(S,\infty) = E_g(S,\infty)$ but $f \not\equiv g$.*

So we observe that deficiencies of poles play a vital role in order to find sufficient conditions for which the conclusions of Theorems F, and G-H holds true.

We naturally raise the following questions.

Question 1: Is there any significant contribution of the deficiencies of the other values in Theorems G and H?

Question 2: What happens if we reduce the degree of the equation defining S in Theorem H?

In this paper we give some affirmative answers to the above questions, which in turn will further improve, generalize and extend Theorems G and H.

The following theorem is the main result of the paper.

Theorem 1.1. *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where n (≥ 6) is an integer, and a, b are two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple roots. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S, m) = E_g(S, m)$. If one of the following conditions is satisfied:*

(i) $m \geq 2$ *and* $\Theta_f + \Theta_g > \max\{\frac{10-n}{2}, \frac{n+1}{n-1}\}$
(ii) $m = 1$ *and* $\Theta_f + \Theta_g > \max\{\frac{11-n}{2}, \frac{n+1}{n-1}\}$
(iii) $m = 0$ *and* $\Theta_f + \Theta_g > \max\{\frac{16-n}{3}, \frac{n+1}{n-1}\}$,

then $f \equiv g$, where $\Theta_f = \Theta(0; f) + \Theta(-a\frac{n-1}{n}; f) + \Theta(\infty; f) + \frac{1}{2}\delta_2(-a; f)$ and Θ_g can be defined similarly.

The examples that follow show that the condition $\Theta_f + \Theta_g > \frac{n+1}{n-1}$ in Theorem 1.1 is sharp, when $n \geq 8$ and $m \geq 2$.

Example 1.2. (Example 2, [11]). Let $f = -a\frac{1-h^{n-1}}{1-h^n}$ and $g = -ah\frac{1-h^{n-1}}{1-h^n}$, where $h = \frac{\alpha^2(e^z-1)}{e^z-\alpha}$, $\alpha = exp(\frac{2\pi i}{n})$ and $n(\geq 3)$ is an integer.

Then we have $T(r, f) = (n-1)T(r, h) + O(1)$; $T(r, g) = (n-1)T(r, h) + O(1)$ and $T(r, h) = T(r, e^z) + O(1)$. Next, we see that $h \neq \alpha, \alpha^2$, and so for any complex number $\gamma \neq \alpha, \alpha^2$ we have $\overline{N}(r, \gamma; h) \sim T(r, h)$. Also, we note that a root of $h = 1$ is not a pole and zero of f and g. Hence $\Theta(\infty; f) = \Theta(\infty; g) = \frac{2}{n-1}$. On the other hand, we have

$$\Theta(0; f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\sum_{k=1}^{n-2} \overline{N}(r, \beta^k; h) + \overline{N}(r, \infty; h)}{(n-1)T(r, h) + O(1)} = 0$$

and

$$\Theta(0; g) = 1 - \limsup_{r\longrightarrow\infty} \frac{\sum_{k=1}^{n-2} \overline{N}(r, \beta^k; h) + \overline{N}(r, 0; h)}{(n-1)T(r, h) + O(1)} = 0,$$

where $\beta = exp\,(\frac{2\pi i}{n-1})$. Also, we have

$$\delta_2(-a; f) = 1 - \limsup_{r\longrightarrow\infty} \frac{2\overline{N}(r, 0; h)}{(n-1)T(r, h) + O(1)} = \frac{n-3}{n-1}$$

and

$$\delta_2(-a; g) = 1 - \limsup_{r\longrightarrow\infty} \frac{2\overline{N}(r, \infty; h)}{(n-1)T(r, h) + O(1)} = \frac{n-3}{n-1}.$$

Observe that the polynomial $(n-1)z^n - nz^{n-1} + 1$ has double zero at the point $z = 1$. Consequently it has $n-1$ distinct zeros, which we denote by u_k, $k = 1, \ldots, n-1$. So, we have

$$\Theta(-a\frac{n-1}{n};f)=1-\limsup_{r\longrightarrow\infty}\frac{\sum_{k=1}^{n-1}\overline{N}(r,u_k;e^z)}{(n-1)T(r,e^z)+O(1)}=0$$

and

$$\Theta(-a\frac{n-1}{n};g)=1-\limsup_{r\longrightarrow\infty}\frac{\sum_{j=1}^{n-1}\overline{N}(r,v_j;e^z)}{(n-1)T(r,e^z)+O(1)}=0,$$

where $v_j=\frac{1}{u_j}, j=1,\ldots,n-1$. Therefore $\Theta_f+\Theta_g=\frac{n+1}{n-1}$. Clearly $E_f(S,\infty)=E_g(S,\infty)$ because $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$ but $f\not\equiv g$.

Example 1.3. Let f and g be as in Example 1.2, where $h=\frac{\alpha(\alpha e^z-1)}{e^z-1}$, $\alpha=exp(\frac{2\pi i}{n})$ and $n(\geq 3)$ is an integer.

Now we give some definitions and notation which are used in the rest of the paper

Definition 1.4. [19] *Let f and g be two non-constant meromorphic functions such that f and g share $(a,0)$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ the reduced counting function of those a-points of f and g where $p>q$, by $N_E^{1)}(r,a;f)$ the counting function of those a-points of f and g where $p=q=1$, and by $\overline{N}_E^{(2}(r,a;f)$ the reduced counting function of those a-points of f and g where $p=q\geq 2$. In the same way we can define the functions $\overline{N}_L(r,a;g)$, $N_E^{1)}(r,a;g)$, $\overline{N}_E^{(2}(r,a;g)$; and the functions $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a\in\mathbb{C}\cup\{\infty\}$.*

Observe that when f and g share (a,m), $m\geq 1$, then $N_E^{1)}(r,a;f)=N(r,a;f\mid=1)$.

Definition 1.5. *We denote by $\overline{N}(r,a;f\mid=k)$ the reduced counting function of those a-points of f whose multiplicities is exactly k, where $k\geq 2$ is an integer.*

Definition 1.6. [8, 9] *Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined as follows.

$$F=\frac{f^{n-1}(f+a)}{-b},\ G=\frac{g^{n-1}(g+a)}{-b}. \tag{2.1}$$

Also, we will use the function H defined as follows:

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{2.2}$$

Lemma 2.1. [13] *Let f be a non-constant meromorphic function and let*

$$R(f)=\sum_{k=0}^{n}a_kf^k\left(\sum_{j=0}^{m}b_jf^j\right)^{-1}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n\neq 0$ and $b_m\neq 0$. Then $T(r,R(f))=dT(r,f)+S(r,f)$, where $d=\max\{n,m\}$.

Lemma 2.2. [19] *Let F, G be two non-constant meromorphic functions such that they share $(1,0)$ and $H \not\equiv 0$, where H is defined by (2.2). Then*

$$N_E^{1)}(r,1;F|=1)=N_E^{1)}(r,1;G|=1)\le N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.3. *Let $S=\{z: z^n+az^{n-1}+b=0\}$, where a, b are nonzero constants such that $z^n+az^{n-1}+b=0$ has no multiple roots, $n\ (\ge 3)$ is an integer, and let F, G be given by (2.1). If for two non-constant meromorphic functions f and g, $E_f(S,0)=E_g(S,0)$ and $H\not\equiv 0$, then*

$$\begin{aligned} N(r,H) \ \le\ & \overline{N}(r,0,f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a;f\,|\ge 2)\\ &+\overline{N}(r,-a;g\,|\ge 2)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,-a\frac{n-1}{n};g)+\overline{N}_*(r,1;F,G)\\ &+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g'), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of those zeros of f', which are not the zeros of $f(f+a)(f+a\frac{n-1}{n})(F-1)$ and $\overline{N}_0(r,0;g')$ is defined similarly.

Proof. Since $E_f(S,0)=E_g(S,0)$ it follows that F and G share $(1,0)$. From (2.1) we have $F'=[nf+(n-1)a]f^{n-2}f'/(-b)$ and $G'=[ng+(n-1)a]g^{n-2}g'/(-b)$. It can easily be verified that the possible poles of H occur at: (i) zeros of f and g, (ii) multiple zeros of $f+a$ and $g+a$, (iii) zeros of $nf+a(n-1)$ and $ng+a(n-1)$, (iv) poles of f and g, (v) those 1-points of F and G with different multiplicities, (vi) zeros of f', which are not the zeros of $f(f+a)\left(f+a\frac{n-1}{n}\right)(F-1)$, (vii) zeros of g', which are not zeros of $g(g+a)\left(g+a\frac{n-1}{n}\right)(G-1)$. Since H has only simple poles, the result follows from above. Lemma 2.3 is proved.

Lemma 2.4. [11]. *Let f, g be two non-constant meromorphic functions. Then $f^{n-1}(f+a)g^{n-1}(g+a)\not\equiv b$, where a, b are nonzero finite constants and $n\ (\ge 5)$ is an integer.*

Lemma 2.5. *Let f, g be two non-constant meromorphic functions such that $\Theta_f+\Theta_g>\frac{n+1}{n-1}$, where Θ_f and Θ_g are as in in the Theorem 1.1. Then $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$ implies $f\equiv g$, where $n\ (\ge 2)$ is an integer and a is a nonzero finite constant.*

Proof. Let

$$f^{n-1}(f+a)\equiv g^{n-1}(g+a) \tag{2.3}$$

and suppose $f\not\equiv g$. We consider two cases:

Case I Let $y=\frac{g}{f}$ be a constant. Then it follows from (2.3) that $y\ne 1$, $y^{n-1}\ne 1$, $y^n\ne 1$ and $f\equiv -a\frac{1-y^{n-1}}{1-y^n}$ is a constant, which is impossible.

Case II Let $y=\frac{g}{f}$ be non-constant. Then

$$f\equiv -a\frac{1-y^{n-1}}{1-y^n}\equiv a\left(\frac{y^{n-1}}{1+y+y^2+\ldots+y^{n-1}}-1\right). \tag{2.4}$$

and

$$f+a\frac{(n-1)}{n}\equiv -a\frac{1-y^{n-1}}{1-y^n}+a\frac{(n-1)}{n}\equiv -a\frac{(n-1)y^n-ny^{n-1}+1}{n(1-y^n)}. \tag{2.5}$$

Assuming $p(z) = (n-1)z^n - nz^{n-1} + 1$, we have $p(0) \neq 0$ and $p(1) = p'(1) = 0$. So from (2.5) we obtain $\sum\limits_{j=1}^{n-1} \overline{N}(r, u_j; y) \leq \overline{N}(r, -a\frac{n-1}{n}; f)$, where u_j, $j = 1, 2, \ldots, n-1$, have the same meaning as in Example *1.2*.

>From (2.4) and Lemma 2.1 we obtain $T(r, f) = (n-1)T(r, y) + S(r, y)$. We first note that the zeros of $1 + y + y^2 + \ldots + y^{n-2}$ contributes to the zeros of both f and g. In addition, the poles of y contributes to the zeros of f and since $g = fy$ the zeros of y contributes to the zeros of g. So from (2.4) we find

$$\sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \overline{N}(r, \infty; y) \leq \overline{N}(r, 0; f), \sum_{k=1}^{n-1} \overline{N}(r, w_k; y) \leq \overline{N}(r, \infty; f),$$

where $w_k = exp\left(\frac{2k\pi i}{n}\right)$ for $k = 1, 2, \ldots, n-1$ and $v_j = exp\left(\frac{2j\pi i}{n-1}\right)$ for $j = 1, 2, \ldots, n-2$. Also, from (2.4) we have $\overline{N}(r, 0; y) \leq \frac{1}{2}N_2(r, -a; f)$.

Hence by the second fundamental theorem we can write

$$\begin{aligned}
(3n-4)\, T(r, y) &\leq \overline{N}(r, \infty; y) + \sum_{i=1}^{n-1} \overline{N}(r, u_i; y) + \sum_{j=1}^{n-2} \overline{N}(r, v_j; y) + \sum_{k=1}^{n-1} \overline{N}(r, w_k; y) \\
&\quad + \overline{N}(r, 0; y) + S(r, y) \\
&\leq \overline{N}(r, 0; f) + \overline{N}(r, -a\frac{n-1}{n}; f) + \overline{N}(r, \infty; f) + \frac{1}{2}N_2(r, -a; f) + S(r, y) \\
&\leq \left(\frac{7}{2} - \Theta(0; f) - \Theta(-a\frac{n-1}{n}; f) - \Theta(\infty; f) - \frac{1}{2}\delta_2(-a; f) + \varepsilon\right) T(r, f) \\
&\quad + S(r, y) \\
&= (n-1)\left(\frac{7}{2} - \Theta_f + \varepsilon\right) T(r, y) + S(r, y),
\end{aligned}$$

implying

$$\frac{3n-4}{n-1}\, T(r, y) \leq \left(\frac{7}{2} - \Theta_f + \varepsilon\right) T(r, y) + S(r, y), \tag{2.6}$$

where $0 < 2\varepsilon < \Theta_f + \Theta_g$. Again putting $y_1 = \frac{1}{y}$ and noting that $T(r, y) = T(r, y_1) + O(1)$, we can use the above arguments to obtain

$$\frac{3n-4}{n-1}\, T(r, y) \leq \left(\frac{7}{2} - \Theta_g + \varepsilon\right) T(r, y) + S(r, y). \tag{2.7}$$

Adding (2.6) and (2.7) we get $\left(\frac{6n-8}{n-1} - 7 + \Theta_f + \Theta_g - 2\varepsilon\right) T(r, y) \leq S(r, y)$, which is a contradiction. Hence $f \equiv g$, and the result follows. Lemma 2.5 is proved.

Lemma 2.6. *Let f be a non-constant meromorphic function and let a_i, $i = 1, 2, \ldots, n$, be finite distinct complex numbers, where $n \geq 2$. Then*

$$N(r, 0; f') \leq T(r, f) + \overline{N}(r, \infty; f) - \sum_{i=1}^{n} m(r, a_i; f) + S(r, f)$$

Proof. Let $F = \sum\limits_{i=1}^{n} \frac{1}{f - a_i}$, then $\sum\limits_{i}^{n} m(r, a_i; f) = m(r, F) + O(1)$. Note that

$$m(r, F) \leq m(r, 0; f') + m(r, \sum_{i=1}^{n} \frac{f'}{f - a_i}) = T(r, f') - N(r, 0; f') + S(r, f).$$

Also, observe that $T(r, f') = m(r, f') + N(r, f') \leq T(r, f) + \overline{N}(r, f) + S(r, f)$ and the result follows. Lemma 2.6 is proved.

3. Proof of Theorem 1.1

We know from the assumption that the zeros of $z^n+az^{n-1}+b$ are simple; we denote them by ω_j, $j=1,2,\dots n$. Let F, G be given by (2.1). Since $E_f(S,m)=E_g(S,m)$ it follows that F, G share $(1,m)$.

Case 1. We first consider the case $H\not\equiv 0$, where H is given by (2.2).

Subcase 1.1. $m\geq 1$. Assuming first that $m\geq 2$ and using Lemma 2.6 with $n=3$, $a_1=0$, $a_2=-a$ and $a_3=-a\frac{n-1}{n}$ we can write

$$
\begin{aligned}
(3.1)\quad & \overline{N}_0(r,0;g')+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G)\leq \overline{N}_0(r,0;g')\\
+\ & \overline{N}(r,1;G\mid\geq 2)+\overline{N}(r,1;G\mid\geq 3)\leq \overline{N}_0(r,0;g')\\
+\ & \sum_{j=1}^{n}\{\overline{N}(r,\omega_j;g\mid=2)+2\overline{N}(r,\omega_j;g\mid\geq 3)\}\leq N(r,0;g'\mid g\neq 0,-a,-a\frac{n-1}{n})\\
\leq\ & N(r,0;g')-N(r,0;g)+\overline{N}(r,0;g)-N(r,-a;g)+\overline{N}(r,-a;g)\\
& -N(r,-a\frac{n-1}{n};g)-\overline{N}(r,-a\frac{n-1}{n};g)\\
=\ & \overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,-a;g)+\overline{N}(r,-a\frac{n-1}{n};g)-2T(r,g)+S(r,g).
\end{aligned}
$$

Hence using (3.1) and Lemmas 2.1 – 2.3, from second fundamental theorem we have for any $\varepsilon>0$

$$
\begin{aligned}
(3.2)\quad & (n+2)\,T(r,f)\leq \overline{N}(r,0;f)+\overline{N}(r,-a;f)+\overline{N}(r,-a\frac{n-1}{n};f)\\
& +\overline{N}(r,\infty;f)+N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\geq 2)-N_0(r,0;f')+S(r,f)\\
\leq\ & \left(7-2\Theta(0,f)-2\Theta(\infty,f)-2\Theta(-a\frac{n-1}{n};f)-\delta_2(-a;f)+\frac{1}{2}\varepsilon\right)T(r,f)\\
& +\left(5-2\Theta(0,g)-2\Theta(\infty,g)-2\Theta(-a\frac{n-1}{n};g)-\delta_2(-a;g)+\frac{1}{2}\varepsilon\right)T(r,g)\\
& +S(r,f)+S(r,g)\leq(12-2\Theta_f-2\Theta_g+\varepsilon)\,T(r)+S(r).
\end{aligned}
$$

In a similar way we can obtain

$$(3.3)\qquad (n+2)\,T(r,g)\leq(11-2\Theta_f-2\Theta_g+\varepsilon)\,T(r)+S(r).$$

Combining (3.2) and (3.3) we conclude that

$$(3.4)\qquad (n-10+2\Theta_f+2\Theta_g-\varepsilon)\,T(r)\leq S(r).$$

Since $\varepsilon>0$, (3.4) leads to a contradiction. As for the case $m=1$, we use Lemma 2.6 to get the following counterpart of formula (3.1):

$$
\begin{aligned}
(3.5)\quad & \overline{N}_0(r,0;g')+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_*(r,1;F,G)\\
\leq\ & \overline{N}_0(r,0;g')+\overline{N}(r,1;G\mid\geq 2)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;F)\\
\leq\ & N(r,0;g'\mid g\neq 0,-a,-a\frac{n-1}{n})+\frac{1}{2}N(r,0;f'\mid f\neq 0,-a,-a\frac{n-1}{n})\\
\leq\ & \overline{N}(r,0;g)+\overline{N}(r,-a;g)+\overline{N}(r,-a\frac{n-1}{n};g)+\overline{N}(r,\infty;g)-2T(r,g)+\frac{1}{2}\{\overline{N}(r,0;f)\\
& +\overline{N}(r,-a;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;f)\Big\}-T(r,f)+S(r,f)+S(r,g).
\end{aligned}
$$

So using (3.5), Lemmas 2.2 and 2.3, and proceeding as in (3.2), from second fundamental theorem we have for any $\varepsilon > 0$

$$(3.6)\qquad (n+2)\,T(r,f) \le 2\left\{\overline{N}(r,0;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;f)\right\}$$
$$+N_2(r,-a;f)2\left\{\overline{N}(r,0;g)+\overline{N}(r,-a\frac{n-1}{n};g)+\overline{N}(r,\infty;g)\right\}+N_2(r,-a;g)$$
$$+\frac{1}{2}\left\{\overline{N}(r,0;f)+\overline{N}(r,-a;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;f)\right\}-2T(r,g)-T(r,f)$$
$$+S(r,f)+S(r,g) \le (11-2\Theta_f-2\Theta_g+\varepsilon)\,T(r)+2T(r)+S(r).$$

Similarly we can obtain

$$(3.7)\qquad (n+2)\,T(r,g) \le (11-2\Theta_f-2\Theta_g+\varepsilon)\,T(r)+2T(r)+S(r).$$

Combining (3.6) and (3.7) we conclude that

$$(3.8)\qquad (n-11+2\Theta_f+2\Theta_g-\varepsilon)\,T(r) \le S(r).$$

Since $\varepsilon > 0$, (3.8) leads to a contradiction.

Subcase 1.2. $m = 0$. Using Lemma 2.6 we observe that

$$
\begin{aligned}
(3.9)\qquad & \overline{N}_0(r,0;g')+\overline{N}_E^{(2}(r,1;F)+2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\le\ & \overline{N}_0(r,0;g')+\overline{N}_E^{(2}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\le\ & \overline{N}_0(r,0;g')+\overline{N}(r,1;G\,|\ge 2)+\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)\\
\le\ & N(r,0;g'\mid g\ne 0,-a,-a\frac{n-1}{n})+\overline{N}(r,1;G\,|\ge 2)+2\overline{N}(r,1;F\,|\ge 2)\\
\le\ & 2\left\{\overline{N}(r,0;g)+\overline{N}(r,\infty;g)+\overline{N}(r,-a;g)+\overline{N}(r,-a\frac{n-1}{n};g)\right.\\
& \left.+\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,-a;f)+\overline{N}(r,-a\frac{n-1}{n};f)\right\}\\
& -4T(r,f)-4T(r,g)+S(r,f)+S(r,g).
\end{aligned}
$$

Hence using (3.9) and Lemmas 2.2 and 2.3, from second fundamental theorem we have for any $\varepsilon > 0$

$$
\begin{aligned}
(3.10)\qquad & (n+2)\,T(r,f)\\
\le\ & \overline{N}(r,0;f)+\overline{N}(r,-a;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;f)+N_E^{1)}(r,1;F)\\
& +\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)-N_0(r,0;f')+S(r,f)\\
\le\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)\right\}+N_2(r,-a;f)\\
& +\overline{N}(r,0;g)+\overline{N}(r,-a\frac{n-1}{n};g)+\overline{N}(r,\infty;g)+\overline{N}(r,-a;g|\ge 2)+\overline{N}_E^{(2}(r,1;F)\\
& +2\overline{N}_L(r,1;G)+2\overline{N}_L(r,1;F)+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\\
\le\ & (16-3\Theta_f-3\Theta_g+\varepsilon)\,T(r)+2T(r)+S(r).
\end{aligned}
$$

In a similar manner we can obtain

$$(n+2)\,T(r,g) \le (16-3\Theta_f-3\Theta_g+\varepsilon)\,T(r)+2T(r)+S(r). \tag{3.11}$$

Combining (3.10) and (3.11) we conclude that

$$(n-16+3\Theta_f+3\Theta_g-\varepsilon)\,T(r) \le S(r). \tag{3.12}$$

Since $\varepsilon > 0$, (3.12) leads to a contradiction.

Case 2. $H \equiv 0$. By integration we get from (2.2)

$$\frac{1}{F-1} \equiv \frac{A}{G-1}+B, \tag{3.13}$$

where A and B are constants and $A \neq 0$. >From (3.13) we obtain

$$F \equiv \frac{(B+1)G+A-B-1}{BG+A-B}. \tag{3.14}$$

Clearly (3.14) together with Lemma 2.1 yields

$$T(r,f) = T(r,g)+O(1). \tag{3.15}$$

Subcase 2.1. Assume that $B \neq 0, -1$.
If $A-B-1 \neq 0$, then from (3.14) we obtain $\overline{N}(r, \frac{B+1-A}{B+1}; G) = \overline{N}(r,0;F)$. Hence using Lemma 2.1 and the second fundamental theorem we obtain

$$\begin{aligned} nT(r,g) &< \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{B+1-A}{B+1};G)+S(r,g) \\ &\le \overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\overline{N}(r,0;g+a)+\overline{N}(r,0;f)+\overline{N}(r,0;f+a)+S(r,g) \\ &\le 2T(r,f)+3T(r,g)+S(r,g), \end{aligned}$$

which, in view of (3.15), leads to a contradiction because $n \geq 6$. Thus $A-B-1=0$, and hence (3.14) reduces to $F \equiv \frac{(B+1)G}{BG+1}$, implying $\overline{N}(r, \frac{-1}{B}; G) = \overline{N}(r,\infty;f)$. Again by Lemma 2.1 and the second fundamental theorem we have

$$\begin{aligned} nT(r,g) &< \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{-1}{B};G)+S(r,g) \\ &\le \overline{N}(r,\infty;g)+\overline{N}(r,0;g)+\overline{N}(r,0;g+a)+\overline{N}(r,\infty;f)+S(r,g) \\ &\le T(r,f)+3T(r,g)+S(r,g), \end{aligned}$$

which, in view of (3.15), leads to a contradiction because $n \geq 6$.

Subcase 2.2. Assume that $B=-1$. From (3.14) we have

$$F \equiv \frac{A}{-G+A+1}. \tag{3.16}$$

If $A+1 \neq 0$, then from (3.17) we obtain $\overline{N}(r, A+1; G) = \overline{N}(r,\infty;f)$. So repeating the arguments used in the Subcase 2.1, we again get a contradiction. Hence $A+1=0$, and from (3.17) we infer $FG \equiv 1$, implying $f^{n-1}(f+a)g^{n-1}(g+a) \equiv b^2$, which is impossible by Lemma 2.4.

Subcase 2.3. Assome that $B = 0$. From (3.14) we obtain

$$F \equiv \frac{G + A - 1}{A}. \tag{3.17}$$

If $A - 1 \neq 0$, then from (3.17) we obtain $\overline{N}(r, 1 - A; G) = \overline{N}(r, 0; F)$. So in the same manner as above we again get a contradiction. So $A = 1$ and hence $F \equiv G$, that is, $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$. Now the assertion of the theorem follows from Lemma 2.5. This completes the proof of Theorem 1.1.

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