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A WEIGHTED TRANSPLANTATION THEOREM FOR LAGUERRE FUNCTION EXPANSIONS

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Abstract. We establish a generalized weighted transplantation theorem for Laguerre function expansions, which extends the corresponding result by G. Garrigós et al. "A sharp weighted transplantation theorem for Laguerre function expansions" (J. Funct Anal. 244 (2007), pp. 247-276).

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1. Introduction

We consider the system of Laguerre functions defined by

$$\mathcal{L}_k^\alpha(y) = c_{k,\alpha} y^{\frac{\alpha}{2}} e^{-\frac{y}{2}} L_k^{(\alpha)}(y), y \in \mathbb{R}_+ = (0,\infty), \; k \in \mathbb{N},$$

where $L_k^{(\alpha)}(y) = (y^{\alpha+k}e^{-y})^{(k)}/(k! y^\alpha e^{-y})$ is the usual Laguerre polynomial of degree k. For $\alpha > -1$ this system forms an orthonormal basis in $L^2(\mathbb{R}_+)$ when we choose the normalizing constants

$$c_{k,\alpha} = \sqrt{\Gamma(k+1)/\Gamma(\alpha+k+1)}, \; k \in \mathbb{N}.$$

This produces a formal expansion $f = \sum_{k=0}^\infty < f, \mathcal{L}_k^\alpha > \mathcal{L}_k^\alpha$, which is convergent in norm at least for $f \in L^2(\mathbb{R}_+)$.

The main object in the theory of Laguerre function expansions is the set of transplantation operators, defined for $\alpha, \beta > -1$ and $f \in L^2(\mathbb{R}_+)$ by

$$T_\beta^\alpha f = \sum_{k=0}^\infty < f, \mathcal{L}_k^\alpha > \mathcal{L}_k^\beta.$$

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The L^p boundedness of such operators was first established by Kanjin [7]. Recently, G. Garrigós et al. [5] extended the Kanjin's result to the power weighted spaces (see also [16]).

The purpose of this paper is to establish a generalized weighted transplantation theorem for Laguerre function expansions, which extends the corresponding result by G. Garrigós et al. [5]. The main result of the paper is the following theorem.

Theorem 1.1. *Let $-1 < \alpha < \beta$ and $1 < p < \infty$. Then the operators T_α^β and T_β^α admit bounded extensions to the weighted space $L^p(\omega)$ whenever $\omega(x) = (1+x)^{p\gamma}x^{p\delta}$ with $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$ and $\gamma \in \mathbb{R}$.*

We remark that in the special case $\gamma = 0$, Theorem 1.1 has been proved by G. Garrigós et al. (see [5], Theorem 1.4). So, our result extends essentially the main result of [5]. Also, the proof of Theorem 1.1 is curried out by using arguments similar to one used in [5].

To prove Theorem 1.1, we need to establish new weighted multiplier theorems for Hermite function expansions in $\mathbb{R}^d$ and Laguerre function expansions in $\mathbb{R}_+$, respectively. Recall that the Hermite functions in $\mathbb{R}^d$ are defined by

$$\eta_{\mathbf{k}}(x) = d_{\mathbf{k},d}e^{-|x|^2/2}\prod_{i=1}^{d} H_{k_i}(x_i),\ \mathbf{k} = (k_1, \cdots, k_n), k_i \in \mathbb{N},$$

where $H_k(t) = (-1)^k e^{t^2} D^{(k)}(e^{-t^2})$ is the usual Hermite polynomial in $\mathbb{R}$ and $\mathbb{N} = \{0, 1, 2, \cdots\}$. Normalizing by $d_{\mathbf{k},d} = \prod_{i=1}^{d}(2^{k_i}k_i!\sqrt{\pi})^{-1/2}$, the system $\{\eta_{\mathbf{k}}\}_{\mathbf{k}}$ becomes an orthonormal basis in $L^2(\mathbb{R}^d)$ and a complete system of eigenvectors for the Hermite operator $-\Delta + |x|^2$.

Theorem 1.2. *Let $1 < p < \infty$ and $m \in l^\infty(\mathbb{N}^d)$ be such that*

$$|\Delta^\alpha m(\mathbf{k})| \le C_\alpha(1 + |\mathbf{k}|)^{-|\alpha|},\ \mathbf{k} \in \mathbb{N}^d,\ \forall\, \alpha \in \mathbb{N}^d, \tag{1.1}$$

where Δ^α is a difference operator. Consider the operator $T_m f = \sum_{\mathbf{k}} m(\mathbf{k}) < f, \eta_{\mathbf{k}} > \eta_{\mathbf{k}}$, defined at least for $f \in L^2(\mathbb{R}^d)$. Then T_m admits a bounded extension to the weighted space $L^p(\omega)$ whenever $\omega(x) = (1+|x|)^\gamma \mu(x)$ with $\mu \in A_p(\mathbb{R}^d)$ and $\gamma \in \mathbb{R}$, where $\mu \in A_p(\mathbb{R}^d)$ stands for the Muckenhoupt class.

We remark that in the special case $\gamma = 0$, Theorem 1.2 has been proved in [5] under weaker conditions (see [5], Theorem 1.6). But, our Theorem 1.2 cannot be deduced from the conditions imposed in [5].

Theorem 1.3. *Let $\alpha > -1$, $1 < p < \infty$ and $m \in C^\infty[0,\infty)$ be such that*

$$|D^l m(\xi)| \leq C_l(1+\xi)^{-l}, \quad \xi \geq 0, l \in \mathbb{N}. \tag{1.2}$$

Consider the operator $T_m f = \sum_{k\geq 0} m(k) < f, L_k^\alpha > L_k^\alpha$, defined at least for $f \in L^2(\mathbb{R}_+)$. Then T_m admits a bounded extension to the weighted space $L^p(\omega)$ whenever $\omega(x) = (1+x)^{p\gamma}x^{p\delta}$ with $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$ and $\gamma \in \mathbb{R}$.

We remark that in the special case $\gamma = 0$, Theorem 1.3 has been proved in [5], Theorem 1.8.

The paper is organized as follows. In Section 2 we prove Theorem 1.2 by using a new class of weights $A_p(\varphi)$. In Section 3 we establish Theorem 1.3. The main result of the paper - Theorem 1.1 is proved in Section 4. Finally, Section 5 is devoted to the applications of Theorems 1.1-1.3 to the boundedness property of the Littlewood-Paley g-functions associated with the Laguerre expansions.

2. Multipliers for Hermite expansions

In this section we prove Theorem 1.2. First we introduce some notation and properties of the new weight function class $A_p(\varphi)$.

Throughout the paper, $Q(x,t)$ denotes a cube centered at x and of the side length t. Given a cube $Q = Q(x,t)$ and a number $\lambda > 0$, we will write λQ for the λ-dilate cube, which is the cube with the same center x and with side length λt. Given a Lebesgue measurable set E and a weight ω, $|E|$ will denote the Lebesgue measure of E and $\omega(E) = \int_E \omega dx$. The symbol $\|f\|_{L^p(\omega)}$ denotes $(\int_{\mathbb{R}^d} |f(y)|^p \omega(y)dy)^{1/p}$ for $0 < p < \infty$, and $\|f\|_{L^{1,\infty}(\omega)}$ denotes $\sup_{\lambda>0} \lambda^{-1}\omega(\{x \in \mathbb{R}^d : |f(x)| > \lambda\})$. The letter C denotes constants that are independent of the main parameters involved, but whose value may vary from line to line. For a measurable set E, by χ_E we denote the characteristic function of E. By $A \sim B$ we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

In this section, we let $\varphi(t) = (1+t)^{\beta_0}$ for $\beta_0 > 0$ and $t \geq 0$.

A weight always means a positive function which is locally integrable. We say that a weight ω belongs to the class $A_p(\varphi)$ for $1 < p < \infty$, if there is a constant C such that for all cubes $Q = Q(x,r)$ with center x and side length r

$$\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega(y)\,dy\right)\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega^{-\frac{1}{p-1}}(y)\,dy\right)^{p-1} \le C.$$

Also, we say that a nonnegative function ω belongs to the class $A_1(\varphi)$ (or satisfies the $A_1(\varphi)$ condition), if there exists a constant C such that for all cubes Q

$$M_\varphi(\omega)(x) \le C\omega(x), \ a.e.\ x \in \mathbb{R}^d.$$

where

$$M_\varphi f(x) = \sup_{x\in Q}\frac{1}{\varphi(|Q|)|Q|}\int_Q |f(y)|\,dy.$$

Since $\varphi(|Q|) \ge 1$, we have $A_p(\mathbb{R}^d) \subset A_p(\varphi)$ for $1 \le p < \infty$, where $A_p(\mathbb{R}^d)$ denotes the class of classical Muckenhoupt weights (see [4]). It is well known that if $\omega \in A_\infty(\mathbb{R}^d) = \bigcup_{p\ge1} A_p(\mathbb{R}^d)$, then $\omega(x)dx$ is a doubling measure, that is, there exist a constant $C > 0$ such that for any cube Q

$$\omega(2Q) \le C\omega(Q).$$

Now we list some properties of weights $\omega \in A_\infty(\varphi) = \bigcup_{p\ge1} A_p(\varphi)$, similar to that of classical Muckenhoupt weights.

Lemma 2.1. *For any cube $Q \subset \mathbb{R}^d$ the following assertions hold:*

(i) *If $1 \le p_1 < p_2 < \infty$, then $A_{p_1}(\varphi) \subset A_{p_2}(\varphi)$.*

(ii) *$\omega \in A_p(\varphi)$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}(\varphi)$, where $1/p + 1/p' = 1$.*

(iii) *If $\omega_1,\ \omega_2 \in A_1(\varphi)$, $p > 1$, then $\omega_1\omega_2^{1-p} \in A_p(\varphi)$.*

(iv) *If $\omega \in A_p$ for $1 \le p < \infty$, then*

$$\frac{1}{\varphi(|Q|)|Q|}\int_Q |f(y)|dy \le C\left(\frac{1}{\omega(Q)}\int_Q |f|^p\omega(y)dy\right)^{1/p}.$$

In particular, if $f = \chi_E$ for any measurable set $E \subset Q$, then

$$\frac{|E|}{\varphi(|Q|)|Q|} \le C\left(\frac{\omega(E)}{\omega(Q)}\right)^{1/p}.$$

Remark 2.1. *It follows from the definition of $A_p(\varphi)$ and Lemma 2.1 (iii), that if $\omega \in A_p(\varphi)$, then $\omega(x)dx$ generally is not a doubling measure. Indeed, let $0 \le \gamma \le \beta_0/d$,* *it is easy to check that* $\omega(x) = (1+|x|)^{-(d+\gamma)} \notin A_\infty(\mathbb{R}^d)$ *and* $\omega(x)dx$ *is not a doubling measure, but* $\omega(x) = (1+|x|)^{-(d+\gamma)} \in A_1(\varphi)$.

It is easy to see that the set of all Schwartz functions, denoted by $\mathcal{S}$, is dense in $L^p(\omega)$ for $\omega \in A_\infty(\varphi)$ and $1 \le p < \infty$. Hence, we always can assume that $f \in \mathcal{S}$ if $f \in L^p(\omega)$ for $1 \le p < \infty$.

Lemma 2.2. *Let* $1 \le p_1 < \infty$ *and* $\omega \in A_{p_1}(\varphi)$. *Then for* $p_1 < p < \infty$ *the inequality holds:*

$$\int_{\mathbb{R}^d} |M_\varphi f(x)|^p \omega(x)dx \le C_p \int_{\mathbb{R}^d} |f(x)|^p \omega(x)dx.$$

Further, let $1 \le p < \infty$, *then* $\omega \in A_p(\varphi)$ *if and only if*

$$\omega(\{x \in \mathbb{R}^d :\ M_\varphi f(x) > \lambda\}) \le \frac{C_p}{\lambda^p} \int_{\mathbb{R}^d} |f(x)|^p \omega(x)dx, \quad \lambda > 0.$$

The dyadic sharp maximal operator $M_\varphi^{\sharp,\triangle} f(x)$ is defined by

$$\begin{aligned} M_\varphi^{\sharp,\triangle} f(x) &:= \sup_{x\in Q, r<1} \frac{1}{|Q|}\int_{Q(x_0,r)} |f(x) - f_Q|\, dx + \sup_{x \in Q, r\ge 1} \frac{1}{\varphi(|Q|)|Q|}\int_{Q(x_0,r)} |f|\, dx \\ &\simeq \sup_{x\in Q, r<1} \inf_C \frac{1}{|Q|}\int_{Q(x_0,r)} |f(y) - C|\, dy + \sup_{x \in Q, r\ge 1} \frac{1}{\varphi(|Q|)|Q|}\int_{Q(x_0,r)} |f|\, dx, \end{aligned}$$

where Q denotes a dyadic cube and $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$. Similarly, we define the sharp maximal operator $M_\varphi^\sharp f(x)$ for an arbitrary cube with sides parallel to the coordinate axes.

Lemma 2.3. *Let* $1 < p < \infty$, $\omega \in A_\infty(\varphi)$ *and* $f \in L^p(\omega)$, *then*

$$\|M_\varphi^\triangle f\|_{L^p(\omega)} \le C \|M_\varphi^{\sharp,\triangle} f\|_{L^p(\omega)}.$$

Here $M_\varphi^\triangle f(x)$ denotes the dyadic maximal operator. Lemmas 2.2 and 2.3 follow from [19].

Note that $|f(x)| \le M_\varphi^\triangle f(x)$ *a.e.* $x \in \mathbb{R}^d$ and $M_\varphi^{\sharp,\triangle} f(x) \le M_\varphi^\sharp f(x)$ for $x \in \mathbb{R}^d$. By Lemma 2.3, we have

Proposition 2.1. *Let* $1 < p < \infty$, $\omega \in A_\infty(\varphi)$ *and* $f \in L^p(\omega)$, *then*

$$\|f\|_{L^p(\omega)} \le \|M_\varphi^\triangle f\|_{L^p(\omega)} \le C\|M_\varphi^\sharp f\|_{L^p(\omega)}.$$

In order to prove Theorem 1.1, we need to introduce some vector-valued spaces. Let X be a Hilbert space with norm $|\cdot|_X$, and let $\|f\|_{L^p_X(\omega)}$ denote $(\int_{\mathbb{R}^d} |f(y)|_X^p \omega(y)dy)^{1/p}$ for $0 < p < \infty$.

Consider the Bochner integral operator T, defined by

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy,$$

where the X-valued kernel K satisfies the following conditions (for $N \geq n\beta_0 + 1$):

(i) $|K(x,z)|_X \leq C_N|x-z|^{-d}(1+|x-z|)^{-N}$,

(ii) $|K(x,z)-K(x_0,z)|_X \leq C_N \frac{|z-z_0|}{(1+|x-z|)^N|z_0-z|^{d+1}}$, if $2|x-x_0| < |x-z|$.

The next result can be deduced from Lemmas 2.2 and 2.3, and Proposition 2.1.

Proposition 2.2. *If the Bochner operator T is bounded from $L^p(\mathbb{R}^d)$ into $L^p_X(\mathbb{R}^d)$, then for any $r > 1$,*

$$M^{\sharp}_{\varphi}(|Tf|_X)(x) \leq CM_{\varphi,r}f(x),$$

where $M_{\varphi,r}f(x) = [M_{\varphi}(|f|^r)(x)]^{1/r}$, and, as a consequence, the inequality

$$\|Tf\|_{L^p_X(\omega)} \leq C\|f\|_{L^p(\omega)}$$

holds for $1 \leq p_1 < p < \infty$ and $\omega \in A_{p_1}(\varphi)$.

For the proofs of the above stated results we refer to [19].

Now we proceed to prove Theorem 1.2. We define the Hermite g-function and g^*-function, respectively, by the following formulas:

$$g_l(f)(x) = \left[\int_0^\infty |s^l\partial_s^l T_s f(x)|^2 \frac{ds}{s}\right]^{1/2}, \; l = 1,2,\cdots,$$

$$g^*_\lambda(f)(x) = \left[\int_{\mathbb{R}^d}\int_0^\infty |s^l \frac{s^{-\frac{d}{2}}}{(1+\frac{|x-y|}{\sqrt{s}})^{d\lambda}} \partial_s^l T_s f(x)|^2 \frac{ds}{s}\right]^{1/2}, \; \lambda > 1,$$

where $T_s = e^{-s(-\Delta+|x|^2)}$ denotes the Hermite heat semigroup.

Denoting by $T_s(y,z)$ the kernel of T_s, we can write

$$s^l\partial_s^l T_s f(y) = \int_{\mathbb{R}^d} s^l \left[\frac{\partial^l T_s(y,z)}{\partial s^l}\right] f(z)dz.$$

For convenience, we change the variable $s = t^2$ in the definition of g and g^*, and denote by $Q_t(y,z)$ the new(normalized) kernels $t^{2l}[\frac{\partial^l T_s(y,z)}{\partial s^l}]|_{s=t^2}$ for $l \geq 1$. It is easy to check that these kernels are symmetric and satisfy the inequalities (see [20], pp. 98-99):

(a) $|Q_t(y,z)| \leq Ct^{-d}e^{-\frac{a}{t^2}|x-y|^2}$, $0 < t < 1$,

(b) $|Q_t(y,z)| \leq C2^{-dt}e^{-b|x-y|^2}$, $t \geq 1$,

(c) $|Q_t(y+h,z)-Q_t(y,z)|+|Q_t(y,z+h)-Q_t(y,z)| \le Cht^{-d-1}e^{-\frac{a}{2}t|x-y|^2}$, for $0<t<1$, $\forall |h|\le t$,

(d) $|Q_t(y+h,z)-Q_t(y,z)|+|Q_t(y,z+h)-Q_t(y,z)| \le Ch2^{-dt}e^{-b|x-y|^2}$, for $t \ge 1, \forall |h| \le t$, where C, a and b are positive constants, independent of x, y, t.

To prove Theorem 1.2 it is convenient to look at the functions g and g^* as vector-valued singular integrals. Let A denote the Hilbert space $L^2(\mathbb{R}_+, dt/t)$, and B denote the Hilbert space $L^2(\mathbb{R}_+ \times \mathbb{R}^d, dtdy/t^{n+1})$.

Consider the operator $G_1 : L^2(\mathbb{R}^d) \to L^2_A(\mathbb{R}^d)$ defined by

$$G_1 f(x) = \int_{\mathbb{R}^d} K_1(x,z) f(z) dz,$$

where $K_1(x,z)$ is the A-valued kernel: $K_1(x,z) := \{Q_t(x,z)\}_t$, and the operator $G_2 : L^2(\mathbb{R}^d) \to L^2_B(\mathbb{R}^d)$ defined by

$$G_2 f(x) = \int_{\mathbb{R}^d} K_2(x,z) f(z) dz,$$

where $K_2(x,z)$ is the B-valued kernel:

$$K_2(x,z) := \left\{ \left(1 + \frac{|x-y|}{t}\right)^{-\frac{d\lambda}{2}} Q_t(y,z) \right\}_{(t,y)}.$$

Observe that $|G_1 f(x)|_A = g_l(x)$ and $|G_2 f(x)|_B = g^*_\lambda(x)$. Therefore, the boundedness of g_l and g^*_λ in $L^p(\omega)$ are equivalent to the boundedness of G_1 from $L^p(\omega)$ into $L^p_A(\omega)$ and G_2 from $L^p(\omega)$ into $L^p_B(\omega)$, respectively. Moreover, boundedness holds for the Muckenhoupt weights for $1<p<\infty$ (see [5]). Hence, in order to apply Proposition 2.2, we need to establish the following lemmas.

Lemma 2.4. *There exist positive constants* c_1 *and* c_2 *such that*

(i) $|K_1(x,y)|_A \le c_1 |x-y|^{-d} e^{-c_2|x-y|^2}$,

(ii) $|K_1(x,y)-K_1(x_0,y)|_A \le c_1 \dfrac{|x-x_0|}{|x-y|^{d+1}} e^{-c_2|x-y|^2}$, *if* $2|x-x_0| < |x-y|$.

Proof. We use the above stated inequalities (a)–(d), and fist prove the assertion (i). Note that

$$\begin{aligned} |K_1(x,y)|_A^2 &= \int_0^\infty |Q_t(x,y)|^2 \frac{dt}{t} \\ &= \int_0^1 |Q_t(x,y)|^2 \frac{dt}{t} + \int_1^\infty |Q_t(x,y)|^2 \frac{dt}{t} \\ &:= I_1 + I_2. \end{aligned}$$

Using the inequality (a), for I_1 we have

$$\begin{aligned} I_1 &\le C\int_0^1 t^{-2d}e^{-a\frac{|x-y|^2}{t^2}}\frac{dt}{t}\\ &= C\int_0^1 t^{-2d}e^{-a\frac{|x-y|^2}{2t^2}}e^{-a\frac{|x-y|^2}{2t^2}}\frac{dt}{t}\\ &\le Ce^{-a\frac{|x-y|^2}{2}}\int_0^1\frac{1}{(t+|x-y|)^{2d+1}}dt\\ &\le C\frac{1}{|x-y|^{2d}}e^{-a\frac{|x-y|^2}{2}}. \end{aligned}$$

Now using the inequality (b), for I_2 we obtain

$$I_2 \le C\int_1^\infty e^{-t}e^{-a|x-y|^2}\frac{dt}{t}\le Ce^{-b|x-y|^2}\le C\frac{1}{|x-y|^{2d}}e^{-b\frac{|x-y|^2}{2}}.$$

To prove the assertion (ii), first note that

$$\begin{aligned} |K_1(x,y) &- K_1(x_0,y)|_A^2\\ &\le \int_0^\infty |Q_t(x,y)-Q_t(x_0,y)|^2\frac{dt}{t}\\ &= \int_0^1 |Q_t(x,y)-Q_t(x_0,y)|^2\frac{dt}{t}+\int_1^\infty |Q_t(x,y)-Q_t(x_0,y)|^2\frac{dt}{t}\\ &:= I_3+I_4. \end{aligned}$$

If $|x-y|\ge 1$, then by the inequality (c), for I_3 we have

$$\begin{aligned} I_3 &\le C\int_0^1\left(\frac{|x-x_0|}{t}\right)^2 t^{-2d}e^{-a\frac{|x-y|^2}{t^2}}\frac{dt}{t}\\ &\le C\frac{|x-x_0|^2}{|x-y|^{2(d+2)}}\int_0^1 e^{-a\frac{|x-y|^2}{2t^2}}t^2\frac{dt}{t}\\ &\le C\frac{|x-x_0|^2}{|x-y|^{2(d+2)}}e^{-a\frac{|x-y|^2}{2}}\int_0^1 tdt\\ &\le C\frac{|x-x_0|^2}{|x-y|^{2(d+1)}}e^{-a\frac{|x-y|^2}{2}}. \end{aligned}$$

If $|x-y|<1$, an application of the inequality (c) yields

$$\begin{aligned} I_3 &\le C\int_0^{|x-y|}\left(\frac{|x-x_0|}{t}\right)^2 t^{-2d}e^{-a\frac{|x-y|^2}{t^2}}\frac{dt}{t}\\ &\quad +C\int_{|x-y|}^1\left(\frac{|x-x_0|}{t}\right)^2 t^{-2d}e^{-a\frac{|x-y|^2}{t^2}}\frac{dt}{t} \end{aligned}$$

$$\begin{aligned}
&\leq C\frac{|x-x_0|^2}{|x-y|^{2(d+2)}}\int_0^{|x-y|} e^{-a\frac{|x-y|^2}{2t^2}}t^2\frac{dt}{t}\\
&\quad +C\frac{|x-x_0|^2}{|x-y|^{2d}}\int_{|x-y|}^1 e^{-a\frac{|x-y|^2}{2t^2}}t^{-2}\frac{dt}{t}\\
&\leq C\frac{|x-x_0|^2}{|x-y|^{2(d+2)}}e^{-a\frac{|x-y|^2}{2}}\int_0^{|x-y|} tdt\\
&\quad +C\frac{|x-x_0|^2}{|x-y|^{2d}}e^{-a\frac{|x-y|^2}{2}}\int_{|x-y|}^1 t^{-3}dt\\
&\leq C\frac{|x-x_0|^2}{|x-y|^{2(d+1)}}e^{-a\frac{|x-y|^2}{2}}.
\end{aligned}$$

To estimate I_4, we apply the inequality (d) to obtain

$$I_4 \leq C\int_1^\infty |x-x_0|^2 e^{-t}e^{-b|x-y|^2}dt \leq C\frac{|x-x_0|^2}{|x-y|^{2(d+1)}}e^{-b\frac{|x-y|^2}{2}}.$$

This completes the proof of Lemma 2.4.

Lemma 2.5. *Let $\lambda > 4$, $N_1 = d(\frac{\lambda}{2}-1)$ and $N_2 = d(\frac{\lambda}{2}-1)-1$, then there exist positive constants C_{N_1}, C_{N_2} such that*

(i) $|K_2(x,z)|_B \leq C_{N_1}|x-z|^{-d}(1+|x-z|)^{-N_1}$,

(ii) $|K_2(x,z)-K_2(x,z_0)|_B \leq C_{N_2}\frac{|z-z_0|}{(1+|x-z|)^{N_2}|z_0-z|^{d+1}}$, *if* $2|z-z_0| < |x-z|$.

The proof is similar to that of Lemma 2.4, and hence, is omitted.

Theorem 2.1. *Let $1 \leq p_1 < p < \infty$ and $\omega \in A_{p_1}(\varphi)$. Then, for $l \geq 1$ there is a constant $C > 0$ so that*

$$\|g_l(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

Obviously, Theorem 2.1 is a consequence of Lemma 2.4 and Proposition 2.2.

As an immediate consequence of Theorem 2.1 we can state the following result.

Corollary 2.1. *Let $1 \leq p_1 < p < \infty$ and $\omega \in A_{p_1}(\varphi)$. Then for $l \geq 1$ there is a constant $C > 0$ so that*

$$C^{-1}\|f\|_{L^p(\omega)} \leq \|g_l(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

Theorem 2.2. *Let $1 \leq p_1 < p < \infty$ and $\omega \in A_{p_1}(\varphi)$. Then for each $\lambda > 2(\beta_0+4)$ there is a constant $C > 0$ so that*

$$\|g^*_\lambda(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}.$$

Proof. Adapting the arguments used in [15], pp. 43-44, and using a duality argument and Lemma 2.5 and Proposition 2.1, we obtain the desired result.

As a consequence of Theorem 2.2, we have the following result.

Corollary 2.2. *Let $1 < p < \infty$ and $\omega(x) = (1 + |x|)^\gamma$ with $|\gamma| < \beta_0$. Then there exists a constant $C > 0$ such that for each $\lambda > 2(\beta_0 + 4)$*

$$\|g_\lambda^* f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Proof. Note that $\omega(x) = (1 + |x|)^\gamma \in A_1(\varphi)$ if $-\beta_0 < \gamma < \beta_0$. Applying Theorem 2.2, we get

$$\|g_\lambda^* f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

This implies

$$\|g_\lambda^* f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Thus, Corollary 2.2 is proved.

Theorem 2.3. *Let $\omega(x) = (1 + |x|)^\gamma \mu(x)$ with $\mu \in A_p(\mathbb{R}^d)$ (Muckenhoupt class) and $\gamma \in \mathbb{R}$. Then there exists a positive constant λ_0 depending on γ and μ such that for each $\lambda > \lambda_0$*

$$\|g_l\|_{L^p(\omega)} + \|g_\lambda^*\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Proof. Using the results from [5], we obtain

$$\|g_l\|_{L^p(\mu)} + \|g_\lambda^*\|_{L^p(\mu)} \le C\|f\|_{L^p(\mu)},$$

where $\mu \in A_p(\mathbb{R}^d)$. By the properties of the Muckenhoupt class $A_p(\mathbb{R}^d)$ (see [4]), there exists $\epsilon > 0$ such that

$$\|g_l\|_{L^p(\mu^{1+\epsilon}))} + \|g_\lambda^*\|_{L^p(\mu^{1+\epsilon}))} \le C\|f\|_{L^p(\mu^{1+\epsilon}))}. \tag{2.1}$$

On the other hand, for $\omega_1(x) = (1 + |x|)^{\gamma(1+\epsilon)/\epsilon}$ and $\lambda_0 = 2(|\gamma| + 4)(1 + \epsilon)/\epsilon$, by Corollaries 2.1 and 2.2, we have

$$\|g_l\|_{L^p(\omega_1)} + \|g_\lambda^*\|_{L^p(\omega_1)} \le C\|f\|_{L^p(\omega_1)}. \tag{2.2}$$

Putting together (2.1) and (2.2), we obtain the desired result.

To prove Theorem 1.2, we also need the following result proved in [5].

Proposition 2.3. *Let $\lambda > 2$ and T_m be as in Theorem 1.2. Then for all $l \geq d\lambda/2+1$ we have*

$$g_l(T_m f)(x) \leq C g_\lambda^*(f)(x), \text{ a.e. } x \in \mathbb{R}^d.$$

Proof of Theorem 1.2. Combining Corollary 2.1, Theorem 2.3 and Proposition 2.3 we have, for $f \in C_c(\mathbb{R}^d)$

$$\|T_m f\|_{L^p(\omega)} \leq C\|g_l(T_m f)\|_{L^p(\omega)} \leq C\|g_\lambda^* f\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)}$$

provided that ω, l and λ satisfy the conditions of Theorem 2.3 and Proposition 2.3. The proof is complete.

Using the same transference principle as in Corollary 3.4 from [6], we obtain a counterpart of Theorem 1.2 for Laguerre expansions when $\alpha = \frac{n}{2} - 1$.

The next two lemmas were stated in [5].

Lemma 2.6. *Let $\alpha = \frac{n-2}{2}$ where $n \in \mathbb{N}_+$. Then for some constants $\alpha_k \in \mathbb{R}$, $k = 1, 2 \cdots$, the following equalities hold:*

$$L_k^\alpha(|z|^2) = \sum_{|\mathbf{k}|=k} \alpha_{\mathbf{k}} \eta_{2\mathbf{k}}(z)|z|^\alpha, \ \forall z \in \mathbb{R}^d, \ k = 1, 2 \cdots.$$

We shall also need the following elementary fact.

Lemma 2.7. *For every $f \in L^1(0, \infty)$ we have*

$$\int_{\mathbb{R}^d} f(|z|^2)|z|^{-(d-2)} dz = c_d \int_0^\infty f(t) dt.$$

Corollary 2.3. *The assertion of Theorem 1.3 remains valid when $\alpha = \frac{n-2}{2}$ and n is a positive integer.*

Proof. Let $m(\xi)$ be as in Theorem 1.3. The function $M(\xi) = m((\xi_1 + \cdots + \xi_d)/2)$ restricted to the lattice $\mathbb{N}^d$ defines a multiplier $\{M(\mathbf{k})\}$ which satisfies the conditions (1.2).

By Lemma 2.6 we have

$$(T_m f)(|z|^2) = \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} m(\mathbf{k}) < f, L_{\mathbf{k}}^\alpha > \alpha_{\mathbf{k}} \eta_{2\mathbf{k}}(z)|z|^\alpha, \ z \in \mathbb{R}^d.$$

Let $\omega(|z|) = (1+|z|)^\gamma |z|^{p\delta}$ for $\gamma \in \mathbb{R}$ and $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$. Observing that $|z|^{(d-2)(\frac{p}{2}-1)}|z|^{p2\delta} \in A_p(\mathbb{R}^d)$, we can use Lemmas 2.4 and 2.5 to apply Theorem 1.1

to obtain

$$\begin{aligned}\|T_m f\|^p_{L^p(\omega)} &= \int_0^\infty |(T_m f)(t)|^p \omega(t)dt \\ &= c_n \int_{\mathbb{R}^d} \left| \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} M(2\mathbf{k}) < f, L^\alpha_{\mathbf{k}} > \alpha_{\mathbf{k}} \eta_{2\mathbf{k}}(z) \right|^p |z|^{\alpha p-(d-2)} \omega(|z|^2) dz \\ &= c_n \int_{\mathbb{R}^d} \left| \sum_{k=0}^{\infty} \sum_{|\mathbf{k}|=k} < f, L^\alpha_{\mathbf{k}} > \alpha_{\mathbf{k}} \eta_{2\mathbf{k}}(z) \right|^p |z|^{(d-2)(\frac{p}{2}-1)} \omega(|z|^2) dz \\ &\le C\|f\|_{L^p(\omega)},\end{aligned}$$

Thus, Corollary 2.3 is proved.

3. Multipliers for Laguerre expansions

In this section we prove Theorem 1.3. The strategy is to deduce the result from the special case discussed in Corollary 2.3, by using interpolation of the following analytic family of operators

$$T^z_m f = \sum_{k=0}^{\infty} m_k < f, L^z_k > L^z_k, \text{ where } z \in \mathbb{C} \text{ and } Rez > -1.$$

We first recall the definition of Kanjin's operators $T^{\alpha+i\theta}_\alpha$ and prove their boundedness for the range of $L^p_{\alpha,\gamma}(\mathbb{R}_+)$.

In this section we will use the following notation from [5, 7]. We denote $M(\theta) := (1+|\theta|)^N e^{c|\theta|}$ for suitably large constants N and c. The constants appearing in the section such as C, c or N may depend on α, p, δ and γ, but are independent of $\theta \in \mathbb{R}$. Finally, it is also convenient to denote the admissible range of indices by

(3.1)
$$\mathcal{A} = \left\{ \left(\frac{1}{p}, \alpha, \delta, \gamma\right) \in (0,1) \times (-1,\infty) \times \mathbb{R} \times \mathbb{R} : -\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2} \right\}.$$

We first state the boundedness of $T^{\alpha+i\theta}_\alpha$ in $L^p_{\delta,\gamma}(\mathbb{R}_+)$ for special values of α.

Observe that (see [7], p. 539), the Laguerre polynomials can be extended to complex parameters $z \in \mathbb{C}$ with $Rez > -1$ by the formula

$$L^{(z)}_k(y) = \frac{D^{(k)}_y[y^{z+k}e^{-y}]}{k! y^z e^{-y}} = \sum_{j=0}^{k} \frac{\Gamma(k+z+1)}{\Gamma(k-j+1)\Gamma(j+z+1)} \frac{(-y)^j}{j!},\ y > 0,$$

and likewise for the corresponding Laguerre functions we have

$$L^z_k(y) = \left(\frac{\Gamma(k+1)}{\Gamma(z+k+1)}\right)^{1/2} y^{1/2} e^{-y/2} L^{(z)}_k(y),\ y > 0.$$

Moreover, the following lemma is true, which was proved in [7].

Lemma 3.1. *Let $\alpha > -1$ and $f \in C_c^\infty(0,\infty)$. Then for each $N \geq 1$ there exist a constant $C > 0$ and a number $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $\theta \in \mathbb{R}$*

$$|< f, L_k^{\alpha+i\theta} >| \leq C(1+|\theta|)^{4N+\alpha} e^{\frac{\pi}{2}|\theta|}(1+k)^{-N}. \tag{3.2}$$

Using this lemma one can define the complex transplantation operators

$$T_\alpha^z f = \sum_{k=0}^{\infty} < f, L_k^z > L_k^\alpha, \ Re z > -1, \ \alpha > -1,$$

at least for functions $f \in C_0^\infty(\mathbb{R}_+)$.

For every $\alpha > -1$ and $\theta \in \mathbb{R}$ we define a multiplier by

$$\lambda(\xi) = \lambda_{\alpha,\theta}(\xi) = \left(\frac{\Gamma(\xi+\alpha+1+i\theta)}{\Gamma(\xi+\alpha+1)}\right)^{1/2}, \ \xi \geq 0. \tag{3.3}$$

Observe that λ is an analytic function of ξ when $Re\xi > -1-\alpha$. The following result has been proved in [5].

Lemma 3.2. *Let $\alpha > -1$. Then the function $\lambda(\xi)$ defined by (3.3) belongs to $C^\infty(0,\infty)$ and satisfies*

$$\sup_{\xi\in[0,\infty)} (1+|\xi|)^l |D^l \lambda(\xi)| \leq C_l(1+|\theta|)^l, \ \forall\, \theta \in \mathbb{R}, \ l = 0,1,2,\cdots,$$

where the constants C_l are independent of θ.

We prove Theorem 1.2 under the following assumption on the indices $(\frac{1}{p},\alpha,\delta,\gamma)$.

Assumption (A). The point $(\frac{1}{p},\alpha,\delta,\gamma)$ is so that the multiplier operator $T_\lambda f = \sum_{k=0}^{\infty} \lambda(k) < f, L_k^\alpha > L_k^\alpha$, with $\lambda = \lambda_{\alpha,\theta}$ as in (3.3), is bounded on $L^p_{\delta,\gamma}(\mathbb{R}_+)$ and satisfies

$$\|T_\lambda f\|_{L^p_{\delta,\gamma}} \leq C(1+|\theta|)^N e^{c|\theta|} \|f\|_{L^p_{\delta,\gamma}}, \ \forall\, \theta \in \mathbb{R},$$

for some constants $C, c, N > 0$, where

$$\|f\|_{L^p_{\delta,\gamma}} = \left(\int_{\mathbb{R}_+} (|f(x)|(1+x)^\gamma x^\delta)^p dx\right)^{1/p}.$$

Remark 3.1. *It follows from Corollary 2.3 and Lemma 3.2 that the Assumption (A) is fulfilled for parameters from the set $\mathcal{A}$ (see (3.1)) of the form $(\frac{1}{p},\frac{n-2}{2},\delta,\gamma)$, whenever $n \in \mathbb{Z}_+$. Moreover, the Assumption (A) also holds for $(\frac{1}{2},\alpha,0,0)$ and for* *all* $\alpha > -1$, *and, by the duality, it holds for a fixed* $(\frac{1}{p}, \alpha, \delta, \gamma)$ *if and only if it is true for* $(\frac{1}{p'}, \alpha, -\delta, -\gamma)$.

In order to prove Theorem 1.3, we also need the following complex interpolation result.

Lemma 3.3. *Let* $P_0 = (\frac{1}{p_0}, \alpha_0, \delta_0, \gamma_0)$ *and* $P_1 = (\frac{1}{p_1}, \alpha_1, \delta_1, \gamma_1)$ *be two fixed points from* $\mathcal{A}$ *for which the assertion of Theorem 1.2 holds. Then the assertion of the theorem must also hold at the points* $P = (\frac{1}{p}, \alpha, \delta, \gamma)$ *of the form*

$$P = (1-t)P_0 + tP_1, \ t \in (0,1). \tag{3.4}$$

Proof. As in Lemma 3.20 from [5], we define

$$\alpha(z) = (1-z)\alpha_0 + z\alpha_1, \ \delta(z) = (1-z)\delta_0 + z\delta_1, \text{ and } \gamma(z) = (1-z)\gamma_0 + z\gamma_1,$$

for complex $z = s + i\theta$ and $0 \le s \le 1$. Recall that $M(\theta) = (1+|\theta|)^N e^{c|\theta|}$ for suitably large constants N and c. By Lemma 3.1, the operator

$$T_m^{\alpha+i\tau} f = \sum_{k=0}^{\infty} m(k) < f, L_k^{\sigma-i\tau} > L_k^{\sigma+i\tau} = (T_\sigma^{\sigma+i\tau})^* T_m^\sigma T_\sigma^{\sigma-i\tau} f$$

is well defined and bounded at least when $f \in L^2(\mathbb{R}_+)$. We define an analytic family of operators by letting

$$S_z F(y) = y^{\delta(z)}(1+y)^{\gamma(z)} T_m^{\alpha(z)}(F(x)x^{-\delta(z)}(1+x)^{-\gamma(z)})(y)$$

at least for $F \in L_c^2(0,\infty)$.

Now we are going to show that $\{S_z\}$ satisfies the conditions of Stein's interpolation theorem (see [2]). To this end, observe first that, given any two subsets E_1, E_2 compactly contained in $(0,\infty)$, the function

$$z \mapsto \Phi(z) = < S_z(\chi_{E_1}), \chi_{E_2} >$$

is well defined whenever $0 \le Rez \le 1$, and satisfies

(3.5)
$$\begin{aligned} |\Phi(z)| &\le \|T_m^{\sigma(z)}(x^{-\delta(z)}(1+x)^{-\gamma(z)}\chi_{E_1})\|_2 \|(y^{-\delta(z)}(1+y)^{-\gamma(z)}\chi_{E_2})\|_2 \\ &\le C_{E_2} \|(T_{\alpha(s)}^{\alpha(s)+i(\alpha_1-\alpha_0)\theta})^* T^{\alpha(s)m} T^{\alpha(s)-i(\alpha_1-\alpha_0)\theta}(x^{-\delta(z)}(1+x)^{-\gamma(z)}\chi_{E_1})\|_2 \\ &\le C_{E_2} M(\theta) \|(x^{-\delta(z)}(1+x)^{-\gamma(z)}\chi_{E_1})\|_2 \\ &\le C_{E_1} C_{E_2} M(\theta), \end{aligned}$$

by the L^2 boundedness of $T_\sigma^{\sigma+i\tau}$, $\forall \, \sigma > -1$.

Next, we show that the function Φ is holomorphic in a neighborhood of the strip $\bar{S} := \{0 \leq Re z \leq 1\}$. Indeed, since $\|T_m^{\sigma(z)}\|_{L^2\to L^2}$ is uniformly bounded in the compact sets of $\bar{S}$, similar to (3.4), it is enough to show the holomorphy of $z \mapsto < S_z F, G >$ for all $F, G \in C_c^\infty(0,\infty)$. Denoting $f(x) = x^{-\delta(z)}(1+x)^{\gamma(z)}F(x)$, $g(y) = y^{-\delta(\bar{z})}(1+y)^{\gamma(\bar{z})}G(y)$ and $\alpha(z) = \sigma + i\tau$, we can write

$$\begin{aligned} < S_z F, G > &= < T_m^{\alpha(z)}(f), g > = < T_m^{(\sigma)} T_\sigma^{\sigma-i\tau}(f), \overline{T_\sigma^{\sigma+i\tau}(g)} > \\ &= \sum_k m_k < f, L_k^{\sigma-i\tau} > \overline{< g, L_k^{\sigma+i\tau} >} \\ &= \sum_k m_k < x^{-\delta(z)}(1+x)^{\gamma(z)}F, L_k^{\alpha(z)} >< y^{-\delta(\bar{z})}(1+y)^{\gamma(\bar{z})}F, L_k^{\alpha(\bar{z})} > . \end{aligned}$$

Since the series converges uniformly when z belongs to a compact set of $\bar{S}$, it is easy to show the holomorphy of the map

$$z \in \bar{S} \mapsto < x^{\pm\delta(z)}(1+x)^{\pm\gamma(z)}F, L_k^{\alpha(\bar{z})} > = \int_0^\infty x^{\pm\delta(z)}(1+x)^{\pm\gamma(z)}F(x)L_k^{\alpha(\bar{z})}(x)dx,$$

for all $F \in C_c^\infty(0,\infty)$.

Combining this with (3.4) we conclude that Φ is holomorphic in the strip $\{0 < Re\, z < 1\}$, continuous in the closure and has admissible growth for complex interpolation. To verify the conditions of Stein's interpolation theorem (see [2]), we only need to show the boundedness of the operator S_z at the limiting bands

$$S_{i\theta} : L^{p_0}(\mathbb{R}_+) \to L^{p_0}(\mathbb{R}_+) \text{ and } S_{1+i\theta} : L^{p_1}(\mathbb{R}_+) \to L^{p_1}(\mathbb{R}_+).$$

When $Rez = 0$ we use the assumption that Theorem 1.2 (and hence Assumption (A)) holds for the point p_0. Then, both $T_m^{\alpha_0}$ and $T_{\alpha_0}^{\alpha_0-i(\alpha_1-\alpha_0)\theta}$ are bounded in $L^{p_0}_{\delta_0,\gamma_0}$ and in $L^{p_0'}_{-\delta_0,-\gamma_0}$, which implies

$$\begin{aligned} \|S_{i\theta}F\|_{p_0} &= \|(T_{\alpha_0}^{\alpha_0+i(\alpha_1-\alpha_0)})^* T_m^{\alpha_0} T_{\alpha_0+i\tau}^{\alpha_0-i(\alpha_1-\alpha_0)}(x^{-\delta(i\theta)}(1+x)^{\gamma(i\theta)}F)\|L^{p_0}_{\delta_0,\gamma_0} \\ &\leq M(\theta)\|x^{-\delta_0-i(\delta_1-\delta_0)\theta}(1+x)^{\gamma_0-i(\gamma_1-\gamma_0)\theta}F(x)\|L^{p_0}_{\delta_0,\gamma_0} \\ &= M(\theta)\|F\|_{p_0}. \end{aligned}$$

When $Re\ z = 1$, we have a similar result. Thus, by Stein's theorem S_s must be bounded in $L^{p_s}(\mathbb{R}_+)$ for $\frac{1}{p_s} = \frac{1-s}{p_0} + \frac{s}{p_1}$ and all $s \in (0,1)$. Letting $s = t$ and using (3.4), we have $p_t = p$, $\alpha(t) = \alpha$ and $\delta(t) = \delta$.

Moreover, such boundedness translates into

$$\begin{aligned} \|T_m^\alpha f\|_{L^p_{\delta,\gamma}} &= \|y^{\delta(t)}(1+y)^{\gamma(t)}T_m^{\alpha(t)}(x^{\delta(t)}(1+x)^{\gamma(t)}f(x)x^{-\delta(t)}(1+x)^{-\gamma(t)})\|_{L^p} \\ &= \|S_t(x^{\delta(t)}(1+x)^{\gamma(t)}f(x))\|_{L^p} \\ &\leq M\|x^{\delta(t)}(1+x)^{\gamma(t)}f(x)\|_{L^p} = M\|f\|_{L^p_{\delta,\gamma}}, \end{aligned}$$

showing that the assertion of Theorem 1.2 holds for the point $P = (\frac{1}{p}, \alpha, \delta, \gamma)$. This completes the proof of Lemma 3.3.

Proof of Theorem 1.3. We need to show that the operator T_m^α is bounded in $L^p_{\delta,\gamma}$ for every fixed $P = (\frac{1}{p}, \alpha, \delta, \gamma) \in \mathcal{A}$. When $\alpha > 0$, $\alpha = \alpha_n := \frac{n-2}{2}$, and n is an integer so that $\alpha_{n-1} < \alpha < \alpha_n$, then we easily find two points from $\mathcal{A}$ of the form $P_0 = (\frac{1}{p}, \alpha_{n-1}, \delta_0, \gamma_0)$, $P_1 = (\frac{1}{p}, \alpha_n, \delta_1, \gamma_1)$ and some $t \in (0,1)$ to satisfy $P = (1-t)P_0 + tP_1$. When $-1 < \alpha < 0$, one can choose a number α_0 close enough to -1, and interpolate between the points $P_0 = (\frac{1}{2}, \alpha_0, 0, 0)$, $P_1 = (\frac{1}{p}, 0, \delta_1, \gamma_1)$. By Corollary 2.3, the assertion of Theorem 1.3 holds for points P_0 and P_1, and therefore, by Lemma 3.3, it must also hold for the point P. Theorem 1.3 is proved.

4. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following result.

Theorem 4.1. *Let $\alpha > -1$, $\gamma \in \mathbb{R}$ and $\theta \in \mathbb{R}$. Then the operator $T_\alpha^{\alpha+i\theta}$ can be boundedly extended to $L^p_{\delta,\gamma}(\mathbb{R}_+)$ for all $1 < p < \infty$ and $-\frac{\alpha}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$. Moreover, there exist constants $C, c > 0$ and a number $N \in \mathbb{N}$(depending only on $\alpha, p, \delta, \gamma$) such that*

$$\|T_\alpha^{\alpha+i\theta} f\|_{L^p_{\delta,\gamma}} \leq C(1+|\theta|)^N e^{c|\theta|} \|f\|_{L^p_{\delta,\gamma}}, \ \forall\, \theta \in \mathbb{R}. \tag{4.1}$$

The proof of the theorem follows the scheme proposed by Garrigós et al. in [5] and Kanjin in [7]. Obviously, under Assumption (A), it is enough to show (4.1) for the operator

$$\widetilde{T}_\alpha^{\alpha+i\theta} f = \sum_{k=0}^{\infty} \left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+1+i\theta)} \right)^{1/2} < f, L_k^{\alpha+i\theta} > L_k^\alpha$$

instead of $T_\alpha^{\alpha+i\theta}$.

Following [5] and [7], we can define for $\epsilon > 0$ the operators

$$G_{\theta,\epsilon}(f) = \sum_{k=0}^{\infty} \left(\frac{\Gamma(k+\alpha+1)}{\Gamma(k+\alpha+1+\epsilon+i\theta)} \right)^{1/2} < f, L_k^{\alpha+\epsilon+i\theta} > L_k^\alpha$$

so that $\widetilde{T}_\alpha^{\alpha+i\theta} f(x) = \lim_{\epsilon\to 0} G_{\theta,\epsilon} f(x)$ for all $x > 0$, at least $f \in C_c^\infty(0,\infty)$ by Lemma 3.1. Moreover, the following remarkable formula holds (see [7]):

$$G_{\theta,\epsilon} f(x) = \frac{1}{\Gamma(\epsilon+i\theta)} \int_x^\infty f(t) e^{-\frac{t-x}{2}} \left(1 - \frac{x}{t}\right)^{\epsilon-1+i\theta} \left(\frac{x}{t}\right)^{\frac{\alpha}{2}} t^{\frac{\epsilon+i\theta}{2}} \frac{dt}{t}.$$

Adapting the arguments used in [5], pp. 260-263, we can obtain the following result.

Proposition 4.1. *Let $\alpha > -1$, $\gamma \in \mathbb{R}$ and let p, δ be such that $\delta > -\frac{1}{p} - \frac{\alpha}{2}$. Then, there exist constants $C, c > 0$ and a number $N \in \mathbb{N}$ (depending only on $\alpha, p, \delta, \gamma$) such that*

$$\|G_{\theta,\epsilon}\|_{L^p_{\delta,\gamma}} \leq C(1+|\theta|)^N e^{c|\theta|} \left(\|f(x)x^{\frac{\epsilon}{2}}\|_{L^p_{\delta,\gamma}} + \|f(x)x^{-\frac{\epsilon}{2}}\|_{L^p_{\delta,\gamma}} \right),$$

for all $\theta \in \mathbb{R}$ and all $0 < \epsilon \leq 1$.

Proof of Theorem 4.1. By Lemma 3.2 all the multipliers $\lambda = \lambda_{\alpha,\theta}$ in (3.3) satisfy the conditions of Theorem 1.3. Hence, Assumption (A) is satisfied for all $(\frac{1}{p}, \alpha, \delta, \gamma) \in \mathcal{A}$, and we can infer Theorem 4.1 immediately from Proposition 4.1 and Fatou's lemma. Indeed, using these facts, for $f \in C_c^\infty(0,\infty)$ and with some constant C (independent of ϵ) we obtain

$$\begin{aligned} \|T_\alpha^{\alpha+i\theta} f\|_{L^p_{\delta,\gamma}} &= \|T_\lambda \widetilde{T}_\alpha^{\alpha+i\theta} f\|_{L^p_{\delta,\gamma}} \\ &\leq CM(\theta)\|\widetilde{T}_\alpha^{\alpha+i\theta} f\|_{L^p_{\delta,\gamma}} \\ &\leq CM(\theta) \lim_{\epsilon\to 0} \|G_{\theta,\epsilon} f\|_{L^p_{\delta,\gamma}} \\ &\leq CM(\theta) \lim_{\epsilon\to 0} \left(\|f(x)x^{\frac{\epsilon}{2}}\|_{L^p_{\delta,\gamma}} + \|f(x)x^{-\frac{\epsilon}{2}}\|_{L^p_{\delta,\gamma}} \right) \\ &\leq CM(\theta)\|f\|_{L^p_{\delta,\gamma}}, \end{aligned}$$

The proof is complete.

We also need the following lemma proved in [5].

Lemma 4.1. *Let $\alpha > -1$ and $z = \sigma + i\tau$ with $\sigma > -1$. Then the operator T_α^z is bounded in $L^2(\mathbb{R}_+)$.*

Proof of Theorem 1.1. We fix $\beta > \alpha_0 > -1$ so that $-\frac{\alpha_0}{2} - \frac{1}{p} < \delta < 1 - \frac{1}{p} + \frac{\alpha_0}{2}$. Hence, we need only to show that $T_{\alpha_0}^\beta$ and $T_\beta^{\alpha_0}$ are bounded operators in $L^p_{\alpha,\gamma}(\mathbb{R}_+)$. We let $P := (\frac{1}{p}, \alpha, \delta, \gamma)$, which clearly belongs to $\mathcal{A}$. It is easy to see that there exist two other points in $\mathcal{A}$ of the form $P_0 = (\frac{1}{p_0}, \alpha_0, \delta_0, \gamma_0)$ and $P_1 = (\frac{1}{2}, \alpha_1, 0, 0)$, and some $t \in (0,1)$ such that $P = (1-t)P_0 + tP_1$. This can be done explicitly if α_1 is chosen sufficiently large, by taking $\delta_0 = \delta/(1-t)$ and $t = \frac{\beta-\alpha_0}{\alpha_1-\alpha_0}$. As in Section 3, we use the notation $\alpha(z) = (1-z)\alpha_0 + z\alpha_1$, $\delta(z) = (1-z)\delta_0$ and $\gamma(z) = (1-z)\gamma_0$ for $z \in \mathbb{C}$.

By Lemma 4.1, we can define the analytic family of operators

$$S_z = y^{\alpha(z)}(1+y)^{\gamma(z)} T_{\alpha_0}^{\alpha(\bar{z})}(F(x)x^{-\alpha(z)}(1+x)^{-\gamma(z)})(y), \; 0 \leq Re z \leq 1,$$

at least for $F \in L_c^\infty(0,\infty)$. Then, arguing as in Section 3, we conclude that S_z satisfies the conditions of Stein's theorem, where the boundedness of the operators

$$S_{i\theta} : L^{p_0}(\mathbb{R}_+) \to L^{p_0}(\mathbb{R}_+) \text{ and } S_{1+i\theta} : L^{p_0}(\mathbb{R}_+) \to L^{p_0}(\mathbb{R}_+)$$

follows from Theorem 4.1 and Lemma 4.1, respectively. Thus, S_t must be bounded in $L^{p_t} = L^p$, which translates into

$$\begin{aligned}\|T_{\alpha_0}^{\beta} f\|_{L^p_{\delta,\gamma}} &= \|S_t(x^{(1-t)\delta_0}(1+x)^{(1-t)\gamma_0} f(x))\|_{L^p} \\ &\le M\|x^{(1-t)\delta_0}(1+x)^{(1-t)\gamma_0} f(x)\|_{L^p} \\ &= M\|f\|_{L^p_{\delta,\gamma}}.\end{aligned}$$

This proves the required $L^p_{\delta,\gamma}$ boundedness for the operators $T_{\alpha_0}^{\beta}$ for any $\beta > \alpha_0 > -1$. The boundedness of $T_{\beta}^{\alpha_0}$ follows by duality. Indeed, if $(\frac{1}{p}, \alpha_0, \delta, \gamma) \in \mathcal{A}$, then an elementary algebraic manipulation shows that $(\frac{1}{p'}, \alpha_0, -\delta, -\gamma) \in \mathcal{A}$ as well, where $\frac{1}{p'} = 1 - \frac{1}{p}$. Then, for all $f \in C_c^\infty(0,\infty)$ we have

$$\begin{aligned}\|T_{\beta}^{\alpha_0} f\|_{L^p_{\delta,\gamma}} &= \sup_{\|g\|_{p'}=1} \left| \int_0^\infty T_{\beta}^{\alpha_0} f(x) x^{\delta}(1+x)^{\gamma} g(x)dx \right| \\ &= \sup_{\|g\|_{p'}=1} \left| \int_0^\infty f(y) T_{\alpha_0}^{\beta} f(x^{\delta}(1+x)^{\gamma} g)dx \right| \\ &\le \|y^{\delta}(1+y)^{\gamma} f(y)\|_{L^p} \sup_{\|g\|_{p'}=1} \|T_{\alpha_0}^{\beta}(x^{\delta}(1+x)^{\gamma} g))\|_{L^p_{-\delta,-\gamma}} \\ &\le \|f\|_{L^p_{\delta,\gamma}} M \sup_{\|g\|_{p'}=1} \|x^{\delta}(1+x)^{\gamma} g)\|_{L^p_{-\delta,-\gamma}} \\ &= M\|f\|_{L^p_{-\delta,-\gamma}}.\end{aligned}$$

Theorem 1.1 is proved.

5. Applications

In this section, we study the Littlewood-Paley g-functions for the Laguerre semigroup. Consider the heat diffusion semigroup e^{-tL} associated with the Laguerre operator $L = L^{(\alpha)}$. Similar to the classical case, treated in [16], g-functions of order $l = 1, 2, \cdots$ can be defined by

$$g_l^{(\alpha)}(f) = \left\{ \int_0^\infty \left| t^l \frac{\partial^l}{\partial t^l}(e^{-tL^{(\alpha)}} f) \right|^2 \frac{dt}{t} \right\}^{1/2}.$$

The main purpose of this section is to extend Theorem 5.4 from [5] to our case. More precisely, we are going to prove the following result.

Theorem 5.1. *Let $\alpha > -1$, $\gamma \in \mathbb{R}$, $1 < p < \infty$, and δ be such that $-1/p - \alpha/2 < \delta < 1 - 1/p + \alpha/2$. Then for every $l = 1, 2, \cdots$, there is a positive constant C such that*

$$\frac{1}{C}\|f\|_{L^p_{\delta,\gamma}} \leq \|g_l^{(\alpha)} f\|_{L^p_{\delta,\gamma}} \leq C\|f\|_{L^p_{\delta,\gamma}}, \; f \in C_c^\infty(0, \infty).$$

Proof. The proof is similar to that of Theorem 5.4 from [5]. So, we only give a sketch of the proof. Since the first inequality follows from the usual polarization argument, we need only to prove the second inequality. We first consider the case $l = 1$. For simplicity we write $g(f) = g_1^{(\alpha)}(f)$, and drop the superscript (α) when reference to such index is clear. Recall that the kernel $h_t(x, y)$ of e^{-tL} is given explicitly by

$$h_t(y,z) = \sum_{k=0}^{\infty} e^{-t(k+\frac{\alpha+1}{2})} L_k^\alpha(y) L_k^\alpha(z) = \frac{r^{1/2}}{1-r} exp\left\{-\frac{1}{2}\frac{1+r}{1-r}(y+z)\right\} I_\alpha\left(\frac{2(ryz)^{1/2}}{1-r}\right),$$

where $r = e^{-t}$, $I_\alpha = i^{-\alpha} J_\alpha(is)$, and J_α is the usual Bessel function of order α (see [10]).

We first claim that the assertion of the theorem is true when $\alpha = \frac{n-2}{2}$. Indeed, denoting $\Phi(x) = |x|^2$, it is easy to see that for $x \in \mathbb{R}^d$ (see [10]):

$$e^{-tL}(f)(|x|^2) = e^{-\frac{t}{4}(-\Delta + |x|^2)}(\frac{f \circ \Phi}{|\cdot|^\alpha}).$$

Hence $g(f)(|x|^2) = 4g_1(\frac{f\circ\Phi}{|\cdot|^\alpha})|x|^\alpha$, where g_1 was defined in Section 2. Following the same lines as in the proof of Corollary 2.1, the claim can be obtained from Proposition 2.1.

To prove the assertion for any index $\alpha > -1$, we split the operator into two parts as follows:

$$g^*(f) = \left\{\int_{t_0}^{\infty} \left|t^l \frac{\partial^l}{\partial t^l}(e^{-tL^{(\alpha)}} f)\right|^2 \frac{dt}{t}\right\}^{1/2} \text{ and } g_*(f) = \left\{\int_0^{t_0} \left|t^l \frac{\partial^l}{\partial t^l}(e^{-tL^{(\alpha)}} f)\right|^2 \frac{dt}{t}\right\}^{1/2},$$

where t_0 is a sufficiently large number to be chosen later. In the remaining part of the proof, we will need the following result, proved in [5]: there exist a small number $r_0 \in (0, r_0)$ and a constant $C = C(\alpha, r_0) > 0$ such that

$$\sup_{0<r\leq r_0} \left|\frac{\partial}{\partial r}[h_{\ln\frac{1}{r}}(y,z)]\right| \leq C r_0^{\frac{\alpha-1}{2}} y^{\alpha/2} z^{\alpha/2} e^{-(y+z)/8}, \; \forall\, y,\; z > 0. \tag{5.1}$$

We begin with the operator $g^*(f)$, and choose t_0 such that $e^{-t_0} = r_0$. By (5.1) we have

$$\begin{aligned} g^*(f)(y) &\leq \left\{\int_{t_0}^{\infty}\left[\int_{\mathbb{R}_+}\left|\frac{\partial}{\partial t}[h_t(y,z)]\right| |f(z)|dz\right]^2 tdt\right\}^{1/2} \\ &\leq \int_{\mathbb{R}_+}\left\{\int_{t_0}^{\infty}\left|\frac{\partial}{\partial t}[h_t(y,z)]\right|^2 tdt\right\}^{1/2} |f(z)|dz \\ &= \int_{\mathbb{R}_+}\left\{\int_0^{r_0}\left|\frac{\partial}{\partial r}[h_{\ln\frac{1}{r}}(y,z)]\right|^2 r\ln rdr\right\}^{1/2} |f(z)|dz \\ &\leq C\int_{\mathbb{R}_+} y^{\alpha/z} z^{\alpha/2} e^{-(y+z)/8}|f(z)|dz. \end{aligned}$$

Hence,

$$\begin{aligned} \|g^*(f)\|_{L^p_{\delta,\gamma}} &\leq C\left[\int_{\mathbb{R}_+} e^{\frac{-py}{8}} y^{(\frac{\alpha}{2}+\delta)p}(1+y)^{\gamma p}dy\right]^{1/p} \\ &\quad\times\left[\int_{\mathbb{R}_+} e^{\frac{-p'z}{8}} z^{(\frac{\alpha}{2}-\delta)p'}(1+z)^{-\gamma p'}dz\right]^{1/p'} \|f\|_{L^p_{\delta,\gamma}}, \end{aligned}$$

and both integrals are finite since $-1/p-\alpha/2<\delta<1-1/p+\alpha/2$ and $\gamma\in\mathbb{R}$.

Now we turn to the operator g_*, which we need to write as a linear vector-valued operator in order to use transplantation. Let H denote the Hilbert space $L^2((0,\infty),\frac{dt}{t})$. Consider the mapping $G: L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+;H)$ defined by

$$G(f) = G^{(\alpha)}(f) = \left\{t\frac{\partial}{\partial t}(e^{-tL^{(\alpha)}}f)\right\}_{t>0},\ f\in L^2(\mathbb{R}_+).$$

Since $g(f) = |G(f)|_H$, the $L^p_{\delta,\gamma}$ boundedness of g is equivalent to the boundedness of G from $L^p_{\delta,\gamma}$ into $L^p_{\delta,\gamma}(\mathbb{R}_+;H)$. Likewise we define

$$G_*(f) = G_*^{(\alpha)} = \left\{t\frac{\partial}{\partial t}(e^{-tL^{(\alpha)}}f)\chi_{(0,t_0]}(t)\right\}_{t>0}.$$

Finally, we denote by $\bar{T}^{\alpha}_{\beta}$ the vector-valued extension of the transplantation operator to $L^2(\mathbb{R}_+;H)$, defined as follows

$$\bar{T}^{\alpha}_{\beta}(\{f_t\}_{t>0}) = \{T^{\alpha}_{\beta}(f_t)\}_{t>0},\ \{f_t\}_{t>0}\in L^2(\mathbb{R}_+;H).$$

By Krivine's theorem (see, e.g., [8]), the vector-valued operator $\overline{T^{\alpha}_{\beta}}$ is bounded in $L^p_{\delta,\gamma}(\mathbb{R}_+;H)$ if and only if T^{α}_{β} is bounded in $L^p_{\alpha,\gamma}(\mathbb{R}_+)$. Denote by $\bar{M}$ the vector-valued extension of the multiplier operator $Mf=\sum_{k\geq 0} m(k) < f, L^{\beta}_k > L^{\beta}_k$, where $m(s) = \frac{2s+\alpha+1}{2s+\beta+1}$. It is easy to see that this multiplier satisfies the conditions of Theorem 1.3.

Given $\alpha > -1$, we choose $\beta = \frac{n}{2} - 1$, for some positive integer n such that $\beta \geq \alpha$. It is known that (see [5], p. 272):

$$G_*^{(\alpha)} = \overline{T_\alpha^\beta} \circ \bar{N}_{\beta-\alpha} \circ \bar{M} \circ G^{(\beta)} \circ T_\beta^\alpha,$$

where

$$\overline{N}_{\beta-\alpha}(\{f_t\}_{t>0}) = \left\{e^{\frac{\beta-\alpha}{2}t}\chi_{(0,t_0]}(t)f_t\right\}_{t>0}, \; \{f_t\}_{t>0} \in L^2(\mathbb{R}_+; H).$$

Applying Theorems 1.1 and 1.3, we can obtain the boundedness of these operators in $L^p_{\delta,\gamma}$ or $L^p_{\delta,\gamma}(\mathbb{R}_+; H)$ when $-\frac{1}{p} - \frac{\alpha}{2} < \delta < 1 - \frac{1}{p} + \frac{\alpha}{2}$ and $\gamma \in \mathbb{R}$. Thus, Theorem 5.1 is proved for $l = 1$.

Now we proceed to prove the $L^p_{\delta,\gamma}$-boundedness of g_l when $l \geq 2$. Observe first that by the previous result we know the boundedness of $G : L^p_{\delta,\gamma} \to L^p_{\delta,\gamma}(\mathbb{R}_+; H)$, which by Krivine's theorem implies the boundedness of the vector-valued extension $\bar{G} : L^p_{\delta,\gamma}(H) \to L^p_{\delta,\gamma}(H \times H)$ given by

$$\{f_s\}_{s>0} \mapsto \{Gf_s\}_{s>0} = \left\{t\frac{\partial}{\partial t}[e^{-tL}f_s]\right\}_{(t,s)}.$$

Thus, we obtain the boundedness for the composition operator $\bar{G} \circ G : L^p_{\delta,\gamma}(H) \to L^p_{\delta,\gamma}(H \times H)$. Note that $|\bar{G} \circ Gf|^2_{H\times H} = \frac{1}{6}g_2(f)^2$ (see [5], p. 273). Combining all the above facts we obtain the desired estimate $\|g_2(f)\|_{\delta,\gamma} \leq C\|f\|_{\delta,\gamma}$. Similar arguments and induction yield the same conclusion for g_l for all $l \geq 1$. This completes the proof Theorem 5.1.

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