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# EXTENDED MEAN FIELD GAMES

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Abstract. In this paper we present a reformulation of the original mean field problem using random variables point of view, which allows us to consider situations where the interaction a player and the mean field also takes into account the collective behavior of the players but not only its state. We prove an existence result for this type of mean field games in the case of quadratic Hamiltonians.

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## 1. Introduction

Mean field games is a recent area of research started by Minyi Huang, Peter E. Caines, and Roland P. Malhamé [9], [10] and Pierre Louis Lions and Jean Michel Lasry [11] – [14] which attempts to understand the limiting behavior of systems involving very large numbers of rational agents which play dynamic games under partial information and symmetry assumptions. Inspired by ideas in statistical physics, these authors introduced a class of models in which the individual player contribution is encoded in a mean field that contains only statistical properties about the ensemble. The literature on mean field games and its applications is growing fast, for a recent survey see [16] and reference therein. Applications of mean field games arise in the study of growth theory in economics [15] or environmental policy [2], for instance, and it is likely that in the future they will play an important rôle in economics and population models. There is also a growing interest in numerical methods for these problems [1] – [3]. Also, the discrete state problem is considered both in discrete time [5] and the continuous time [6]. Several problems have been worked out in detail in [7], [8].

This paper is structured as follows: we start in section 2 to discuss the original formulation by Lions and Lasry of mean field games as a coupled system of transport equations with a Hamilton-Jacobi equation. Then we present a reformulation of this problem (similar to one used in [10]) as a coupled system of an ordinary differential equation in an $L^2$ space, together with a Hamilton-Jacobi equation. Section 3 concerns an extensions of the original mean-field problem where a player and the mean-field also takes into account the collective behaviour of the players not only its state. Finally in the section 4 we prove the existence of solutions to the extended mean field system for quadratic Hamiltonian of special form.

## 2. Two formulations of deterministic mean field games

In this section we review the original formulation for deterministic mean-field games from Lions-Lasry which is inspired by some of the material in the lectures by Lions at College de France. Then we discuss a reformulation in terms of random variables. This set up is essentially the one considered by Minyi Huang, Peter E. Caines, and Roland P. Malhamé, in [10] and is particularly suited to the extensions we consider in this paper.

In the standard mean field game setting one considers a population of players where each individual has a state given by his position $x \in \mathbb{R}^d$. We denote by $\mathcal{P}(\mathbb{R}^d)$ the set of probability measures in $\mathbb{R}^d$. This set is a metric space endowed with the Wasserstein metric $W_2$, see for instance [17]. Population of players is described at each time $t$ by a probability measure $\theta(t) \in \mathcal{P}(\mathbb{R}^d)$. Given an individual player who, for some reason knows the $\theta(t)$ for all time, his or her objective is to minimize a certain performance criterion. For this let $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be a running cost, $\psi : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be a terminal cost. We suppose that both $L$ and $\psi$ satisfy standard hypothesis for optimal control problems, that is:

- $L$ and $\psi$ are continuous functions and bounded below, without loss of generality we can assume $L, \psi \geq 0$.
- $\psi$ is Lipschitz in the first coordinate.
- $L$ is coercive:
$$\frac{L(v, x, \theta)}{|v|} \xrightarrow{|v| \to \infty} \infty, \text{ uniformly in } x.$$
- $L$ is uniformly convex in $v$.

A typical example is

$$L(v,x,\theta)=\frac{|v|^2}{2}-\int_{\mathbb{R}^d}V(x,y)d\theta(y), \tag{2.1}$$

for some function $V\colon \mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$.

The value function from the point of view of a player at the point $x$ at time $t$ is defined through the optimal control problem:

$$u(x,t)=\inf_{\mathbf{x}}\int_t^T L(\dot{\mathbf{x}}(s),\mathbf{x}(s),\theta(s))ds+\psi(\mathbf{x}(T),\theta(T)).$$

For $(p,x,\theta)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)$ we define the Hamiltonian

$$H(p,x,\theta)=\sup_{v\in\mathbb{R}^d}-v\cdot p-L(v,x,\theta).$$

Then it is well known that $u$ is the unique viscosity solution of the Hamilton-Jacobi equation

$$-u_t+H(D_xu,x,\theta)=0 \tag{2.2}$$

satisfying the terminal condition $u(x,T)=\psi(x,\theta(T))$. It is also known that if $u$ is trajectory for this ps a classical solution to (2.2) the optimal trajectory is given by

$$\dot{\mathbf{x}}(s)=-D_pH(D_xu(\mathbf{x}(s),s),\mathbf{x}(s),\theta(s)). \tag{2.3}$$

The mean-field game hypothesis consists in assuming that all players have access to the same information and act in a rational way. Therefore each one of them follows the optimal trajectories (2.3). This then implies that the probability distribution of players is transported by the vector field $-D_pH(D_xu(x,t),x,\theta(t))$. Therefore $\theta$ is a (weak) solution of the equation

$$\theta_t-\operatorname{div}(D_pH(D_xu,x,\theta)\theta)=0,$$

together with an initial condition for $\theta$, $\theta(0)=\theta_0\in\mathcal{P}(\mathbb{R}^d)$, which encodes the initial distribution of players. This leads to the system

$$\begin{cases}-u_t+H(D_xu,x,\theta)=0\\ \theta_t-\operatorname{div}(D_pH(D_xu,x,\theta)\theta)=0.\end{cases} \tag{2.4}$$

subjected to the initial-terminal conditions

$$\begin{cases}u(x,T)=\psi(x,\theta(T))\\ \theta(x,0)=\theta_0.\end{cases} \tag{2.5}$$

The system (2.4) and its second order analogue was first introduced and studied by Pierre Louis Lions and Jean Michel Lasry in [12]. More detailed proofs of existence and uniqueness to those systems for quadratic Hamiltonians can be found in notes by Pierre Cardaliaguet from P.-L. Lions lectures at College de France.

2.1. **Random variables point of view.** Let $(\Omega, \mathcal{F}, P)$ be a probability space, where $\Omega$ is an arbitrary nonempty set, $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$ and $P$ is a probability measure. We recall that a $\mathbb{R}^d$ valued random variable $X$ is a function $X : \Omega \to \mathbb{R}^d$. We denote by $L^q(\Omega, \mathbb{R}^d)$ the set of $\mathbb{R}^d$ valued random variable whose norm is in $L^q(\Omega)$. The law $\mathcal{L}(X)$ of a $\mathbb{R}^d$ valued random variable is the probability measure in $\mathbb{R}^d$ defined by

$$\int\limits_{\mathbb{R}^d} \varphi(x) d\mathcal{L}(X)(x) = E\varphi(X).$$

Note that since all relevant random variables are $\mathbb{R}^d$ valued, we write $L^q(\Omega)$ instead of $L^q(\Omega, \mathbb{R}^d)$, to simplify the notation. We can reformulate the mean field game problem by replacing the probability $\theta(t)$ encoding the distribution of players by a random variable $\mathbf{X}(t) \in L^q(\Omega)$ such that $\mathcal{L}(\mathbf{X}(t)) = \theta(t)$. Of course for each measure $\theta$ there are many possible random variables with law $\theta$, however this will not create any problem. Each outcome of the random variable $\mathbf{X}$ represents the position of random player chosen according to the probability $\theta$.

We say that a function $f \colon L^q(\Omega) \to \mathbb{R}$ depends only on the law if for any $X, \tilde{X} \in L^q(\Omega)$ with the same law, i.e., $\mathcal{L}(X) = \mathcal{L}(\tilde{X})$, we have $f(X) = f(\tilde{X})$. Let $\eta : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$, $\eta(\theta)$, we can define a function, $\tilde{\eta} : L^q(\Omega; \mathbb{R}^d) \to \mathbb{R}$, $\tilde{\eta}(X)$, which depends only on the law of $X$, by

$$\tilde{\eta}(X) = \eta(\mathcal{L}(X)).$$

This allows us to identify functions in $\mathcal{P}(\Omega)$ with functions in $L^q(\Omega)$ which depend only on the law. To make the presentation more intuitive, we use the same notation for functions whether they are written in terms of the probability measure $\theta$ or in terms of a random variable $X$ with $\mathcal{L}(X) = \theta$, i.e. we omit the tilde and write simply $\eta(X)$ or $\eta(\theta)$, according to the previous identification.

In this new setting, the Lagrangian is a function $L : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \to \mathbb{R}$ that we denote by $L(v, x, X)$, which in the last coordinate depends only on the law. For example the Lagrangian (2.1) can be now written as

$$L(v, x, X) = \frac{|v|^2}{2} - EV(x, X).$$

As before, suppose an individual player knows the distribution of players which is now encoded on a trajectory $\mathbf{X}(t) \in L^q(\Omega)$ for all times. His or her objective is to minimize a certain performance criterion, determined as before by a running cost

$L : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega;\mathbb{R}^d) \to \mathbb{R}$, and a terminal cost $\psi : \mathbb{R}^d \times L^q(\Omega;\mathbb{R}^d) \to \mathbb{R}$. We assume that both $L(v,x,X)$ and $\psi(x,X)$ depend only on the law of $X$.

Then the value function from the point of view of a reference player which is at a point $x$ at time $t$ is

$$u(x,t) = \inf_{\mathbf{x}} \int_t^T L(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{X}(s))ds + \psi(\mathbf{x}(T), \mathbf{X}(T)).$$

As before, for $(p,x,X) \in \mathbb{R}^d \times \mathbb{R}^d \times L^2(\Omega)$, the Hamiltonian is given by

$$H(p,x,X) = \sup_{v \in \mathbb{R}^d} -v \cdot p - L(v,x,X).$$

The Hamiltonian $H$ which is a function $H : \mathbb{R}^d \times \mathbb{R}^d \times L^2(\Omega) \to \mathbb{R}$, that we denote by $H(p,x,X)$, depends only on the last coordinate through its law, i.e., if $X, \tilde{X} \in L^2(\Omega)$ have the same law, i.e., $\mathcal{L}(X) = \mathcal{L}(\tilde{X})$ then

$$H(p,x,X) = H(p,x,\tilde{X}).$$

Then $u$ is the unique viscosity solution of the Hamilton-Jacobi equation

$$-u_t(x,t) + H(D_x u(x,t), x, \mathbf{X}(t)) = 0$$

with the terminal condition $u(x,T) = \psi(x, \mathbf{X}(T))$.

Because of the rationality hypothesis, the dynamics of a typical player at position $\mathbf{X}(s)(\omega)$, $\omega \in \Omega$, is then given by

$$\dot{\mathbf{X}}(s)(\omega) = -D_p H(D_x u(\mathbf{X}(s)(\omega), s), \mathbf{X}(s)(\omega), \mathbf{X}(s)).$$

This yields the following alternative formulation of the mean field game (2.4)

$$\begin{cases} -u_t(x,t) + H(D_x u(x,t), x, \mathbf{X}(t)) = 0 \\ \dot{\mathbf{X}}(t) = -D_p H(D_x u(\mathbf{X},t), \mathbf{X}(t), \mathbf{X}(t)). \end{cases}$$

where the initial-terminal condition (2.5) is replaced by

$$\begin{cases} u(x,T) = \psi(x, \mathbf{X}(T)) \\ \mathbf{X}(0) = X_0, \end{cases}$$

where $\mathcal{L}(X_0) = \theta_0$.

The connection between the two formulations is an easy consequence of the following well known result:

**Proposition 2.1.** *Let $b\colon \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ be a smooth, globally Lipschitz vector field over $\mathbb{R}^d$ and let $\mathbf{X}(t) : [0,T] \times \Omega \to \mathbb{R}^d$ be a solution to*

$$\dot{\mathbf{X}} = b(\mathbf{X}, t). \tag{2.6}$$

*With the law $\theta = \mathcal{L}(\mathbf{X})$ which is absolutely continuous with a smooth density $\theta(x,t)$. Then $\theta(x,t)$ is a solution to*

$$\theta_t(x,t) + \operatorname{div}(b(x,t)\theta(x,t)) = 0.$$

*with initial condition $\theta(0) = \mathcal{L}(\mathbf{X}(0))$.*

*Proof.* We have

$$\int\limits_{\mathbb{R}^d} \phi(x)\theta_t(x,t)dx = \frac{d}{dt}\int\limits_{\mathbb{R}^d} \phi(x)\theta(x,t)dx = \frac{d}{dt}E\phi(\mathbf{X}(t)) = ED\phi(\mathbf{X}(t))\dot{\mathbf{X}}(t) =$$

$$= ED\phi(\mathbf{X})b(\mathbf{X},t) = \int\limits_{\mathbb{R}^d} D\phi(x)b(x,t)\theta dx = -\int\limits_{\mathbb{R}^d} \phi(x)\operatorname{div}(b(x,t)\theta)dx$$

for every $\phi \in C_c^1(\mathbb{R}^d)$. Thus $\theta_t + div(b(x,t)\theta) = 0$. □

## 3. Extended mean-field games

In many applications one must consider mean field games where the pay-off of each player depends not only on the statistical information or state of the remaining players but also on the actions the other players take. In the random variables point of view this corresponds to the running costs that depend on $\dot{\mathbf{X}}$. As before we assume that the distribution of players is represented by a random variable $\mathbf{X}(t) \in L^q(\Omega)$, which we suppose to be differentiable with derivative $\dot{\mathbf{X}}(t) \in L^q(\Omega)$. Many other alternative spaces could be used here and this is not essential for this part of discussion, for instance we could consider $\dot{\mathbf{X}} \in L^q([0,t], L^q(\Omega))$. As before, the objective of an individual player is to minimize a certain performance criterion. For this let $L : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega) \to \mathbb{R}$ be a Lagrangian, $\psi : \mathbb{R}^d \times L^q(\Omega) \to \mathbb{R}$ be a terminal cost. We assume that $L(v,x,X,Z)$ depends only on the joint law of $(X,Z)$ i.e., if $X, Z, \tilde{X}, \tilde{Z} \in L^q(\Omega)$ satisfy $\mathcal{L}(X,Z) = \mathcal{L}(\tilde{X},\tilde{Z})$ then

$$L(v,x,X,Z) = L(v,x,\tilde{X},\tilde{Z}),$$

and that $\psi(x,X)$ depend only on the law of $X$.

We suppose that both $L$ and $\psi$ satisfy standard hypothesis for optimal control problems, that is:

- $L$ and $\psi$ are continuous functions and bounded below.
- $\psi$ is Lipschitz in the first coordinate.
- $L$ is coercive:

$$\frac{L(v,x,X,Z)}{|v|} \xrightarrow{|v|\to\infty} \infty, \text{ uniformly in } x.$$

- $L$ is convex in $v$.

The value function from the point of view of a player at the point $x$ at time $t$ is

$$u(x,t) = \inf_{\mathbf{x}} \int_t^T L(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{X}(s), \dot{\mathbf{X}}(s))ds + \psi(\mathbf{x}(T), \mathbf{X}(T)), \tag{3.1}$$

where the infimum is taken over absolutely continuous trajectories. As before the Hamiltonian is given by

$$H(p,x,X,Z) = \sup_{v\in\mathbb{R}^d} -v\cdot p - L(v,x,X,Z).$$

The Hamiltonian $H$ is a function $H : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega) \times L^q(\Omega) \to \mathbb{R}$, denoted by $H(p,x,X,Z)$, that in the last two coordinates depends only on its joint law. Given that the trajectory $\mathbf{X}(t)$ is known for every player we get that the value function $u$ satisfies the Hamilton-Jacobi equation:

$$-u_t + H(D_x u, x, \mathbf{X}, \dot{\mathbf{X}}) = 0,$$

then by mean field hypothesis the player should follow the optimal trajectories

$$\dot{\mathbf{x}} = -D_p H(D_x u(\mathbf{x},t), \mathbf{x}, \mathbf{X}, \dot{\mathbf{X}}).$$

This leads to the extended mean field system

$$\begin{cases} -u_t + H(D_x u, x, \mathbf{X}, \dot{\mathbf{X}}) = 0 \\ \dot{\mathbf{X}} = -D_p H(D_x u(\mathbf{X},t), \mathbf{X}, \mathbf{X}, \dot{\mathbf{X}}), \end{cases} \tag{3.2}$$

with

$$\begin{cases} u(x,T) = \psi(x, \mathbf{X}(T)) \\ \mathbf{X}(0) = X_0. \end{cases}$$

An important example of a Lagrangian with velocity dependence is

$$L(v,x,X,Z) = \frac{|v+\beta EZ|^2}{2} - V(x,X), \tag{3.3}$$

to which corresponds the Hamiltonian

$$H(p,x,X,Z) = \frac{|p|^2}{2} + \beta p\cdot EZ + V(x,X). \tag{3.4}$$

Mean field games under running costs of type (3.3) are quite interesting to study from the applications point of view since they represent simple models of traffic jams. Indeed, an individual player that plays mean field game with the running cost (3.3) tends to move with a velocity close to $-\beta EZ$, that is to move away from the average of the whole population of the players.

## 4. Existence of solutions to extended mean field games

In this section we prove the existence of the solutions to the extended mean field games system (3.2) for Hamiltonian of the form

$$H(p,x,X,Z) = \frac{|p|^2}{2} + \beta p E Z - V(x,X),$$

where $\beta > 0$ and $V : \mathbb{R}^d \times L^2(\Omega) \to \mathbb{R}$ depends only in the law of $X$. The corresponding Lagrangian is

$$L(v,x,X,Z) = \frac{|\beta E Z + v|^2}{2} + V(x,X).$$

To have the existence of the solutions to (3.2) we suppose further assumptions on $X_0, \psi$ and $V$:

a) $X_0 \in L^2(\Omega)$ and has an absolutely continuous law.
b) $\psi$ is bounded, smooth in $x$ and Lipschitz in both variables.
c) $V$ is twice differentiable in $x$ with $\|V(\cdot,X)\|_{\mathcal{C}^2} \leq C$,
d) $V(x,\cdot)$ and $D_xV(x,\cdot)$ are uniformly Lipschitz: there exist a constant $C > 0$ such that

$$|V(x,X) - V(x,Y)| \leq C\|X - Y\|_{L^2(\Omega)},$$

and

$$|D_xV(x,X) - D_xV(x,Y)| \leq C\|X - Y\|_{L^2(\Omega)}.$$

We intend to prove the existence of the solution to (3.2) by a fixed point argument, for this we take any function $\Phi \in C(\mathbb{R}^d)$ which is bounded, Lipschitz and consider the following system of ODEs in $L^2(\Omega)$

$$\begin{cases} \dot{\mathbf{X}} = -D_pH(\mathbf{P},\mathbf{X},\mathbf{X},\dot{\mathbf{X}}) = -\mathbf{P} - \beta E\dot{\mathbf{X}} \\ \dot{\mathbf{P}} = D_xH(\mathbf{P},\mathbf{X},\mathbf{X},\dot{\mathbf{X}}) = D_xV(\mathbf{X},\mathbf{X}) \\ \mathbf{X}(0) = X_0,\ \mathbf{P}(0) = D_x\Phi(X_0). \end{cases} \tag{4.1}$$

which if we solve the first equation in $\dot{\mathbf{X}}$, can be written as

$$\begin{cases} \dot{\mathbf{X}} = -\mathbf{P} + \frac{\beta}{1+\beta}E\mathbf{P} \\ \dot{\mathbf{P}} = D_xV(\mathbf{X},\mathbf{X}) \\ \mathbf{X}(0) = X_0,\ \mathbf{P}(0) = D_x\Phi(X_0). \end{cases} \tag{4.2}$$

Since $\Phi$ is Lipschitz $D_x\Phi$ exists almost everywhere and since $X_0$ has an absolutely continuous law $\mathbf{P}(0)$ is well defined on a set of full measure. Furthermore the Lipschitz conditions on $D_xV$ guarantee that the right hand sides of the equations in (4.2) are Lipschitz in $\mathbf{X},\mathbf{P}$, hence there exists a unique solution $(\mathbf{X},\mathbf{P})$ to (4.1). It is easy to see from the equation that $\mathbf{P} \in C^1([0,T]; L^2(\Omega))$ and $\mathbf{X} \in C^2([0,T]; L^2(\Omega))$. We define the function $\tilde{u}(x,t)$ as the solution to the optimal control problem

$$\tilde{u}(x,t) = \inf_{\mathbf{x}} \int_t^T L(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{X}, \dot{\mathbf{X}}) + \psi(\mathbf{x}(T), \mathbf{X}(T)) \tag{4.3}$$

where the infimum is taken over all Lipschitz trajectories $\mathbf{x}(s)$ starting at $\mathbf{x}(t) = x$.

**Lemma 4.1.** *Let $\Phi \in C(\mathbb{R}^d)$ any bounded, Lipschitz function then $\tilde{u}(x,t)$ defined by (4.3) is uniformly bounded and Lipschitz in $x$. Furthermore for any $t, t < T$, $\tilde{u}$ is semi-concave in $x$. More specifically we have*

(1) $|\tilde{u}| \leq \|V\|_\infty \cdot (T-t) + \|\psi\|_\infty$ *for all* $x \in \mathbb{R}^d, 0 \leq t \leq T$.
(2) $|\tilde{u}(x+y,t) - \tilde{u}(x,t)| \leq (C + (T-t)C)|y|$ *for all* $x, y \in \mathbb{R}^d, 0 \leq t \leq T$.
(3) $\tilde{u}(x+y,t) + \tilde{u}(x-y,t) - 2\tilde{u}(x,t) \leq \left(C(T-t) + \frac{C}{T-t}\right)|y|^2$ *for all* $x, y \in \mathbb{R}^d, 0 \leq t < T$.

*Where the constants $C$ depend only on $L, \psi$.*

*Proof.*

- **$\tilde{u}$ is bounded**

  From assumptions on $V$ we have

$$\tilde{u}(x,t) \geq -\|V\|_\infty (T-t) - \|\psi\|_\infty.$$

  On the other hand if we take $\mathbf{x}(s) = x - \beta \int_t^s E\dot{\mathbf{X}}(\tau) d\tau$ from (4.3) yields

$$\tilde{u}(x,t) \leq \int_t^T V(\mathbf{X}(\tau), \mathbf{X}(\tau)) d\tau + \psi(\mathbf{x}(T), \mathbf{X}(T)) \leq (T-t)\|V\|_\infty + \|\psi\|_\infty.$$

- **$\tilde{u}$ is Lipschitz**

  To prove the point 2 of the Lemma we take $x, y \in \mathbb{R}^d$, and any $\varepsilon > 0$. Let $x^\varepsilon$ be an $\varepsilon$-suboptimal trajectory at a point $(x,t)$, i.e.

$$\tilde{u}(x,t) \geq \int_t^T L(\dot{\mathbf{x}}^\varepsilon(s), \mathbf{x}^\varepsilon(s), \mathbf{X}(s), \dot{\mathbf{X}}(s)) ds + \psi(\mathbf{x}^\varepsilon(T), \mathbf{X}(T)) - \varepsilon. \tag{4.4}$$

  We have then

$$\tilde{u}(x+y,t) \leq \int_t^T L(\dot{\mathbf{x}}^\varepsilon(s), \mathbf{x}^\varepsilon(s) + y, \mathbf{X}(s), \dot{\mathbf{X}}(s)) ds + \psi(\mathbf{x}^\varepsilon(T) + y, \mathbf{X}(T)). \tag{4.5}$$

Thus

$$\tilde{u}(x+y,t)-\tilde{u}(x,t)\le\int\limits_t^T[L(\dot{\mathbf{x}}^\varepsilon,\mathbf{x}^\varepsilon+y,\mathbf{X},\dot{\mathbf{X}})-L(\dot{\mathbf{x}}^\varepsilon,\mathbf{x}^\varepsilon,\mathbf{X},\dot{\mathbf{X}})]ds$$

$$+\psi(\mathbf{x}^\varepsilon(T)+y,\mathbf{X}(T))-\psi(\mathbf{x}^\varepsilon(T),\mathbf{X}(T))+\varepsilon=\int\limits_t^T[V(\mathbf{x}^\varepsilon+y,\mathbf{X})-V(\mathbf{x}^\varepsilon,\mathbf{X})]ds$$

$$+\psi(\mathbf{x}^\varepsilon(T)+y,\mathbf{X}(T))-\psi(\mathbf{x}^\varepsilon(T),\mathbf{X}(T))+\varepsilon.$$

This gives

$$\tilde{u}(x+y,t)-\tilde{u}(x,t)\le(C(T-t)+C)|y|+\varepsilon,$$

where the constant $C$ depends only on Lipschitz constants of $\psi$ and $V$. Since $x,y$ and $\varepsilon>0$ are arbitrary we conclude that

$$|\tilde{u}(x+y,t)-\tilde{u}(x,t)|\le(C(T-t)+C)|y|.$$

- **$\tilde{u}$ is semi-concave**

For the semi-concavity we take any $x,y\in\mathbb{R}^d$, $\varepsilon>0$ and $\mathbf{x}^\varepsilon$ as above. Let

$$\mathbf{y}(s)=y\frac{T-s}{T-t},$$

then inequalities

$$\tilde{u}(x\pm y,t)\le\int\limits_t^T L(\dot{\mathbf{x}}^\varepsilon\pm\dot{\mathbf{y}},\mathbf{x}^\varepsilon\pm\mathbf{y},\mathbf{X},\dot{\mathbf{X}})ds+\psi(\mathbf{x}^\varepsilon(T),\mathbf{X}(T))$$

together with (4.4) yield

$$\tilde{u}(x+y,t)+\tilde{u}(x-y,t)-2\tilde{u}(x,t)\le\int\limits_t^T[L(\dot{\mathbf{x}}^\varepsilon+\dot{\mathbf{y}},\mathbf{x}^\varepsilon+\mathbf{y}(s),\mathbf{X},\dot{\mathbf{X}})+$$

$$L(\dot{\mathbf{x}}^\varepsilon-\dot{\mathbf{y}},\mathbf{x}^\varepsilon-\mathbf{y},\mathbf{X},\dot{\mathbf{X}})-2L(\dot{\mathbf{x}}^\varepsilon,\mathbf{x}^\varepsilon,\mathbf{X},\dot{\mathbf{X}})]ds+2\varepsilon=\int\limits_t^T[|\dot{\mathbf{y}}|^2+V(\mathbf{x}^\varepsilon+\mathbf{y},\mathbf{X})$$

$$+V(\mathbf{x}^\varepsilon-\mathbf{y},\mathbf{X})-2V(\mathbf{x}^\varepsilon,\mathbf{X})]ds+2\varepsilon\le\left(C(T-t)+\frac{C}{T-t}\right)|y|^2+2\varepsilon,$$

where we used the bounds on $D^2_{xx}V$ and the fact that $\dot{\mathbf{y}}(s)=-\frac{y}{T-t}$. Now sending $\varepsilon$ to zero we get the result. $\square$

Let $\mathcal{A}$ be the set of functions $\Phi\in C(\mathbb{R}^d)$ which satisfy following conditions

(1) $|\Phi(x)|\le T\|V\|_\infty+\|\psi\|_\infty$ for all $x\in\mathbb{R}^d$,
(2) $|\Phi(x+y)-\Phi(x)|\le(C+TC)|y|$ for all $x,y\in\mathbb{R}^d$,
(3) $\Phi(x+y)+\Phi(x-y)-2\Phi(x)\le\left(CT+\frac{C}{T}\right)|y|^2$ for all $x,y\in\mathbb{R}^d$,

where the constants $C$ are defined as in Lemma 4.1. The space $C(\mathbb{R}^d)$ endowed with the topology of locally uniformly convergence is a topological vector space and according to the Arzelà- Ascoli theorem the set $\mathcal{A}$ is its compact, convex subset.

**Lemma 4.2.** *The mapping*

$$F\colon \Phi(x) \mapsto \tilde{u}(x,0). \tag{4.6}$$

*is a continuous and compact mapping from $\mathcal{A}$ into itself.*

The Lemma4.1 shows that the mapping 4.6 maps the set $\mathcal{A}$ into itself. We prove the continuity of the mapping $F$ by assuming the contrary: there exist $\Phi_n \to \Phi$ locally uniformly in $\mathbb{R}^d$ such that $F(\Phi_n) \nrightarrow F(\Phi)$ locally uniformly in $\mathbb{R}^d$. Then since $F(\Phi_n) \in \mathcal{A}$ and $\mathcal{A}$ is compact we can assume without loss of generality that $\tilde{u}_n = F(\Phi_n) \to \bar{u} \neq F(\Phi)$ locally uniformly. Since $\Phi_n$ are uniformly semi-concave we can assume (passing to a further subsequence if necessary) that $D\Phi_n \to D\Phi$ a.e.. Let $(\mathbf{X}_n, \mathbf{P}_n)$ solve the equation

$$\begin{cases} \dot{\mathbf{X}}_n = -\mathbf{P}_n + \frac{\beta}{1+\beta} E\mathbf{P}_n \\ \dot{\mathbf{P}}_n = D_x V(\mathbf{X}_n, \mathbf{X}_n) \\ \mathbf{X}_n(0) = X_0,\, \mathbf{P}_n(0) = D_x\Phi_n(X_0). \end{cases}$$

Then the assumptions c), d) imply that $D_x V$ is Lipschitz in both variables, hence substracting the equations for $(\mathbf{X}, \mathbf{P})$ from the equations for $(\mathbf{X}_n, \mathbf{P}_n)$ we get

$$E(|\dot{\mathbf{X}}_n(t) - \dot{\mathbf{X}}(t)|^2 + |\dot{\mathbf{P}}_n(t) - \dot{\mathbf{P}}(t)|^2) \leq C(E|\mathbf{X}_n(t) - \mathbf{X}(t)|^2 + |\mathbf{P}_n(t) - \mathbf{P}(t)|^2).$$

Therefore the Gronwall's inequality then yields:

$$\begin{aligned} &E(|\mathbf{X}_n(t) - \mathbf{X}(t)|^2 + |\mathbf{P}_n(t) - \mathbf{P}(t)|^2) \leq C(E|\mathbf{X}_n(0) - \mathbf{X}(0)|^2 + |\mathbf{P}_n(0) - \mathbf{P}(0)|^2) \\ &= CE|D\Phi_n(X_0) - D\Phi(X_0)|^2. \end{aligned}$$

By the dominated convergence theorem the right hand side here goes to zero. Consequently $\mathbf{X}_n \to \mathbf{X}$ and $\mathbf{P}_n \to \mathbf{P}$ in $L^\infty([0,T]; L^2(\Omega))$, the equation (4.2) then implies that $\dot{\mathbf{X}}_n \to \dot{\mathbf{X}}$. Thus using assumptions b), c) and d) we get

$$H(p, x, \mathbf{X}_n, \dot{\mathbf{X}}_n) \to H(p, x, \mathbf{X}, \dot{\mathbf{X}}) \text{ locally uniformly in } x, p,$$

and

$$\psi(x, \mathbf{X}_n(T)) \to \psi(x, \mathbf{X}(T)) \text{ locally uniformly in } x.$$

From the definition of $\tilde{u}_n = F(\Phi_n)$ and standard results in optimal control we know that $\tilde{u}_n$ solves the Hamilton-Jacobi equation with Hamiltonian $H(p, x, \mathbf{X}_n, \dot{\mathbf{X}}_n)$ and terminal value $\psi(x, \mathbf{X}_n(T))$. Since $\tilde{u}_n \to \bar{u}$ locally uniformly the stability of viscosity solutions ([4]) implies that $\bar{u}$ is a viscosity solution to the Hamilton-Jacobi equation with the Hamiltonian $\widetilde{H}(p,x,t) = H(p,x,\mathbf{X}(t),\dot{\mathbf{X}}(t))$ and terminal value $\psi(x,\mathbf{X}(T))$:

$$\begin{cases} -\bar{u}_t(x,t) + \widetilde{H}(D_x\bar{u}(x,t),x,t) = 0, \\ \bar{u}(x,T) = \psi(x,\mathbf{X}(T)) \end{cases} \tag{4.7}$$

on the other hand from the definition of $\tilde{u} = F(\Phi)$ and standard optimal control theory results we have that $\tilde{u}$ also is a viscosity solution to the (4.7). Since $\mathbf{X}$ and $\dot{\mathbf{X}}$ are Lipschitz continuous in $t$, Lipschitz bounds on $V$ and some standard estimates give

$$|\widetilde{H}(p,x,t) - \widetilde{H}(q,y,s)| \leq R|p-q| + |\beta E\dot{\mathbf{X}}(t)||p-q| + \beta RE|\dot{\mathbf{X}}(t) - \dot{\mathbf{X}}(s)|$$
$$+ C|x-y| + C\|\mathbf{X}(t) - \mathbf{X}(s)\|_{L^2(\Omega)} \leq C(T,R)(|p-q| + |x-y| + |t-s|)$$

for any $R > 0$ and $t,s \in [0,T]$ and all $x,y,p,q \in \mathbb{R}^d$ with $|p|,|q| \leq R$. This condition implies the uniqueness of the viscosity solutions to the equation (4.7)( [4]), thus $\bar{u} = \tilde{u}$ which is a contradiction. Hence the mapping $F$ is continuous, the compactness of $\mathcal{A}$ then implies that $F$ is also compact.

**Theorem 4.1.** *Under the above conditions on $H$,$\psi$ and $X_0$ there exist a continuous Lipschitz semi-concave function $u$ on $\mathbb{R}^d$ and a random variable $\mathbf{X} \in C^2([0,T];L^2(\Omega))$, such that the couple $(u,\mathbf{X})$ solves the system of extended mean field equations* (3.2) *in the sense that $u \in C([0,T] \times \mathbb{R}^d)$ is a viscosity solution to the Hamilton-Jacobi equation:*

$$\begin{cases} -u_t + H(D_xu,x,\mathbf{X},\dot{\mathbf{X}}) = 0, \ \ \textit{in } [0,T] \times \mathbb{R}^d \\ u(x,T) = \psi(x,\mathbf{X}(T)), \end{cases}$$

*$u$ is differentiable at every point $(\mathbf{X}(t),t)t > 0$, and $\mathbf{X} \in C^{1,1}([0,T];L^2(\Omega))$ is a classical solution to the ODE:*

$$\begin{cases} \dot{\mathbf{X}} = -D_pH(D_xu(\mathbf{X},s),\mathbf{X},\mathbf{X},\dot{\mathbf{X}}), \ \ \textit{in } [0,T] \times \Omega \\ \mathbf{X}(0) = X_0. \end{cases}$$

*Proof.* By Lemma (4.2) and Schauder's fixed point theorem there exists $u \in C(\mathbb{R}^d)$ such that

$$u(x) = F(u) = \tilde{u}(x,0),$$

where $\tilde{u}(x,t)$ is defined as in 4.3. Let us denote it by $u(x,t) := \tilde{u}(x,t)$. Then $u$ solves the Hamilton-Jacobi equation

$$\begin{cases} -u_t(x,t) + H(D_xu(x,t),x,\mathbf{X}(t),\dot{\mathbf{X}}(t)) = 0 \\ u(x,T) = \psi(x,\mathbf{X}(T)). \end{cases} \tag{4.8}$$

From optimal control theory (see [4]) we know that for almost every $x$ we have the existence of the optimal trajectories which are given by the Hamiltonian flow

$$\begin{cases} \dot{\mathbf{x}}(x,t) = -D_pH(\mathbf{p},\mathbf{x},\mathbf{X},\dot{\mathbf{X}}) \\ \dot{\mathbf{p}}(x,t) = D_xH(\mathbf{p},\mathbf{x},\mathbf{X},\dot{\mathbf{X}}) \\ \mathbf{x}(x,0) = x,\ \mathbf{p}(x,0) = D_xu(x,0). \end{cases} \tag{4.9}$$

We also know that $\mathbf{p}(x,t) = Du(\mathbf{x}(x,t),t)$ and that $Du$ exists for all points $(\mathbf{x}(x,t),t)$ with $t > 0$. Now put $\mathbf{Y}(t) = \mathbf{x}(X_0,t)$ and $\mathbf{Q}(t) = \mathbf{p}(X_0,t)$, then from (4.9)

$$\begin{cases} \dot{\mathbf{Y}} = -D_pH(\mathbf{Q},\mathbf{Y},\mathbf{X},\dot{\mathbf{X}}) \\ \dot{\mathbf{Q}} = D_xH(\mathbf{Q},\mathbf{Y},\mathbf{X},\dot{\mathbf{X}}) \\ \mathbf{Y}(0) = X_0,\ \mathbf{Q}(0) = D_xu(X_0). \end{cases} \tag{4.10}$$

Since $D_pH, D_xH$ are Lipschitz in $p, x$ the uniqueness of solutios to the system (4.10) of ordinary differential equations in $L^2(\Omega)$ yields $\mathbf{X}(t) = \mathbf{Y}(t)$ and $\mathbf{P}(t) = \mathbf{Q}(t) = \mathbf{p}(X_0,t) = Du(\mathbf{Y}(t),t)$ for all $t \in [0,T]$, thus

$$\begin{cases} \dot{\mathbf{X}}(t) = -D_pH(Du(\mathbf{X}(t),t),\mathbf{X}(t),\mathbf{X}(t),\dot{\mathbf{X}}(t)) \\ \mathbf{X}(0) = X_0. \end{cases}$$

$\square$

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