Известия НАН Армении. Математика, том 48, н. 2, 2013, стр. 43-50.

ON AN EFFECTIVE SOLUTION OF THE RIEMANN PROBLEM FOR THIRD ORDER IMPROPERLY ELLIPTIC EQUATIONS

S. M. ALI RAEISIAN

Institute of Mathematics, Armenian National Academy of Sciences

E-mail: *s_ma_raissian@yahoo.com*

Abstract. The paper considers a boundary value problem for a third order improperly elliptic equation. A numerical method that reduces this problem into six uniquely solvable problems is developed, and the finite differences method is applied to solve the resulting problems.

MSC2010 numbers: 35G45, 35G15, 35J25, 35J57, 65N06, 65N20.

Keywords: improperly elliptic equation; boundary value problem; Riemann-Dirichlet type problem; finite differences method.

1. Introduction

Let $D = \{(x, y) : \ a < x < b, \ c < y < dd\}$ be a rectangle in a complex plane with boundary $\Gamma = \partial D$. In this paper we consider the elliptic equation

$$\frac{\partial^3}{\partial \overline{z}^2 \partial z} u(x, y) := \frac{1}{8} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)^2 \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) u(x, y) = 0, \quad (x, y) \in D. \tag{1.1}$$

This equation is improperly elliptic because the characteristic equation has two roots in the upper half-plane and one root in the lower half-plane. So, the classical boundary value problems are not correct for this equation, and hence, the boundary conditions are taken in the following form (see [1]):

$$u\big|_\Gamma = f_0(x, y), \quad (x, y) \in \Gamma, \tag{1.2}$$

$$\Re \left(\frac{\partial u}{\partial N} \right) \Big|_\Gamma = f_1(x, y), \quad (x, y) \in \Gamma, \tag{1.3}$$

where $f_0 \in C^{(1,\alpha)}(\Gamma)$ and $f_1 \in C^{(\alpha)}(\Gamma)$. Here $C^{(\alpha)}(\Gamma)$ (respectively, $C^{(1,\alpha)}(\Gamma)$) stands for the class of those functions defined on Γ, which satisfy Hölder condition with exponent α (respectively, the first order derivatives satisfy Hölder condition with exponent α).

We are looking for a solution of the problem (1.1) – (1.3) in the class of functions $C^{(3)}(D) \cap C^{(1,\alpha)}(D \cup \Gamma)$.

The general form of the problem (1.1) – (1.3), known as Riemann-Dirichlet type problem for improperly elliptic equations, was considered by Tovmasyan in [1]. More precisely, in [1] it was considered the following equation:

$$\frac{\partial^n}{\partial\overline{z}^p\partial z^q}u(x,y) \equiv \frac{1}{2^n}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^p\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^q u(x,y)=0, \quad (x,y)\in D_1, \tag{1.4}$$

$(p \geq q)$ with boundary conditions

$$\left.\frac{\partial^k u}{\partial N^k}\right|_\Gamma = f_k, \quad k=0,1,...,q-1, \tag{1.5}$$

$$\left.\Re\left(\frac{\partial^k u}{\partial N^k}\right)\right|_\Gamma = f_k, \quad k=q,...,p-1, \tag{1.6}$$

where $D_1=\{(x,y):\ x^2+y^2<1\}$ is the unit disk in a complex plane with boundary $\Gamma=\partial D_1$. In [1] the solvability of non-homogeneous problem and the general solution of the corresponding homogeneous problem was found, and it was shown that the number of linearly independent solutions of the homogeneous problem is $(p-q)^2$. So the number of corresponding linearly independent solutions of the homogeneous problem in our case is equal to one.

Observe that direct application of the finite differences method is effective for solving boundary value problems for elliptic equations with real coefficients (see, for example, [2], [3]), which is not the case for complex coefficients. In [4] it was shown that direct application of the finite differences method can be considered as an effective method of solution for first order improperly and second order properly elliptic equations. However, this is not the case for the problem (1.1) – (1.3), due to complex plane and non uniqueness of the solution of (1.1) – (1.3).

The purpose of this paper is to develop an effective numerical method of solution of the problem (1.1) – (1.3): we reduce this problem into six uniquely solvable problems, and then apply the finite differences method to solve the resulting problems.

2. Description of the algorithm

The general solution of the equation (1.1) can be represented in the form

$$u(x,y)=\overline{z}\Phi_1(z)+\Phi_0(z)+\Omega_1(\overline{z}), \quad z=x+iy, \quad (x,y)\in D, \tag{2.1}$$

where Φ_0, Φ_1 are arbitrary analytic functions in D, and Ω_1 is analytic in $\overline{D}=\{\overline{z}:z\in D\}$. We can write (2.1) in terms of three analytic functions Φ, Ψ and Ω as follows:

$$\begin{aligned} u(x,y) &= \overline{z}\Phi_1(z)+\Phi_0(z)+\overline{\Omega_1(\overline{z})}+\left(\Omega_1(\overline{z})-\overline{\Omega_1(\overline{z})}\right) \\ &= \overline{z}\Phi(z)+\Psi(z)-i\Im(\Omega(z)), \end{aligned} \tag{2.2}$$

where $\Omega(z) = \frac{1}{2}\overline{\Omega_1(\overline{z})}$. So we have to determine the analytic in D functions Φ, Ψ and Ω by the conditions (1.2), (1.3).

From (1.1) we have

$$\frac{\partial^3}{\partial z^2 \partial \overline{z}} \overline{u} = 0.$$

Assuming $U = \Re(u) = \frac{1}{2}(u + \overline{u})$ and using (1.1) – (1.3), it is easy to see that U satisfies the problem

$$\begin{aligned} &\Delta^2 U = 0, \\ &U\big|_\Gamma = \Re(f_0), \\ &\left(\frac{\partial U}{\partial N}\right)\bigg|_\Gamma = f_1. \end{aligned} \tag{2.3}$$

This is a Dirichlet problem for biharmonic equation which is uniquely solvable (see [5]).

It follows from (2.2) that

$$\begin{aligned} U &= \Re(u) = \Re\left(\overline{z}\Phi(z) + \Psi(z) + i\Im(\Omega(z))\right) \\ &= \Re\left(\overline{z}\Phi(z) + \Psi(z)\right) = \overline{z}\Phi(z) + \Psi(z) + z\overline{\Phi(z)} + \overline{\Psi(z)}. \end{aligned} \tag{2.4}$$

So, applying Laplace operator we get

$$\Delta U = \Re(\Phi'(z)).$$

Since $\Phi(z)$ is an analytic function, we can use the Cauchy-Riemann condition

$$\Delta U = \frac{\partial}{\partial x}(\Re(\Phi(z))) = \frac{\partial}{\partial y}(\Im(\Phi(z))). \tag{2.5}$$

We represent the analytic function $\Phi(z)$ as

$$\Phi(z) = u_1 + iv_1, \tag{2.6}$$

where

$$u_1 = \Re(\Phi(z)), \quad v_1 = \Im(\Phi(z)).$$

From the first part of equation (2.5) and the solution of problem (2.3) we can set a Poincare problem for u_1:

$$\begin{aligned} &\Delta u_1 = 0, \\ &\left(\frac{\partial u_1}{\partial x}\right)\bigg|_\Gamma = \Delta U\big|_\Gamma = F_1, \end{aligned} \tag{2.7}$$

where F_1 is assumed to be known on the boundary.

In order to find some fixed (unique) solution u_1^0 for (2.7), we add two conditions:

$$u_1(x_0, y_0) = u_{10}, \quad u_1(x_1, y_1) = u_{11}, \tag{2.8}$$

where u_{10}, u_{11} are arbitrary constants and $(x_0, y_0) \in D$, $(x_1, y_1) \in D$.

Similarly, we can set a Poincare problem for v_1:

$$\Delta v_1 = 0,$$
$$\left(\frac{\partial v_1}{\partial y}\right)\bigg|_\Gamma = \Delta U\big|_\Gamma = F_2, \tag{2.9}$$

with two additional conditions $v_1(x_0, y_0) = v_{10}$, $v_1(x_1, y_1) = v_{11}$, where v_{10}, v_{11} are arbitrary constants and F_2 is assumed to be known on the boundary. So we can find some fixed (unique) solution v_1^0 for (2.9).

Then the general solution of (2.7), (2.9) can be written in the form

$$u_1 = u_1^0 + c_1 + c_2 y, \quad v_1 = v_1^0 + c_3 + c_4 x, \tag{2.10}$$

where c_i, $i = 1, ..., 4$ are arbitrary real constants.

From Cauchy-Riemann conditions for u_1, v_1:

$$(u_1)_y = -(v_1)_x$$

we get $c_2 = -c_4$. Substituting (2.10) into (2.6), we obtain

$$\Phi(z) = u_1^0 + iv_1^0 + (c_1 + ic_3) - ic_2(x + iy) = \Phi^0 + A - ic_2 z, \tag{2.11}$$

where A and c_2 are arbitrary complex and real constants, respectively, and $\Phi^0 = u_1^0 + iv_1^0$, which is determined uniquely.

From (2.4) and (2.11) we have

$$U = \Re(u) = \Re\left(\overline{z}\Phi^0(z) + A\overline{z} - ic_2 z\overline{z} + \Psi(z)\right).$$

So

$$\Re\left(\Psi(z) + \overline{A}z\right) = U - \Re\left(\overline{z}\Phi^0(z)\right). \tag{2.12}$$

It follows from the solution of (2.3), (2.7) and (2.9) that the right-hand side of (2.12) is known on the boundary points Γ, while $\Psi(z) + \overline{A}z$ is an analytic function. Hence we have $\Psi(z) + \overline{A}z = \Psi^0(z) + ic_5$, or equivalently,

$$\Psi(z) = \Psi^0(z) - \overline{A}z + ic_5, \tag{2.13}$$

where c_5 is an arbitrary real constant, and $\Psi^0(z)$ is determined uniquely. From (2.2), (2.11) and (2.13) we obtain

$$\begin{aligned} u &= \overline{z}\left(\Phi^0(z) + A - ic_2 z\right) + \left(\Psi^0(z) - \overline{A}z + ic_5\right) + i\Im(\Omega(z)) \\ &= \overline{z}\Phi^0(z) + \Psi^0(z) - ic_2 z\overline{z} + A\overline{z} - \overline{A}z + ic_5 + i\Im(\Omega(z)) \\ &= \overline{z}\Phi^0(z) + \Psi^0(z) - ic_2 z\overline{z} + 2i(c_3 x - c_1 y) + ic_5 + i\Im(\Omega(z)) \\ &= w^0(z) + i\big(\Im(\Omega(z)) + (c_3 x - c_1 y + c_5) - c_2 z\overline{z}\big). \end{aligned}$$

Therefore

$$u = w^0(z) + i\big(H(x,y) - c_2 z\overline{z}\big), \tag{2.14}$$

where the function $w^0(z) = \overline{z}\Phi^0(z) + \Psi^0(z)$ is determined uniquely, and

$$H(x,y) = \Im(\Omega(z)) + (c_3 x - c_1 y + c_5)$$

is a harmonic function.

Next, we determine the harmonic function H. From (1.2) we have

$$\Im(u)\big|_\Gamma = \Im(f_0)\big|_\Gamma = \big(\Im(w^0(z)) + H(x,y) - c_2 z\overline{z}\big)\big|_\Gamma.$$

Therefore,

$$(H(x,y))\big|_\Gamma = \Im(f_0 - w^0(z)) + c_2\,(z\overline{z})\big|_\Gamma. \tag{2.15}$$

So, first we find the unique solution of a Dirichlet problem for Laplace equation:

$$\begin{aligned} &\Delta S(x,y) = 0, \quad (x,y) \in D, \\ &S\big|_\Gamma = z\overline{z}\big|_\Gamma\,. \end{aligned} \tag{2.16}$$

Next, from (2.15) we get

$$H\big|_\Gamma = (H_0 + c_2 S)\big|_\Gamma,$$

where H_0 is the unique solution of

$$\begin{aligned} &\Delta H_0 = 0, \\ &H_0\big|_\Gamma = \Im(f_0 - w^0)\big|_\Gamma. \end{aligned} \tag{2.17}$$

From (2.15) – (2.17) we have

$$H = H_0 + c_2 S \quad \text{(because } H, S \text{ are harmonic).}$$

Therefore, we can write (2.14) as

$$u = w^0 + i\big(H_0 + c_2 S - c_2 z\overline{z}\big) = w^0 + iH_0 + ic_2\,(S - z\overline{z})\,,$$

where c_2 is an arbitrary real constant.

Thus, our problem is solvable and the corresponding homogeneous problem has one linearly independent solution $u_0 = i\,(S - z\overline{z})$, where S is a harmonic function satisfying the boundary condition $S\big|_\Gamma = z\overline{z}|_\Gamma$.

3. Numerical solution

In this section we develop an effective numerical method of solution of the problem (1.1) – (1.3).

Notice that numerical methods of solution of various boundary value problems (Dirichlet, Neumann, etc.) for biharmonic and biharmonic type equations were considered in the literature (see, e.g., [6], [7] – [14], and references therein).

First, we consider the problem (2.3). Let us divide the rectangle D by $m+n-2$ straight lines, parallel to coordinate axes and, for simplicity consider equidistant nodes, denoted by

$$x_k = a + kh, \quad h = \frac{b-a}{m}, \quad y_j = c + jh,$$
$$h = \frac{d-c}{n}, \quad k = \overline{0,m},\ j = \overline{0,n}.$$

By the finite differences method, we find the approximate values of the function U at the mesh points:

$$U_i^j \approx U(x_i, y_j),\ i = \overline{1,m}, j = \overline{1,n},$$

where $U(x,y) = \Re(u(x,y))$, and $\{U_i^j\} = U_h$.

Then we use the discrete analogue of the Laplace operator:

$$\Delta_h U_h = \frac{1}{h^2}(U_{i+1}^j + U_{i-1}^j + U_i^{j+1} + U_i^{j-1} - 4U_i^j)$$

to discretize the biharmonic equation in (2.3) as follows:

$$\begin{aligned} \Delta_h\Delta_h U_h =& \frac{20}{h^4}U_i^j - \frac{8}{h^4}(U_{i+1}^j + U_{i-1}^j + U_i^{j-1} + U_i^{j+1}) \\ &+ \frac{2}{h^4}(U_{i+1}^{j+1} + U_{i-1}^{j+1} + U_{i+1}^{j-1} + U_{i-1}^{j-1}) \\ &+ \frac{1}{h^4}(U_{i+2}^j + U_{i-2}^j + U_i^{j-2} + U_i^{j+2}), \end{aligned} \tag{3.1}$$
$$U\big|_\Gamma = (\Re(f_0))_h, \quad (\delta_h U_h)\big|_\Gamma = (f_1)_h,$$

where δ_h, $\Re(f_0)_h$ and $(f_1)_h$ stand for the difference analogue of the operator $\frac{\partial}{\partial N}$ and the values of functions $\Re(f_0)$, f_1 at the boundary points of mesh, respectively.

The discrete problem (3.1) approximates problem (2.3) (see, e.g., [15]). Therefore, from the stability of (3.1), we can deduce the convergence of the grid function to $\{U(x_i, y_j)\}$ (see [3], Theorem 2.5).

From the last two equations in (3.1) we can find the values of function U_h at the points (x_i, y_j) for $i = 0, 1, m-1, m; j = \overline{0, n}$ and $i = \overline{0, m}; j = 0, 1, n-1, n$. If the values U_i^j are on the interior nodes, we find them from the linear system with symmetric pentadiagonal matrix. Since this matrix is positive definite, we can prove the stability of (3.1). An algorithm of solution of this system can be found in [17].

For numerical solution of Poincare problem (2.7) with additional condition (2.8), it is enough to solve it in the domain with boundary Γ^*, where

$$\Gamma^* = \{(x_i, y_j) : i = 1, m-1, j = \overline{1, n-1} \quad \text{or} \quad i = \overline{1, m-1},\ j = 1, n-1\}.$$

Without loss of generality, we can assume that the points (x_0, y_0) and (x_1, y_1) from (2.8) are located on the top left and down right corner of the boundary Γ^*, that is, $(x_0, y_0) = (a+h, d-h)$ and $(x_1, y_1) = (b-h, c+h)$. Therefore

$$u_1(x_0, y_0) = u_{10} = 0, \quad u_1(x_1, y_1) = u_{11} = 0.$$

Next, from the grid boundary conditions

$$\frac{1}{h}[(u_1)_{i+1}^j - (u_1)_i^j] = \frac{1}{4}(\Delta_h U_h)_i^j, \quad i = \overline{1, m-1}, y = \overline{1, n-1}$$

we can find the values of the grid points on the sides of Γ^* parallel to x-axis.

Finally, the values of $(u_1)_i^j$ inside Γ^* and on the sides of Γ^* parallel to y-axis can be found from a system of linear equations whose main matrix can be reduced to the tridiagonal form:

$$T = \begin{pmatrix} A & B & 0 & 0 & \cdots & 0 \\ B & A & B & 0 & \cdots & 0 \\ 0 & B & A & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} \frac{-1}{h} & \frac{1}{h} & 0 & \cdots & 0 & 0 \\ \frac{1}{h^2} & \frac{-4}{h^2} & \frac{1}{h^2} & \cdots & 0 & 0 \\ 0 & \frac{1}{h^2} & \frac{-4}{h^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-4}{h^2} & \frac{1}{h^2} \\ 0 & 0 & 0 & \cdots & \frac{-1}{h} & \frac{1}{h} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The matrix T is diagonally dominant, hence the corresponding linear system is uniquely solvable. This implies unique solvability of the considered modified Poincare problem. Applying the maximum principle (see [16]), we get the unique solvability of our system.

Similarly, the Dirichlet problem for Laplace equation (2.16), (2.17) has a unique solution (see [2]).

Список литературы

[1] N. E. Tovmasyan, Non-Regular Differential Equations and Calculations of Electromagnetic Fields, World Scientific (1998).

[2] J. W. Thomas, Numerical Partial Differential Equations Conservation Laws and Elliptic Equations, Springer, New York (1991).

[3] C. Grossmann, H.-G. Roos, M. Stynes, Numerical Treatment of Partial Differential Equations, Springer, Berlin (2007).

[4] Guo Chun Wen, Approximate Methods and Numerical Analysis for Elliptic Complex Equations, Gordon and Breach, Amsterdam (1999).

[5] H. Begehr, “Boundary value problems in complex analysis”, I. Bol.Asoc.Mat.Venez., **12** (1), 65 – 85 (2005), II. Bol.Asoc.Mat.Venez. 12(2), 217-250 (2005).

[6] P. Peisker, “On the numerical solution of the first order biharmonic equation”, Math. Modelling and Numer. Analysis, **22** (4), 655 – 676 (1988).

[7] Q. A. Dang, “Boundary operator method for approximate solution of biharmonic type equation”, Vietnam J. Math., **22** (1-2), 114 – 120 (1994).

[8] Q. A. Dang, “Mixed boundary-domain operator in approximate solution of biharmonic type equation”, Vietnam J. Math., **26** (3), 243 – 252 (1998).

[9] Q. A. Dang, “Iterative method for solving the Neumann boundary value problem for biharmonic type equation”, J. Comp. Appl. Math., **196**, 634 – 643 (2006).

[10] P. Bjorstad, “Fast numerical solution of thew biharmonic Dirichlet problem on rectangles”, SIAM J. on Numerical Analysis, **20**, 59 – 71 (1983).

[11] Shidong Jiang, Bo Ren, P. Tsuji, Lexing Ying, Second kind integral equations for the first kind Dirichlet problem of the biharmonic equation in three dimensions, Elsevier (2010).

[12] I. N. Vekua, New Methods for Solving Elliptic Equations, J.Wiley, New York (1967).

[13] J. Cohen, J. Gosselin, “The Dirichlet problem for the biharmonic equation in a C^1 domain in the plane”, Indiana Univ. Math. J., **32** (5), 635 – 685 (1983).

[14] S. G. Mikhlin, Integral Equations, Pergamon Press, New York (1957).

[15] L. Collatz, The Numerical Treatment of Differential Equations, Springer, New York (1966).

[16] A. A. Samarskii, Theory of Difference Schemes, Marcel Dekker, New York (2001).

[17] M. H. Koulaei, F. Toutounian, “Factored sparse approximate inverse of block tridiagonal and block pentadiagonal matrices”, Applied Math. and Computation, **184**, 223 – 234 (2007).

Поступила 25 августа 2011