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SOME PROPERTIES OF CERTAIN CLASSES OF $p-$VALENT FUNCTIONS DEFINED BY THE HADAMARD PRODUCT

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Abstract. In this paper we obtain sandwich type theorems, inclusion relationships, convolution properties and coefficient estimates of certain classes of $p-$valent analytic functions defined by a convolution. Several other new results are also obtained.

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1. Introduction

Let H be the class of functions analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a,n]$ be the subclass of H consisting of function of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \ (z \in U).$$

Let $A(p)$ denote the class of all analytic functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1,2,3,\ldots\};\ z \in U). \tag{1.1}$$

We set $A(1) = A$. If $f(z)$ and $g(z)$ are analytic in U functions, we say that $f(z)$ is subordinate to $g(z)$, or equivalently, $g(z)$ is superordinate to $f(z)$, and write $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $\omega(z)$, which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ $(z \in U)$. It is known that

$$f(z) \prec g(z) \implies f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence (see [5], [18] and [19]): $f(z) \prec g(z) \Longleftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$. For functions $f(z)$ given by (1.1) and

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad (p \in \mathbb{N}; z \in U), \tag{1.2}$$

the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} = (g * f)(z).$$

For functions $f,g \in A(p)$, we define the linear operator $D^n_{\lambda,p} : A(p) \to A(p)$ $(\lambda \geq 0, p \in \mathbb{N}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ by: $D^0_{\lambda,p}(f*g)(z) = (f*g)(z)$, and

$$D^1_{\lambda,p}(f*g)(z) = D_{\lambda,p}(f*g)(z) = (1-\lambda)(f*g)(z) + \frac{\lambda z}{p}\left((f*g)(z)\right)',$$

and (in general)

$$\begin{aligned} D^n_{\lambda,p}(f*g)(z) &= D_{\lambda,p}(D^{n-1}_{\lambda,p}(f*g)(z)) \\ &= z^p + \sum_{k=1}^{\infty}\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}b_{k+p}z^{k+p} \quad (\lambda \geq 0). \end{aligned} \tag{1.3}$$

From (1.3), we can easily deduce that

$$\frac{\lambda}{p}z\left(D^n_{\lambda,p}(f*g)(z)\right)' = D^{n+1}_{\lambda,p}(f*g)(z) - (1-\lambda)D^n_{\lambda,p}(f*g)(z) \ (\lambda > 0). \tag{1.4}$$

The linear operator $D^n_{\lambda,1}(f*g)(z) = D^n_{\lambda}(f*g)(z)$ was introduced by Aouf and Seoudy in [3]. Observe that the operator $D^n_{\lambda,p}(f*g)(z)$ reduces to known operators for specific choices of g, n and λ. Some of them follows.

(*i*) For $\lambda = 1$ and $g(z) = \dfrac{z^p}{1-z}$ we have $D^n_{p,1}(f*g)(z) = D^n_p f(z)$, where D^n_p is the $p-$valent Salagean operator introduced and studied by Kamali and Orhan.

(*ii*) For $n = 0$ and $g(z) = z^p + \sum_{k=1}^{\infty}\left[\frac{p+l+\lambda k}{p+l}\right]^s z^k$ $(\lambda > 0; p \in; l, s \in \mathbb{N}_0)$, we get

$$D^0_{\lambda,p}(f*g)(z) = (f*g)(z) = I^s_p(\lambda,l)f(z),$$

where $I^s_p(\lambda,l)$ is the generalized multiplier transformation, which was introduced by Cătas [7]. Notice that the operator $I^s_p(\lambda,l)$ contains, as special cases, the multiplier transformation (see [8]), the generalized Sălăgeăn operator introduced and studied by Al-Oboudi [1], which in turn, contains as special case the Sălăgeăn operator ([24]).

(*iii*) For $n = 0$ and

$$g(z) = z^p + \sum_{k=1}^{\infty}\frac{(\alpha_1)_k...(\alpha_l)_k}{(\beta_1)_k...(\beta_s)_k(1)_k}z^{k+p},$$

where $\alpha_i, \beta_j \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $(i = 1,2,...l)$, $(j = 1,2,...s)$, $l \leq s+1, l,s \in \mathbb{N}_0$, we obtain

$$D^0_{\lambda,p}(f*g)(z) = (f*g)(z) = H_{p,l,s}(\alpha_1)f(z),$$

where $H_{p,l,s}(\alpha_1)$ is the Dziok-Srivastava operator introduced and studied in [9] (see also [10] and [11]). The operator $H_{p,l,s}(\alpha_1)$, in turn contains a number of other interesting operators such as, the Hohlov linear operator (see [12]), the Carlson-Shaffer linear operator (see [6] and [23]), the Ruscheweyh derivative operator (see [22]), the Bernardi-Libera-Livingston operator (see [4], [14] and [15]), and the Owa-Srivastava fractional derivative operator (see [20]).

Using the linear operator $D^n_{\lambda,p}(f*g)$, we define a new subclass $\mathcal{C}^n_{\lambda,p}(f,g;\alpha,A,B)$ of the class $A(p)$ as follows:

Definition 1.1. Let $g \in A(p)$ be defined by (1.3). A function $f \in A(p)$ is said to be in the class $\mathfrak{C}_{\lambda,p}^{n}(f,g;\alpha,A,B)$ if it satisfies the following subordination condition:

$$(1-\alpha)\frac{D_{\lambda,p}^{n}(f*g)(z)}{z^{p}}+\alpha\frac{D_{\lambda,p}^{n+1}(f*g)(z)}{z^{p}}\prec\frac{1+Az}{1+Bz}$$
$$(p\in\mathbb{N};n\in\mathbb{N}_{0};\lambda>0;\alpha\in\mathbb{C};-1\leq B<A\leq1;z\in U).$$

Let $\mathfrak{C}_{\lambda,p}^{n}(f,g;0,A,B)=\mathfrak{C}_{\lambda,p}^{n}(f,g;A,B)$, and $\mathfrak{C}_{\lambda,p}^{n}(f,g;\alpha,1-2\beta,-1)=\mathfrak{C}_{\lambda,p}^{n}(f,g;\alpha,\beta)$, where $\mathfrak{C}_{\lambda,p}^{n}(f,g;\alpha,\beta)$ denotes the class of functions from $A(p)$ satisfying

$$\Re\left\{(1-\alpha)\frac{D_{\lambda,p}^{n}(f*g)(z)}{z^{p}}+\alpha\frac{D_{\lambda,p}^{n+1}(f*g)(z)}{z^{p}}\right\}>\beta,$$

$p\in\mathbb{N};n\in\mathbb{N}_{0};\lambda>0;\alpha\in\mathbb{C};0\leq\beta<1;z\in U$. We set $\mathfrak{C}_{\lambda,p}^{n}(f,g;0,\beta)=\mathfrak{C}_{\lambda,p}^{n}(f,g;\beta)$.

In the present paper we establish subordination and superordination properties, convolution properties, inclusion relationships and embedding properties for the class $\mathfrak{C}_{\lambda,p}^{n}(f,g;\alpha,A,B)$. Several other new results are also obtained.

2. Preliminary results

In order to state and prove our main results, we need the following definition and a number of known lemmas.

Definition 2.1. [18]. *Define Q as the set of all functions $f(z)$ that are analytic and injective on $\bar{U}\backslash E(f)$, where*

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and satisfy $f'(\zeta)\neq0$ for $\zeta\in\bar{U}\backslash E(f)$.

Lemma 2.1. [19]. *Let $h(z)$ be an analytic and convex (univalent) function in U with $h(0)=1$. Suppose also that the function $\phi(z)$ given by*

$$\phi(z)=1+c_{1}z+c_{2}z^{2}+... \tag{2.1}$$

is analytic in U. If $\phi(z)+\frac{z\phi'(z)}{\gamma}\prec h(z)$ $\quad(\Re(\gamma)>0;\gamma\neq0)$, then

$$\phi(z)\prec\psi(z)=\gamma z^{-\gamma}\int_{0}^{z}h(t)t^{\gamma-1}\,dt\prec h(z),$$

and $\psi(z)$ is the best dominant.

Lemma 2.2. [25]. *Let $q(z)$ be a convex univalent function in U and let $\sigma\in\mathbb{C},\eta\in\mathbb{C}^{*}=\mathbb{C}\backslash\{0\}$ with*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0,-\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function $\phi(z)$ is analytic in U and $\sigma\phi(z)+\eta z\phi'(z)\prec\sigma q(z)+\eta zq'(z)$, then $\phi(z)\prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.3. [18]. *Let $q(z)$ be a convex univalent function in U and $\kappa \in \mathbb{C}$. Further assume that $\Re(\bar{\kappa}) > 0$. If $\phi(z) \in H[q(0), 1] \cap Q$, and $\phi(z) + \kappa z\phi'(z)$ is univalent in U, then $q(z) + \kappa zq'(z) \prec \phi(z) + \kappa z\phi'(z)$ implies $q(z) \prec \phi(z)$ and $q(z)$ is the best subdominant.*

Lemma 2.4. [16]. *Let $\mathfrak{F}$ be an analytic and convex function in U. If $f, g \in A$ and $f, g \prec \mathfrak{F}$ then $\lambda f + (1-\lambda) g \prec \mathfrak{F}$ $(0 \le \lambda < 1)$.*

Lemma 2.5.[21]. *Let $f(z) = 1 + \sum_{k=1} a_k z^k$ be analytic in U and $g(z) = 1 + \sum_{k=1} b_k z^k$ be analytic and convex in U. If $f \prec g$, then $|a_k| \le |b_1|$ $(k \in \mathbb{N})$.*

The next lemma contains three well-known identities for the Gauss hypergeometric function ${}_2F_1$ defined by

$$(2.2)\quad {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k \ \left(a,b,c \in \mathbb{C}; c \notin \mathbb{Z}_0^- = \{0,-1,-2,...\}; z \in U\right).$$

Notice that the series in (2.2) converges absolutely for $z \in U$, and hence ${}_2F_1$ represents an analytic function in U (for details we refer [26, Chapter 14]).

Lemma 2.6. [26]. For real or complex parameters a, b, and c $(c \notin Z_0^-)$, the following identities hold $(\Re(c) > \Re(b) > 0)$:

$$(2.3)\qquad \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z);$$

$$(2.4)\qquad {}_2F_1(a,b;c;z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right);$$

$$(2.5)\qquad {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)\ .$$

3. Main results

In what follows, unless otherwise stated, we assume that $p \in \mathbb{N}$, $n \in \mathbb{N}_0$, $-1 \le B < A \le 1$, $\lambda > 0$ and $g(z)$ is the function given by (1.2).

Our first result concerns subordination property.

Theorem 3.1. *Let $f \in \mathfrak{C}^n_{\lambda,p}(f, g; \alpha, A, B)$ with $\Re\{\alpha\} > 0$. Then*

$$(3.1)\qquad \frac{D^n_{\lambda,p}(f*g)(z)}{z^p} \prec \psi(z) \prec \frac{1+Az}{1+Bz},$$

where the function $\psi(z)$ given by

$$(3.2)\quad \psi(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1, 1; \frac{p}{\alpha\lambda} + 1; \frac{Bz}{Bz+1}\right), & if\ B \neq 0, \\ 1 + \frac{p}{\alpha\lambda + p} Az, & if\ B = 0. \end{cases}$$

is the best dominant of (3.1). Furthermore,

$$\Re\left\{\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}\right\} > \eta \quad (z\in U), \tag{3.3}$$

where $\eta = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{p}{\alpha\lambda}+1;\frac{B}{B-1}\right), & if\ B\neq 0,\\ 1-\frac{p}{\alpha\lambda+p}A, & if\ B=0.\end{cases}$

The estimate (3.3) *is the best possible.*

Proof. Consider the function

$$\phi(z) = \frac{D^n_{\lambda,p}(f*g)(z)}{z^p} \quad (z\in U), \tag{3.4}$$

and observe that $\phi(z)$ is of the form (2.1) and is analytic in U. Differentiating (3.5) with respect to z and using the identity (1.5), we get

$$(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p} + \alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p} = \phi(z) + \frac{\alpha\lambda}{p}z\phi'(z) \prec \frac{1+Az}{1+Bz} \quad (z\in U).$$

Now, using Lemma 2.1 for $\gamma = \frac{\alpha\lambda}{p}$, we obtain

$$\begin{aligned}\frac{D^n_{\lambda,p}(f*g)(z)}{z^p} &\prec \psi(z) = \frac{p}{\alpha\lambda} z^{-\frac{p}{\alpha\lambda}} \int_0^z \frac{1+At}{1+Bt} t^{\frac{p}{\alpha\lambda}-1}dt \\ &= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1;\frac{p}{\alpha\lambda}+1;\frac{Bz}{Bz+1}\right), & if B\neq 0,\\ 1+\frac{p}{\alpha\lambda+p}Az, & if B=0,\end{cases}\end{aligned}$$

where we have made a change of variable followed by the use of identities (2.3) - (2.5) with $a=1,\ b=\frac{p}{\alpha\lambda}$ and $c=b+1$. This proves the assertion (3.1).

Next, in order to prove (3.3), it is enough to show that $\inf_{|z|<1}\{\Re(\psi(z))\} = \psi(-1)$. Indeed, we have for $|z|\leq r<1$,

$$\Re\left(\frac{1+Az}{1+Bz}\right) \geq \frac{1-Ar}{1-Br}.$$

Setting $G(z,s) = \frac{1+Asz}{1+Bsz}$ and $dv(s) = \frac{p}{\alpha\lambda}s^{-\frac{p}{\alpha\lambda}}\,ds\ (0\leq s\leq 1)$, which is a positive measure on $[0,1]$,we get $\psi(z) = \int_0^1 G(z,s)\,dv(s)$, so that

$$\Re\{\psi(z)\} \geq \int_0^1 \frac{1-Asr}{1-Bsr}dv(s) = \psi(-r) \quad (|z|\leq r<1).$$

Letting $r\to 1^-$ in the last inequality, we obtain the assertion (3.3).

Finally, the estimate (3.3) is the best possible because $\psi(z)$ is the best dominant of (3.1). Theorem 3.1 is proved.

Taking $\alpha=1$ in Theorem 3.1, we obtain the following result.

Corollary 3.1. The following inclusion property holds true for the class $C^n_{\lambda,p}(f,g;A,B)$:

$$\mathfrak{C}^{n+1}_{\lambda,p}(f,g,A,B) \subset C^n_{\lambda,p}(f,g;\sigma) \subset \mathfrak{C}^n_{\lambda,p}(f,g;A,B),$$

where $\sigma = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\frac{p}{\lambda}+1;\frac{B}{B-1}\right), & if\ B \neq 0, \\ 1 - \frac{p}{\lambda+p}A, & if\ B = 0. \end{cases}$

The result is best possible.

Taking $A = 1 - 2\beta\,(0 \leq \beta < 1)$ and $B = -1$ in Corollary 3.1, we obtain the following

Corollary 3.2. The following inclusion holds: $C_{\lambda,p}^{n+1}(f,g;\beta) \subset C_{\lambda,p}^{n}(f,g;\sigma) \subset C_{\lambda,p}^{n}(f,g;\beta)$, where $\sigma = \beta + (1-\beta)\left\{ {}_2F_1\left(1,1;\frac{p}{\lambda}+1;\frac{1}{2}\right) - 1\right\}$. The result is best possible.

Theorem 3.2. *For $f \in C_{\lambda,p}^{n}(f,g;A,B)$ the function $F_{\delta,p}(f)$ defined by* (see [15])

$$F_{\delta,p}(f)(z) = \frac{\delta+p}{z^{\delta}} \int_0^z t^{\delta-1} f(t)\,dt \quad (\delta > -p) \tag{3.5}$$

belongs to the class $C_{\lambda,p}^{n}(f,g;A,B)$ and satisfies

$$\frac{D_{\lambda,p}^{n}(F_{\delta,p}(f) * g)(z)}{z^p} \prec k(z) \prec \frac{1+Az}{1+Bz},$$

where the function $k(z)$ given by

$$k(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1;\delta+p+1;\frac{Bz}{Bz+1}\right), & if B \neq 0, \\ 1 + \frac{\delta+p}{\delta+p+1}Az, & if B = 0, \end{cases} \tag{3.6}$$

is the best dominant of (3.7). Furthermore,

$$\Re\left\{\frac{D_{\lambda,p}^{n}(F_{\delta,p}(f) * g)(z)}{z^p}\right\} > \chi \quad (z \in U), \tag{3.7}$$

where

$$\chi = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\delta+p+1;\frac{B}{B-1}\right), & if\ B \neq 0, \\ 1 - \frac{\delta+p}{\delta+p+1}A & if\ B = 0. \end{cases} \tag{3.8}$$

The estimate (3.7) is the best possible.

Proof. From (3.5) we have

$$z\left(D_{\lambda,p}^{n}(F_{\delta,p}(f) * g)(z)\right)' = (\delta+p)\,D_{\lambda,p}^{n}(f * g)(z) - \delta D_{\lambda,p}^{n}(F_{\delta,p}(f) * g)(z). \tag{3.9}$$

We define

$$\phi(z) = \frac{D_{\lambda,p}^{n}(F_{\delta,p}(f) * g)(z)}{z^p} \quad (z \in U), \tag{3.10}$$

and observe that the function $\phi(z)$ is of the form (2.1) and is analytic in U. Differentiating (3.9) with respect to z and using the identity (3.8), we get

$$\frac{D_{\lambda,p}^{n}(f * g)(z)}{z^p} = \phi(z) + \frac{z\phi'(z)}{\delta+p} \prec \frac{1+Az}{1+Bz}.$$

The rest of the proof is similar to that of Theorem 3.1. Theorem 3.2 is proved.

Theorem 3.3. *If $f \in C_{\lambda,p}^{n}(f,g;\beta)\,(0 \leq \beta < 1)$, then $f \in C_{\lambda,p}^{n}(f,g;\alpha,\beta)\,(0 \leq \beta < 1, \alpha > 0)$ for $|z| < R$, where*

$$R=\sqrt{1+\left(\frac{\alpha\lambda}{p}\right)^{2}}-\frac{\alpha\lambda}{p}. \tag{3.11}$$

The bound R is the best possible.

Proof. Since $f\in C^{n}_{\lambda,p}(f,g;\beta)$, we can write $\frac{D^{n}_{\lambda,p}(f*g)(z)}{z^{p}}=\beta+(1-\beta)\,u(z)\ (z\in U)$. It is easy to see that the function $u(z)$ is of the form (2.1), is analytic and has a positive real part in U. Differentiating (3.7) with respect to z and using (1.4), we obtain

$$\frac{1}{1-\beta}\left\{(1-\alpha)\frac{D^{n}_{\lambda,p}(f*g)(z)}{z^{p}}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^{p}}-\beta\right\}=u(z)+\frac{\lambda\alpha}{p}zu'(z). \tag{3.12}$$

Using the following well-known estimate (see, e.g., [17]):

$$\frac{\left|zu'(z)\right|}{\Re\{u(z)\}}\leq\frac{2r}{1-r^{2}}\quad(|z|=r<1)$$

in view of (3.11) we obtain

$$\Re\left(\frac{1}{1-\beta}\left\{(1-\alpha)\frac{D^{n}_{\lambda,p}(f*g)(z)}{z^{p}}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^{p}}-\beta\right\}\right)$$

$$\geq\ \Re\{u(z)\}\left(1-\frac{2\lambda\alpha r}{p(1-r^{2})}\right). \tag{3.13}$$

It is easy to see that the right-hand side of (3.12) is positive, provided that $r<R$, where R is given by (3.10). In order to show that the bound R is the best possible, we consider the function $f\in A(p)$ defined by

$$\frac{D^{n}_{\lambda,p}(f*g)(z)}{z^{p}}=\beta+(1-\beta)\frac{1+z}{1-z}\quad(z\in U).$$

By noting that

$$\frac{1}{1-\beta}\left\{(1-\alpha)\frac{D^{n}_{\lambda,p}(f*g)(z)}{z^{p}}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^{p}}-\beta\right\}=\frac{1+z}{1-z}+\frac{2\lambda\alpha z}{p(1-z)^{2}}=0$$

for $|z|=R$, we conclude that the bound R is the best possible. Theorem 3.3 is proved.

Theorem 3.4. *Let $q(z)$ be a univalent function in U* and $\alpha\in\mathbb{C}^{*}$. *Suppose also that $q(z)$ satisfies*

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\left\{0,-\frac{p}{\lambda}\Re\left(\frac{1}{\alpha}\right)\right\}.$$

If $f\in A(p)$ satisfies the following subordination condition

$$(1-\alpha)\frac{D^{n}_{\lambda,p}(f*g)(z)}{z^{p}}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^{p}}\prec q(z)+\frac{\alpha\lambda}{p}zq'(z), \tag{3.14}$$

then $\frac{D^{n}_{\lambda,p}(f*g)(z)}{z^{p}}\prec q(z)$, and $q(z)$ is the best dominant.

Proof. Let the function $\phi(z)$ be defined by (3.5). From (3.13) we find that

$$\phi(z)+\frac{\alpha\lambda}{p}z\phi'(z)\prec q(z)+\frac{\alpha\lambda}{p}zq'(z). \tag{3.15}$$

By using Lemma 2.2 and (3.14), we easily get the assertion of Theorem 3.4.

Taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem 3.4, we get the following result.

Corollary 3.3. *Let $\alpha\in\mathbb{C}^*$ and $-1\le B\le A<1$. Suppose also that*

$$\Re\left(\frac{1-Bz}{1+Bz}\right)>\max\left\{0,-\frac{p}{\lambda}\Re\left(\frac{1}{\alpha}\right)\right\}.$$

If $f\in A(p)$ satisfies the following subordination condition:

$$(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}\prec\frac{1+Az}{1+Bz}+\frac{\alpha\lambda}{p}\frac{(A-B)z}{(1+Bz)^2},$$

then $\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}\prec\frac{1+Az}{1+Bz}$,and the function $\frac{1+Az}{1+Bz}$ is the best dominant.

Theorem 3.5. *Let $q(z)$ be a convex univalent function in U and $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$. Also let $\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}\in H[q(0),1]\cap Q$ and $(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}$ be univalent in U. If*

$$q(z)+\frac{\alpha\lambda}{p}zq'(z)\prec(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p},$$

then $q(z)\prec\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}$, and the function $q(z)$ is the best subdominant.

Proof. Let the function $\phi(z)$ be defined by (3.2). Then

$$q(z)+\frac{\alpha\lambda}{p}zq'(z)\prec(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}=\phi(z)+\frac{\alpha\lambda}{p}z\phi'(z).$$

An application of Lemma 3.3 yields the assertion. Theorem 3.5 is proved.

Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 5, we get the following result.

Corollary 3.4. *Let $q(z)$ be a convex univalent function in U and $-1\le B<A\le 1$, $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$. Also let $\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}\in H[q(0),1]\cap Q$, and*

$$(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}$$

be univalent in U. If

$$\frac{1+Az}{1+Bz}+\frac{\alpha\lambda}{p}\frac{(A-B)z}{(1+Bz)^2}\prec(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p},$$

then $\frac{1+Az}{1+Bz}\prec\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}$, and the function $\frac{1+Az}{1+Bz}$ is the best subdominant.

Combining the above results of subordination and superordination, we easily get the following "sandwich-type" result.

Corollary 3.5. *Let $q_1(z)$ be a convex univalent and $q_2(z)$ be a univalent functions in U, and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. Assume also that $q_2(z)$ satisfies (3.5). If*

$$\frac{D^n_{\lambda,p}(f*g)(z)}{z^p} \in H[q(0),1] \cap Q,$$

and $(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p} + \alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}$ is univalent in U, and also

$$q_1(z)+\frac{\alpha\lambda}{p}zq_1'(z) \prec (1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p} \prec q_2(z)+\frac{\alpha\lambda}{p}zq_2'(z),$$

then $q_1(z) \prec \frac{D^n_{\lambda,p}(f*g)(z)}{z^p} \prec q_2(z)$, and $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant, respectively.

Taking $q_1(z) = \frac{1+A_1z}{1+B_1z}$ and $q_2(z) = \frac{1+A_2z}{1+B_2z}$ $(-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1)$ in Corollary 3.5, we get the following result.

Corollary 3.6. *Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, and let $\Re\left(\frac{1-B_2z}{1+B_2z}\right) > \max\left\{0, -\Re\left(\frac{p}{\alpha\lambda}\right)\right\}$. If $\frac{D^n_{\lambda,p}(f*g)(z)}{z^p} \in H[q(0),1] \cap Q$, and$(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p} + \alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}$ is univalent in U, and also*

$$\begin{aligned}\frac{1+A_1z}{1+B_1z}+\frac{\alpha\lambda}{p}\frac{(A_1-B_1)z}{(1+B_1z)^2} &\prec (1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p} \\ &\prec \frac{1+A_2z}{1+B_2z}+\frac{\alpha\lambda}{p}\frac{(A_2-B_2)z}{(1+B_2z)^2},\end{aligned}$$

then $\frac{1+A_1z}{1+B_1z} \prec \frac{D^n_{\lambda,p}(f*g)(z)}{z^p} \prec \frac{1+A_2z}{1+B_2z}$, and the functions $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are the best subordinant and best dominant, respectively.

Theorem 3.6. *Let $\alpha_2 \ge \alpha_1 \ge 0$ and $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$. Then*

$$C^n_{\lambda,p}(f,g;\alpha_2,A_2,B_2) \subset C^n_{\lambda,p}(f,g;\alpha_1,A_1,B_1). \tag{3.16}$$

Proof. Assuming that $f \in C^n_{\lambda,p}(f,g;\alpha_2,A_2,B_2)$, we get

$$(1-\alpha_2)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha_2\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p} \prec \frac{1+A_2z}{1+B_2z}.$$

Since $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$, we easily find that

$$(1-\alpha_2)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha_2\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p} \prec \frac{1+A_2z}{1+B_2z} \prec \frac{1+A_1z}{1+B_1z}, \tag{3.17}$$

implying $f \in C^n_{\lambda,p}(f,g;\alpha_2,A_1,B_1)$. Thus Theorem 3.6 holds for $\alpha_2 = \alpha_1 \ge 0$. If $\alpha_2 > \alpha_1 \ge 0$, then by Theorem 3.1 and (3.21), we infer $f \in C^n_{\lambda,p}(f,g;A_1,B_1)$, implying

$$\frac{D^n_{\lambda,p}(f*g)(z)}{z^p} \prec \frac{1+A_1z}{1+B_1z}. \tag{3.18}$$

At the same time, we have

$$(1-\alpha_1)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha_1\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}=\left(1-\frac{\alpha_1}{\alpha_2}\right)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}$$

$$+\frac{\alpha_1}{\alpha_2}\left[(1-\alpha_2)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha_2\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}\right]. \quad (3.19)$$

Moreover, since $0\le\frac{\alpha_1}{\alpha_2}<1$, and the function $\frac{1+A_1z}{1+B_1z}$ $(-1\le B_1<A_1\le 1; z\in U)$ is analytic and convex in U, by (3.16) - (3.18) and Lemma 2.5, we find

$$(1-\alpha_1)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha_1\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}\prec\frac{1+A_1z}{1+B_1z},$$

that is, $f\in C^n_{\lambda,p}(f,g;\alpha_1,A_1,B_1)$, which implies (3.15). Theorem 3.6 is proved.

Theorem 3.7. A necessary and sufficient condition for $f\in C^n_{\lambda,p}(f,g;\alpha,A,B)$ is that

$$\sum_{k=1}\frac{p+\lambda\alpha k}{p(A-B)}\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}b_{k+p}\ne e^{i\theta}\quad(0<\theta<2\pi). \quad (3.20)$$

Proof. Observe that a function $f(z)\in C^n_{\lambda,p}(f,g;\alpha,A,B)$ if and only if

$$(1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p}+\alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p}\ne\frac{1+Ae^{i\theta}}{1+Be^{i\theta}}\quad(z\in U;\ 0<\theta<2\pi),$$

which is equivalent to the following

$$\frac{1}{z^p}\left[(1+Be^{i\theta})\left\{(1-\alpha)D^n_{\lambda,p}(f*g)(z)+\alpha D^{n+1}_{\lambda,p}(f*g)(z)\right\}-(1+Ae^{i\theta})\right]$$

$$=\ (1+Be^{i\theta})\left(1+\sum_{k=1}\frac{p+\lambda\alpha k}{p(A-B)}\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}b_{k+p}z^k\right)-(1+Ae^{i\theta})\ne 0,$$

which easily implies the convolution property (3.19). Theorem 3.7 is proved.

Theorem 3.8. A function $f(z)$ belongs to the class $C^n_{\lambda,p}(f,g;\alpha,A,B)$ $(\alpha>0)$ if its coefficients satisfy the condition:

$$\sum_{k=1}(p+\lambda\alpha k)\left(\frac{p+\lambda k}{p}\right)^n|a_{k+p}b_{k+p}|<p(A-B).$$

Proof. By Theorem 3.7, $f(z)\in C^n_{\lambda,p}(f,g;\alpha,A,B)$ if and only if

$$\sum_{k=1}\frac{p+\lambda\alpha k}{p(A-B)}\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}b_{k+p}\ne e^{i\theta}\quad(0<\theta<2\pi).$$

Thus $\left|\sum_{k=1}\frac{p+\lambda\alpha k}{p(A-B)}\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}b_{k+p}\right|<\left|e^{i\theta}\right|=1$, and the result follows.

Theorem 3.9. *Let*

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}\in C^n_{\lambda,p}(f,g;\alpha,A,B). \quad (3.21)$$

Then

$$(3.22) \qquad |a_{k+p}b_{k+p}| \leq \frac{p(A-B)}{p+\lambda\alpha k}\left(\frac{p}{p+\lambda k}\right)^n.$$

The inequality (3.21) is sharp, with the extremal function given by

$$(3.23) \qquad D^n_{\lambda,p}(f*g)(z) = \frac{p}{\alpha\lambda}z^p\int_0^1 u^{\frac{p}{\alpha\lambda}-1}\frac{1+Azu}{1+Bzu}du.$$

Proof. Combining (1.3) and (3.20), we obtain

$$(3.24) \quad (1-\alpha)\frac{D^n_{\lambda,p}(f*g)(z)}{z^p} + \alpha\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{z^p} \prec \frac{1+Az}{1+Bz} = 1+(A-B)z+\dots.$$

An application of Lemma 2.5 to (3.22) yields

$$(3.25) \qquad \left|\left(\frac{p+\lambda\alpha k}{p}\right)\left(\frac{p+\lambda k}{p}\right)^n a_{k+p}b_{k+p}\right| \leq A-B.$$

This and (3.23) imply (3.21). Theorem 3.9 is proved.

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