Известия НАН Армении. Математика, том 48, н. 4, 2013, стр. 63-80.

POSITIVE SOLUTIONS FOR MULTI-POINT BOUNDARY VALUE PROBLEMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we study the problem of existence of positive solution to the following boundary value problem: $D^{\sigma}_{0+}u''(t) - g(t)f(u(t)) = 0$, $t \in (0,1)$, $u''(0) = u''(1) = 0$, $au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, $cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, where D^{σ}_{0+} is the Riemann-Liouville fractional derivative of order $1 < \sigma \le 2$ and f is a lower semi-continuous function. Using Krasnoselskii's fixed point theorems in a cone, the existence of one positive solution and multiple positive solutions for nonlinear singular boundary value problems is established.

MSC2010 numbers: 34A08; 35B09; 47H10

Keywords: Cone; Multi point boundary value problem; Fixed point theorem; Riemann-Liouville fractional derivative.

1. Introduction

The purpose of this paper is to study the problem of existence of positive solutions for the following m-point boundary value problem for fractional differential equation

$$\begin{cases} D^{\sigma}_{0+}u''(t) - g(t)f(u(t)) = 0, & t \in (0,1), \\ u''(0) = u''(1) = 0, \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{1.1}$$

where D^{σ}_{0+} is the Riemann-Liouville fractional derivative of order $1 < \sigma \le 2$, $m > 2$ ($m \in \mathbb{N}$), $a, b, c, d \ge 0$, $\rho = ac + bc + ad > 0$, $\xi_i \in (0,1)$, $a_i, b_i \in (0, +\infty)$ ($i = 1, 2, \ldots, m-2$), $g \in C((0,1); [0, +\infty))$ and $0 < \int_0^1 g(r)dr < \infty$, and f is a nonnegative, lower semi-continuous function defined on $[0, +\infty)$.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and

N. N. thanks Razi University for support.

the applications of such constructions in various scientific fields, such as physics, mechanics, chemistry, engineering, etc. For details we refer to [5, 8, 9] and references therein.

The solution of differential equations of fractional order is much involved. Some analytical methods have been developed, such as the popular Laplace transform method [21, 22], the Fourier transform method [16], the iteration method [23], and Green function method [15, 24]. Numerical schemes for solving fractional differential equations also were introduced (see, e.g. [3, 4, 18]). A great deal of effort has been expended over the last years in attempting to find robust and stable numerical as well as analytical methods for solving fractional differential equations of physical interest. The Adomian decomposition method [20], homotopy perturbation method [19], homotopy analysis method [2], differential transformation method [17] and variational method [6] are relatively new approaches to provide analytical approximate solutions to linear and nonlinear fractional differential equations.

The problem of existence of solutions of initial value problems for fractional order differential equations have been studied in the literature (see [1, 11, 21, 23, 27] and the references therein).

In [13], Liu and Jia have investigated existence of multiple solutions for the problem:

$$\begin{cases} {}^cD^\sigma_{0+}(p(t)u'(t)) + q(t)f(t,u(t)) = 0, \quad t > 0, \ \ 0 < \sigma < 1, \\ p(0)u'(0) = 0, \\ \lim_{t\to\infty} u(t) = \int_0^{+\infty} g(t)u(t)dt, \end{cases}$$

where ${}^cD^\sigma_{0+}$ stands for the standard Caputo's derivative of order σ. Some existence results for the problem (1.1) with $\sigma = 2$ were obtained by Yanga et al. [25] and Zhao et al. [28].

In [12], Liu has considered existence of positive solutions for the following generalized Sturm-Liouville four-point boundary value problem:

$$\begin{cases} u''(t) + g(t)f(u(t)) = \theta, \qquad t \in (0,1), \\ au(0) - bu'(0) = a_1u(\xi_1), \\ cu(1) + du'(1) = b_1u(\xi_2), \end{cases}$$

by using the fixed points of strict-set-contractions.

In [26], Zhou and Chua have studied the following fractional differential equation with multi-point boundary conditions

$$\begin{cases} {}^cD^{\sigma}_{0+}u(t) = f(t, u(t), (Ku)(t), (Hu)(t)), \quad t \in (0,1), \\ au(0) - bu'(0) = d_1u(\xi_1), \\ cu(1) + du'(1) = d_2u(\xi_2), \end{cases}$$

where D^{σ}_{0+} is the Caputo's fractional derivative of order $1 < \sigma \leq 2$. By using the contraction mapping principle and the Krasnoselskii's fixed point theorem, the existence of solutions was established.

In this paper, motivated by the above-mentioned works, and using Krasnoselskii's fixed point theorems in a cone, we show that the problem (1.1) has positive solutions.

The remainder of the paper is organized as follows. In Section 2 we state some preliminary facts needed in the proofs of the main results. We also state a version of the Krasnoselakii's fixed point theorem. In Section 3, we state the main results of the paper, that establish existence of at least one or multiple positive solutions for the problem (1.1). Finally, in Section 4 we discuss an example that illustrates the main results of the paper.

2. Preliminaries

In this section, we present some notations and preliminary lemmas that will be used in the proofs of the main results.

We work in the space $C([0,1])$ with the norm $||u|| = \max_{0\leq t\leq 1}|u(t)|$. We make the following assumptions:

(H1) $f \in C([0,+\infty); [0,+\infty))$;

(H1*) f is a nonnegative, lower semi-continuous function defined on $[0,+\infty)$, i.e., there exist $I \subset [0,+\infty)$ such that for all $x_n \in I$, $x_n \to x_0$ as $n \to \infty$, one has $f(x_0) \leq \underline{\lim}_{n\to+\infty} f(x_n)$. Moreover, f has only a finite number of discontinuity points in each compact subinterval of $[0,+\infty)$.

(H2) $g \in C((0,1); [0,+\infty))$ and $0 < \int_0^1 g(r)dr < +\infty$. Moreover, $g(t)$ does not vanish identically on any subinterval of $[0,1]$;

(H3) $a, b, c, d \geq 0$, $\rho = ac + bc + ad > 0$, $\xi_i \in (0,1)$, $a_i, b_i \in (0,+\infty)$ $(i = 1,2,\dots,m-2)$, $\rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) > 0$, $\rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) > 0$ and $\Delta < 0$, where

$$\Delta = \begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix},$$

and for $t \in [0,1]$

$$\psi(t) = b + at \quad \text{and} \quad \varphi(t) = c + d - ct \tag{2.1}$$

are linearly independent solutions of the equation $x''(t) = 0$, $t \in [0,1]$. Observe that ψ is non-decreasing on $[0,1]$ while φ is non-increasing on $[0,1]$.

Definition 2.1. *Let X be a real Banach space. A non-empty closed set $P \subset X$ is called a cone of X if it satisfies the following conditions:*
(1) $x \in P, \mu \geq 0$ implies $\mu x \in P$,
(2) $x \in P, -x \in P$ implies $x = 0$.

Definition 2.2. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in L^1(\mathbb{R}^+)$ is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. *The Riemann-Liouville fractional derivative of order α ($n-1 < \alpha < n$, $n \in \mathbb{N}$) is defined as*

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.1. *([7]). The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L^1(0,1)$.*

Lemma 2.2. *([7]). Let $\alpha > 0$. Then the differential equation*

$$D_{0+}^{\alpha} u = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $n-1 < \alpha \leq n$.

Lemma 2.3. *([7]). Let $\alpha > 0$. Then the following equality holds for $u \in L^1(0,1)$, $D_{0+}^{\alpha} u \in L^1(0,1)$;*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, and $n-1 < \alpha \leq n$.

Now we present the Green function for a boundary value problem involving fractional differential equation.

Observe first that for $y(t) = u''(t)$ the problem

$$\begin{cases} D^{\sigma}_{0+}u''(t) - g(t)f(u(t)) = 0, & t \in (0,1), \\ u''(0) = u''(1) = 0, \end{cases}$$

becomes into the problem

$$\begin{cases} D^{\sigma}_{0+}y(t) - g(t)f(u(t)) = 0, & t \in (0,1), \\ y(0) = y(1) = 0, \end{cases} \tag{2.2}$$

Lemma 2.4. *If (H1) and (H2) are satisfied, then the boundary value problem (2.2) has a unique solution given by*

$$y(t) = -\int_0^1 H(t,s)g(s)f(u(s))ds, \tag{2.3}$$

where

$$H(t,s) = \begin{cases} \frac{t^{\sigma-1}(1-s)^{\sigma-1}-(t-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \le s \le t \le 1, \\ \frac{t^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Proof. According to Lemma 2.3 we can write

$$\begin{aligned} y(t) &= I^{\sigma}_{0+}\left(g(t)f(u(t))\right) - c_1 t^{\sigma-1} - c_2 t^{\sigma-2} \\ &= \tfrac{1}{\Gamma(\sigma)}\textstyle\int_0^t (t-s)^{\sigma-1}g(s)f(u(s))ds - c_1 t^{\sigma-1} - c_2 t^{\sigma-2}. \end{aligned}$$

Since $\sigma - 2 \le 0$, in view of the boundary condition $y(0) = 0$, we must set $c_2 = 0$ if $\sigma = 2$, and if $\sigma < 2$ then in order to have $c_2 t^{\sigma-2}$ well defined we must choose $c_2 = 0$. Also, using the boundary condition $y(1) = 0$ we must set $c_1 = \frac{1}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-1}g(s)f(u(s))ds$.

Thus, the unique solution of problem (2.2) is given by

$$y(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}g(s)f(u(s))ds - \frac{t^{\sigma-1}}{\Gamma(\sigma)}\int_0^1 (1-s)^{\sigma-1}g(s)f(u(s))ds$$

$$= -\int_0^t \frac{t^{\sigma-1}(1-s)^{\sigma-1} - (t-s)^{\sigma-1}}{\Gamma(\sigma)}g(s)f(u(s))ds - \int_t^1 \frac{t^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}g(s)f(u(s))ds$$

$$= -\int_0^1 H(t,s)g(s)f(u(s))ds.$$

This completes the proof.

Lemma 2.5. *If (H3) holds, then for $y \in C[0,1]$ the boundary value problem*

$$\begin{cases} u''(t) = y(t), & t \in (0,1), \\ au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \\ cu(1) + du'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{2.5}$$

has a unique solution given by

$$u(t) = -\Big[\int_0^1 G(t,s)y(s)ds + A(y(s))\psi(t) + B(y(s))\varphi(t)\Big], \tag{2.6}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases} \varphi(t)\psi(s), & 0 \le s \le t \le 1, \\ \varphi(s)\psi(t), & 0 \le t \le s \le 1, \end{cases} \tag{2.7}$$

$$A(y(s)) = \frac{1}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)y(s)ds & \rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)y(s)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i) \end{vmatrix}, \tag{2.8}$$

$$B(y(s)) = \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)y(s)ds \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)y(s)ds \end{vmatrix}. \tag{2.9}$$

Proof. The proof is similar to that of Lemma 5.5.1 in [14], and it is omitted. □

We assume that $\theta \in (0, \frac{1}{2})$, and for convenience, we set

$$\Lambda_1 = \min\Big\{\frac{\varphi(1-\theta)}{\varphi(0)}, \frac{\psi(\theta)}{\psi(1)}\Big\}, \qquad \Gamma = \min\{\Lambda_1, \frac{\Lambda_2}{\Lambda_3}\},$$
$$\Lambda_2 = \min\{\min_{\theta\le t\le 1-\theta}\varphi(t), \min_{\theta\le t\le 1-\theta}\psi(t), 1\}, \qquad \Lambda_3 = \max\{1, \|\varphi\|, \|\psi\|\}.$$

Lemma 2.6. *Let $\rho, \Delta \neq 0$ and $\theta \in (0,\frac{1}{2})$, then the following inequalities hold:*

$$0 \le G(t,s) \le G(s,s), \quad \text{for } t,s \in [0,1], \tag{2.10}$$

and

$$G(t,s) \ge \Lambda_1 G(s,s), \quad \text{for } t \in \big[\theta, 1-\theta\big] \text{ and } s \in [0,1]. \tag{2.11}$$

Proof. The inequality (2.10) is obvious. So, we have to verify only the inequality (2.11). To this end, observe that for $t \in [\theta, 1-\theta]$ and $s \in [0,1]$ we have

$$\begin{aligned} \frac{G(t,s)}{G(s,s)} &= \begin{cases} \frac{\varphi(t)}{\varphi(s)}, & 0 \le s \le t \le 1-\theta, \\ \frac{\psi(t)}{\psi(s)}, & \theta \le t \le s \le 1, \end{cases} \\ &\ge \begin{cases} \frac{\varphi(1-\theta)}{\varphi(0)}, & 0 \le s \le t \le 1-\theta, \\ \frac{\psi(\theta)}{\psi(1)}, & \theta \le t \le s \le 1, \end{cases} \ge \Lambda_1. \end{aligned}$$

This completes the proof.

Proposition 2.1. *For $t, s \in [0,1]$ we have*

$$0 \le H(t,s) \le H(s,s) \le \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{\sigma-1}.$$

Proposition 2.2. *Let $\theta \in (0, \frac{1}{2})$, then there exists a positive function $\varrho \in C(0,1)$ such that*

$$\min_{\theta \le t \le 1-\theta} H(t,s) \ge \varrho(s) H(s,s), \quad s \in (0,1).$$

Proof. For $\theta \in (0, \frac{1}{2})$ we define

$$\begin{aligned} g_1(t,s) &= t^{\sigma-1}(1-s)^{\sigma-1} - (t-s)^{\sigma-1}, \quad & 0 \le s \le t \le 1, \\ g_2(t,s) &= t^{\sigma-1}(1-s)^{\sigma-1}, & 0 \le t \le s \le 1. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{d}{dt} g_1(t,s) &= (\sigma-1)\Big(t^{\sigma-2}(1-s)^{\sigma-1} - (t-s)^{\sigma-2}\Big) \\ &= (\sigma-1)t^{\sigma-2}\Big((1-s)^{\sigma-1} - (1-\frac{s}{t})^{\sigma-2}\Big) \\ &\le (\sigma-1)t^{\sigma-2}\Big((1-s)^{\sigma-1} - (1-s)^{\sigma-2}\Big), \end{aligned}$$

implying that $g_1(\cdot, s)$ is non-increasing for all $s \in (0,1]$. Also, taking into account that $g_2(\cdot, s)$ is non-decreasing for all $s \in (0,1)$, we can write

$$\begin{aligned} \min_{\theta \le t \le 1-\theta} H(t,s) &= \begin{cases} \frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, & s \in (0,\theta], \\ \min\{\frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, \frac{g_2(\theta,s)}{\Gamma(\sigma)}\}, & s \in [\theta, 1-\theta], \\ \frac{g_2(\theta,s)}{\Gamma(\sigma)}, & s \in [1-\theta, 1). \end{cases} \\ &= \begin{cases} \frac{g_1(1-\theta,s)}{\Gamma(\sigma)}, & s \in (0,\mu], \\ \frac{g_2(\theta,s)}{\Gamma(\sigma)}, & s \in [\mu, 1). \end{cases} \\ &= \begin{cases} \frac{(1-\theta)^{\sigma-1}(1-s)^{\sigma-1} - (1-\theta-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in (0,\mu], \\ \frac{\theta^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, & s \in [\mu, 1), \end{cases} \end{aligned}$$

where $\theta < \mu < 1-\theta$ is a solution of the equation

$$(1-\theta)^{\sigma-1}(1-\mu)^{\sigma-1} - (1-\theta-\mu)^{\sigma-1} = \theta^{\sigma-1}(1-\mu)^{\sigma-1}.$$

It follows from the monotonicity of g_1 and g_2 that

$$\max_{0 \le t \le 1} H(t,s) = H(s,s) = \frac{s^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)}, \quad s \in (0,1).$$

Therefore, setting

$$\varrho(s) = \begin{cases} \frac{(1-\theta)^{\sigma-1}(1-s)^{\sigma-1} - (1-\theta-s)^{\sigma-1}}{s^{\sigma-1}(1-s)^{\sigma-1}}, & s \in (0,\mu], \\ \left(\frac{\theta}{s}\right)^{\sigma-1} & s \in [\mu, 1), \end{cases}$$

we complete the proof.

Remark 2.1. *It follows from Lemmas 2.4 and 2.5 that* $u(t)$ *is a solution of the problem (1.1) if and only if*

$$u(t) = \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t), \tag{2.12}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Lemma 2.7. *Let (H1), (H2) and (H3) be fulfilled. Then the solution* u *of the problem (1.1) satisfies the following conditions:*

(i) $u(t) \geq 0$ *for* $t \in [0,1]$,

(ii) $\min_{\theta \leq t \leq 1-\theta} u(t) \geq \Gamma ||u||$.

Proof. (i) By Lemma 2.6, Proposition 2.1, formulas (2.3) and (2.6)-(2.9), we have

$$G(t,s) \geq 0, \quad W(s) \geq 0, \quad A(W(s)) \geq 0, \quad B(W(s)) \geq 0,$$

implying that $u(t) \geq 0$ for $t \in [0,1]$.

(ii) By Lemma 2.6 and formula (2.12) for $t \in [\theta, 1-\theta]$ we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)W(s)ds + \frac{\Lambda_2}{\Lambda_3} \cdot \Lambda_3[A(W(s)) + B(W(s))] \\ &\geq \Gamma\Big[\int_0^1 G(s,s)W(s)ds + \Lambda_3[A(W(s)) + B(W(s))]\Big] \\ &\geq \Gamma ||u||. \end{aligned}$$

This implies $\min_{\theta \leq t \leq 1-\theta} u(t) \geq \Gamma ||u||$. Lemma 2.7 is proved.

Next, for $\theta \in (0, \frac{1}{2})$ we choose a cone $K = K_\theta$ in $C^1([0,1])$ by setting

$$K = K_\theta = \{u \in C[0,1] \mid u(t) \geq 0, \min_{\theta \leq t \leq 1-\theta} u(t) \geq \Gamma \|u\|\},$$

and define an operator T by

$$(Tu)(t) = \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t), \tag{2.13}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

It is clear that the existence of a positive solution for the system (1.1) is equivalent to the existence of nontrivial fixed point of T in K.

Lemma 2.8. *Suppose that the conditions (H1) and (A1) hold, then* $T(K) \subseteq K$ *and* $T : K \to K$ *is completely continuous.*

Proof. By (2.13), for any $u \in K$ we have $(Tu)(t) \geq 0$, and for $t \in [0,1]$ we can write

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\leq \int_0^1 G(s,s)W(s)ds + \Lambda_3[A(W(s)) + B(W(s))].\end{aligned}$$

Thus,

$$||Tu|| \leq \int_0^1 G(s,s)W(s)ds + \Lambda_3[A(W(s)) + B(W(s))].$$

On the other hand for $t \in [\theta, 1-\theta]$ we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t) \\ &\geq \Lambda_1 \int_0^1 G(s,s)W(s)ds + \frac{\Lambda_2}{\Lambda_3} \cdot \Lambda_3[A(W(s)) + B(W(s))] \\ &\geq \Gamma\Big[\int_0^1 G(s,s)W(s)ds + \Lambda_3[A(W(s)) + B(W(s))]\Big] \\ &\geq \Gamma||Tu||.\end{aligned}$$

This implies $TK \subseteq K$. Using standard arguments and Arzela-Ascoli theorem it can be easily verified that $T : K \to K$ is completely continuous, so we omit the details. Thus, Lemma 2.8 is proved.

As it was mentioned above, our approach to the existence of positive solutions for boundary value problems for fractional differential equations is based on the Krasnoselskii's fixed point theorems in a cone. For completeness of the presentation here we state the following Guo-Krasnoselskii fixed point theorem in a cone (see [10]).

Theorem 2.1. *Let E be a Banach space and $K \subseteqq E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K\bigcap(\overline{\Omega}_2\backslash\Omega_1) \to K$ be a completely continuous operator. Then under each of the following conditions the operator T has a fixed point in $K \cap (\overline{\Omega}_2\backslash\Omega_1)$:*

(A) $\|Tu\| \leq \|u\|, \forall u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, \forall u \in K \cap \partial\Omega_2$;

(B) $\|Tu\| \geq \|u\|, \forall u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, \forall u \in K \cap \partial\Omega_2$.

3. Main results

We define $\Omega_l = \{u \in K : \|u\| < l\}$ and $\partial\Omega_l = \{u \in K : \|u\| = l\}$, where $l > 0$. Observe that if $u \in \partial\Omega_l$ for $t \in [\theta, 1-\theta]$, then we have $\Gamma l \leq u \leq l$. Also, for convenience, we introduce the following notation:

$$\begin{aligned} f_l &= \inf\Big\{\frac{f(u)}{l}\Big|u \in [\Gamma l, l]\Big\}, \qquad f^l = \sup\Big\{\frac{f(u)}{l}\Big|u \in [0, l]\Big\},\\ f_\varrho &= \liminf_{u\to\varrho} \frac{f(u)}{u}, \quad f^\varrho = \limsup_{u\to\varrho} \frac{f(u)}{u}; \quad (\varrho := 0^+ \text{ or } +\infty),\\ \eta &= \min_{\theta\leq s\leq 1-\theta} \varrho(s),\\ \frac{1}{\omega} &= \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\Big[\Big(\int_0^1 G(s,s)ds\Big)\Big(\int_0^1 g(\tau)d\tau\Big) + \Lambda_3\widetilde{A} + \Lambda_3\widetilde{B}\Big],\\ \frac{1}{M} &= \frac{\eta}{\Gamma(\sigma)}\theta^{2(\sigma-1)}\Big[\frac{\Lambda_1}{\rho}\varphi(1-\theta)\psi(\theta)\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big) + \Lambda_2\widehat{A} + \Lambda_2\widehat{B}\Big]. \end{aligned}$$

In the theorems that follow, we always assume that the assumption (H1) is fulfilled.

Theorem 3.1. *Suppose that there exist constants $r, R > 0$ with $r < \Gamma R$ for $r < R$, such that the following two conditions are satisfied:*

(H4) $f^r \leq \omega$,

(H5) $f_R \geq M$.

Then the problem (1.1) has at least one positive solution $u \in K$, such that

$$0 < r \leq \|u\| \leq R.$$

Proof. Case 1. We prove the result assuming that (H1) is satisfied. Also, without loss of generality, we can assume that $r < \Gamma R$ for $r < R$.

By (H4), Proposition 2.1, and formulas (2.8) and (2.9), for $u \in \Omega_r$ we have

$$\begin{aligned} A(W) &\leq \frac{\big(\frac{1}{\Gamma(\sigma)}\big)\big(\frac{1}{4}\big)^{(\sigma-1)}\omega r}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)(\int_0^1 g(\tau)d\tau)ds & \rho - \sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s)(\int_0^1 g(\tau)d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix},\\ &= \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\widetilde{A}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} B(W) &\leq \frac{\big(\frac{1}{\Gamma(\sigma)}\big)\big(\frac{1}{4}\big)^{(\sigma-1)}\omega r}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)(\int_0^1 g(\tau)d\tau)ds \\ \rho - \sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s)(\int_0^1 g(\tau)d\tau)ds\end{vmatrix}\\ &= \Big(\frac{1}{\Gamma(\sigma)}\Big)\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\widetilde{B}. \end{aligned} \tag{3.2}$$

Therefore, by (H4), Lemma 2.6, and formulas (2.13) – (3.2), for $t \in [0,1]$ and $u \in \Omega_r$ we can write

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t)\\ &\le \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\Big(\int_0^1 G(s,s)ds\Big)\Big(\int_0^1 g(\tau)d\tau\Big)\\ &\quad+\frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\widetilde{A}\psi(t) + \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\widetilde{B}\varphi(t)\\ &\le \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\omega r\Big[\Big(\int_0^1 G(s,s)ds\Big)\Big(\int_0^1 g(\tau)d\tau\Big) + \Lambda_3\widetilde{A} + \Lambda_3\widetilde{B}\Big]\\ &= r = \|u\|.\end{aligned}$$

This implies that $\|Tu\| \le \|u\|$ for $u \in \Omega_r$.

On the other hand, by (H5), Proposition 2.2 and formulas (2.8), (2.9) and (2.13), for $u \in \Omega_R$ we have

$$\begin{aligned}A(W) &\ge \frac{(\frac{\eta}{\Gamma(\sigma)})\theta^{2(\sigma-1)}MR}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_\theta^{1-\theta} G(\xi_i,s)(\int_\theta^{1-\theta} g(\tau)d\tau)ds & \rho-\sum_{i=1}^{m-2}a_i\varphi(\xi_i)\\ \sum_{i=1}^{m-2} b_i\int_\theta^{1-\theta} G(\xi_i,s)(\int_\theta^{1-\theta} g(\tau)d\tau)ds & -\sum_{i=1}^{m-2}b_i\varphi(\xi_i)\end{vmatrix},\\ &= \Big(\frac{\eta}{\Gamma(\sigma)}\Big)^{q-1}\theta^{2(\sigma-1)(q-1)}MR\widehat{A}, \qquad (3.3)\end{aligned}$$

and

$$\begin{aligned}B(W) &\ge \frac{(\frac{\eta}{\Gamma(\sigma)})\theta^{2(\sigma-1)}MR}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2}a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_\theta^{1-\theta} G(\xi_i,s)(\int_\theta^{1-\theta} g(\tau)d\tau)ds\\ \rho-\sum_{i=1}^{m-2}b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_\theta^{1-\theta} G(\xi_i,s)(\int_\theta^{1-\theta} g(\tau)d\tau)ds\end{vmatrix}\\ &= \Big(\frac{\eta}{\Gamma(\sigma)}\Big)\theta^{2(\sigma-1)}MR\widehat{B}. \qquad (3.4)\end{aligned}$$

Therefore, by (H5), Lemma 2.6 and formulas (2.13), (3.3) and (3.4), for $t \in [0,1]$ and $u \in \Omega_R$ we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t)\\ &\ge \frac{\eta}{\Gamma(\sigma)}\theta^{2(\sigma-1)}MR\Big[\frac{\Lambda_1}{\rho}\varphi(1-\theta)\psi(\theta)\Big(\int_\theta^{1-\theta} g(\tau)d\tau\Big) + \Lambda_2\widehat{A} + \Lambda_2\widehat{B}\Big]\\ &= R = \|u\|.\end{aligned}$$

This implies that $\|Tu\| \ge \|u\|$ for $u \in \Omega_R$.

Therefore, by Theorem 2.1, it follows that T has a fixed point u in $K \cap (\overline{\Omega_R} \setminus \Omega_r)$. This means that the problem (1.1) has at least one positive solution $u \in K$ satisfying $0 < r \le \|u\| \le R$.

Case 2. When (H1*) holds, by applying the linear approaching method on the domain of discontinuous points of f we can construct a sequence $\{f_j\}_{j=1}^{\infty}$ satisfying the following two conditions

(i) $f_j \in C[0,\infty)$ and $0 \le f_j \le f_{j+1}$ on $[0,\infty)$,

(ii) $\lim_{j\to\infty} f_j = f$, $j = 1,2,\dots$, is pointwisely convergent on $[0,\infty)$.

According to the Case 1, for $f = f_j$ the problem (1.1) has a positive solution $u_j(t)$ given by

$$\begin{aligned}
u_j(t) &= \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds \\
&+\frac{\psi(t)}{\Delta}\left|\begin{matrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{matrix}\right| \\
&+\frac{\varphi(t)}{\Delta}\left|\begin{matrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds \\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau)ds\end{matrix}\right| \\
&= \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds + \psi(t)A_j + \varphi(t)B_j,
\end{aligned}$$

for all $t \in [0,1]$ and $r \le \|u_j\| \le R$, where r and R are independent of j.

By uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$, and $\varphi(t)$, $\psi(t)$ on $[0,1]$, for any small enough $\epsilon > 0$ there exists $\delta > 0$ such that for $t_1,t_2 \in [0,1]$ and $|t_1-t_2| < \delta$, one has $|G(t_1,s)-G(t_2,s)| < \epsilon$, $|\varphi(t_1)-\varphi(t_2)| < \epsilon$ and $|\psi(t_1)-\psi(t_2)| < \epsilon$. Thus, for $t_1,t_2 \in [0,1]$ and $|t_1-t_2| < \delta$ we can write

$$\begin{aligned}
|u_j(t_1)-u_j(t_2)| &\le \int_0^1 |G(t_1,s)-G(t_2,s)|\cdot\Big(\int_0^1 H(s,\tau)g(\tau)f_j(u_j(\tau))d\tau\Big)ds \\
&\quad +A_j|\psi(t_1)-\psi(t_2)| + B_j|\varphi(t_1)-\varphi(t_2)| \\
&\le \frac{1}{\Gamma(\sigma)}\Big(\frac{1}{4}\Big)^{(\sigma-1)}\cdot \max_{\|u_j\|\le R} f_j(u_j)\cdot\Big(\int_0^1 g(\tau)d\tau\Big)\cdot\epsilon + A_j\cdot\epsilon + B_j\cdot\epsilon.
\end{aligned}$$

Thus, $\{u_j\}_{j=1}^{\infty}$ are equicontinuous on $[0,1]$, and hence by Arzela-Ascoli theorem there exists a convergent subsequence of $\{u_j\}_{j=1}^{\infty}$. For convenience, we denote this convergent subsequence by $\{u_j\}_{j=1}^{\infty}$, and without loss of generality, we assume that $\lim_{j\to\infty} u_j(t) = u(t)$, $\forall t \in [0,1]$, and $r \le \|u\| \le R$. By Fatou's Lemma and Lebesgue dominated convergence theorem we have

$$\begin{aligned}\lim_{j\to\infty} u_j(t) \geq& \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau\Big)ds\\ &+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i)\\ \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix}\\ &+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds\\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)\lim_{j\to\infty} f_j(u_j(\tau))d\tau)ds\end{vmatrix},\end{aligned}$$

implying

$$u(t) \geq \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t), \tag{3.5}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

On the other hand, by the conditions (i) and (ii) we have

$$\begin{aligned}u_j(t) \leq& \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau\Big)ds\\ &+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i)\\ \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u_j(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix}\\ &+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u(\tau))d\tau)ds\\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f_j(u(\tau))d\tau)ds\end{vmatrix}.\end{aligned}$$

By the lower semi-continuity of f, we can pass to the limit in the above inequality as $j\to\infty$ to obtain

$$\begin{aligned}u(t) \leq& \int_0^1 G(t,s)\Big(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau\Big)ds\\ &+\frac{\psi(t)}{\Delta}\begin{vmatrix} \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds & \rho-\sum_{i=1}^{m-2} a_i\varphi(\xi_i)\\ \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds & -\sum_{i=1}^{m-2} b_i\varphi(\xi_i)\end{vmatrix}\\ &+\frac{\varphi(t)}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\psi(\xi_i) & \sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds\\ \rho-\sum_{i=1}^{m-2} b_i\psi(\xi_i) & \sum_{i=1}^{m-2} b_i\int_0^1 G(\xi_i,s)(\int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau)ds\end{vmatrix}.\end{aligned}$$

Therefore

$$u(t) \leq \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t), \tag{3.6}$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Finally, by (3.5) and (3.6) we obtain

$$u(t) = \int_0^1 G(t,s)W(s)ds + A(W(s))\psi(t) + B(W(s))\varphi(t),$$

where $W(s) = \int_0^1 H(s,\tau)g(\tau)f(u(\tau))d\tau$.

Therefore $u(t)$ is a positive solution of the problem (1.1). This completes the proof of Theorem 3.1.

Similarly, we can prove the following theorem.

Theorem 3.2. *Assume that there exist constants $r, R > 0$ with $r < \Gamma R$ for $r < R$, such that the following two conditions are satisfied:*

(H4)* $f^r < \omega$,

(H5)* $f_R > M$.

Then the problem (1.1) has at least one positive solution $u \in K$ such that

$$0 < r < ||u|| < R.$$

Theorem 3.3. *Assume that one of the following two conditions is satisfied:*

(H6) $f^0 \leq \omega, \qquad f_\infty \geq \frac{M}{\Gamma}$,

(H7) $f_0 \geq \frac{M}{\Gamma}, \qquad f^\infty \leq \omega$

Then the problem (1.1) has at least one positive solution.

Proof. It is enough to prove the assertion of the theorem for nonnegative and continuous on $[0,\infty)$ functions. Then using the arguments of the proof of Theorem 3.1 we can extend the result to the case of nonnegative and lower semi-continuous on $[0,\infty)$ functions.

We show that (H6) implies (H4) and (H5). Suppose that (H6) holds, then there exist r and R with $0 < r < \gamma R$, such that

$$\frac{f(u)}{u} \leq \omega, \quad 0 < u \leq r \qquad \text{and} \qquad \frac{f(u)}{u} \geq \frac{M}{\Gamma}, \quad u \geq \Gamma R.$$

Hence

$$f(u) \leq \omega u \leq \omega r, \quad 0 < u \leq r$$

and

$$f(u) \geq \frac{M}{\Gamma} u \geq \frac{M}{\Gamma} \Gamma R = MR, \quad u \geq \Gamma R,$$

implying (H4) and (H5). Therefore, by Theorem 3.1 the problem (1.1) has at least one positive solution.

Now suppose that (H7) holds, then there exist $0 < r < R$ with $Mr < \omega R$ such that

$$\frac{f(u)}{u} \geq \frac{M}{\Gamma}, \quad 0 < u \leq r. \tag{3.7}$$

and

$$\frac{f(u)}{u} \leq \omega, \quad u \geq R. \tag{3.8}$$

By (3.7), it follows that

$$f(u) \geq \frac{M}{\Gamma}u \geq \frac{M}{\Gamma}\Gamma r = Mr, \quad \Gamma r \leq u \leq r.$$

So, the condition (H5) holds for $r > 0$. As for (H4), we consider two cases.

(i) If $f(u)$ is bounded, then there exists a constant $D > 0$ such that $f(u) \leq D$ for $0 \leq u < \infty$. By (3.8) there exists a constant $\lambda \geq R$ with $Mr < \omega R \leq \lambda\omega$ satisfying $\lambda \geq \max\{R, \frac{D}{\omega}\}$, such that $f(u) \leq D \leq \lambda\omega$ for $0 \leq u \leq \lambda$, implying (H4).

(ii) If $f(u)$ is unbounded, then there exist $\lambda_1 \geq R$ with $Mr < \omega R \leq \lambda_1\omega$ such that $f(u) \leq f(\lambda_1)$ for $0 \leq u \leq \lambda_1$. This yields $f(u) \leq f(\lambda_1) \leq \lambda_1\omega$ for $0 \leq u \leq \lambda_1$. Thus, condition (H4) holds for λ_1.

Therefore, by Theorem 3.1, the problem (1.1) has at least one positive solution. Theorem 3.3 is proved.

Remark 3.1. *It is easy to see that the assertion of Theorem 3.3 remains valid under each of the following conditions: either $f^0 = 0$ and $f_\infty = +\infty$ or $f_0 = +\infty$ and $f^\infty = 0$.*

Now we are going to give some conclusions about the existence of multiple positive solutions. In the theorems that follow we assume that the assumptions (H1*), (H2) and (H3) are fulfilled.

Theorem 3.4. *Assume that one of the following conditions is satisfied:*

(H8) $f^r < \omega$,

(H9) $f_0 \geq \frac{M}{\Gamma}$ and $f_\infty \geq \frac{M}{\Gamma}$.

Then the problem (1.1) has at least two positive solutions satisfying

$$0 < ||u_1|| < r < ||u_2||.$$

Proof. By the proof of Theorem 3.3, we can take $0 < r_1 < r < \Gamma r_2$ such that $f(u) \geq r_1 M$ for $\Gamma r_1 \leq u \leq r_1$ and $f(u) \geq r_2 M$ for $\Gamma r_2 \leq u \leq r_2$. Therefore, by Theorems 3.2 and 3.3, it follows that problem (1.1) has at least two positive solutions satisfying $0 < ||u_1|| < r < ||u_2||$. □

Theorem 3.5. *Assume that one of the following conditions is satisfied:*

(H10) $f_R > M$,

(H11) $f^0 \leq \omega$ *and* $f^\infty \leq \omega$.

Then the problem (1.1) has at least two positive solutions satisfying

$$0 < \|u_1\| < R < \|u_2\|.$$

Theorem 3.6. *Assume that (H6) (or (H7)) holds, and there exist constants* $r_1, r_2 > 0$ *with* $r_1 M < r_2\omega$ *(or* $r_1 < \Gamma r_2$*) such that (H8) holds for* $r = r_2$ *(or* $r = r_1$*) and (H10) holds for* $R = r_1$ *(or* $R = r_2$*). Then the problem (1.1) has at least three positive solutions satisfying*

$$0 < \|u_1\| < r_1 < \|u_2\| < r_2 < \|u_3\|.$$

Theorem 3.7. *Let* $n = 2k+1$, $k \in \mathbb{N}$. *Assume (H6) (or (H7)) holds. If there exist constants* $r_1, r_2, \ldots, r_{n-1} > 0$ *with* $r_{2i} < \Gamma r_{2i+1}$, *for* $1 \leq i \leq k-1$ *and* $r_{2i-1}M < r_{2i}\omega$ *for* $1 \leq i \leq k$ *(or with* $r_{2i-1} < \Gamma r_{2i}$, *for* $1 \leq i \leq k$ *and* $r_{2i}M < r_{2i+1}\omega$ *for* $1 \leq i \leq k-1$*) such that (H10) (or (H8)) holds for* r_{2i-1}, $1 \leq i \leq k$ *and (H8) (or (H10)) holds for* r_{2i}, $1 \leq i \leq k$. *Then the problem (1.1) has at least n positive solutions* $u_1, \ldots, u_n$ *satisfying*

$$0 < \|u_1\| < r_1 < \|u_2\| < r_2 < \cdots < \|u_{n-1}\| < r_{n-1} < \|u_n\|.$$

The proofs of Theorems 3.5 - 3.7 are similar to that of Theorem 3.4, and so are omitted.

4. An Example.

In this section we discuss an example that illustrates the main results of the paper.

Example. Consider the following singular boundary value problem

$$\begin{cases} D_{0+}^{\frac{3}{2}}(u''(t)) - t^{-\frac{1}{2}}f(u(t)) = 0, & t \in (0,1), \\ u''(0) = u''(1) = 0, \\ u(0) - u'(0) = \frac{1}{2}u(\frac{1}{2}), \\ u(1) + u'(1) = \frac{1}{2}u(\frac{1}{2}), \end{cases} \tag{4.1}$$

where

$$f(u) = \begin{cases} \frac{e^{-u}}{3}, & 0 \leq u \leq 10, \\ (n+1)e^{-u}, & n < u \leq n+1, \quad n = 10, 11, \ldots, 20, \\ e\sqrt{u}, & u > 21. \end{cases}$$

We note that

$$a = b = c = d = 1, \quad \rho = 3, \quad m = 3, \quad \xi_1 = \frac{1}{2}, \quad \sigma = \frac{3}{2},$$
$$a_1 = b_1 = \frac{1}{2}, \quad f_0 = +\infty, \quad f_\infty = +\infty, \quad \Delta = -\frac{9}{2}, \quad g(t) = t^{-\frac{1}{2}}.$$

Let $\theta = \frac{1}{3}$, then we have

$$\Lambda_1 = \frac{2}{3} = 1, \quad \Lambda_2 = 1, \quad \Lambda_3 = 2, \quad \Gamma = \frac{1}{2},$$
$$\omega = \frac{9\pi}{131}, \quad M = \frac{729\pi}{944(3 - 2\sqrt{2})\eta^2},$$

where $\eta = \min_{\frac{1}{3} \le s \le \frac{2}{3}} H(s, s)$.

By calculating, we can let $\mu = \frac{2\sqrt{2}-1}{2\sqrt{2}}$. So, $f_\infty > \frac{M}{\Gamma}$ and $f_0 > \frac{M}{\Gamma}$. Choosing $r = 10$, we get

$$f^r = \sup\left\{\frac{f(u)}{r}\Big| u \in [0, r]\right\} = 0.105409 < 0.2157519 = \omega,$$

showing that (H8) and (H9) are fulfilled. It is easy to see that (H1*), (H2) and (H3) are satisfied as well. So, we can apply Theorem 3.4 to conclude that the problem (4.1) has at least two positive solutions $u_1, u_2 \in K$ satisfying $0 < ||u_1|| < 4 < ||u_2||$.

Acknowledgments. The author would like to thank the anonymous referees for valuable suggestions and comments.

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Поступила 26 декабря 2011