Известия НАН Армении. Математика, том 48, н. 1, 2013, стр. 37- 52.

PARALLEL X-RAY TOMOGRAPHY OF CONVEX DOMAINS AS A SEARCH PROBLEM IN TWO DIMENSIONS

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Abstract. In 1961, at A.M.S. Symposium on Convexity, P.C. Hammer proposed the following problem: how many X-ray pictures of a convex planar domain **D** must be taken to permit its exact reconstruction? Richard Gardner writes in his fundamental 2006 book [4] that X-rays in four different directions would do the job. The present paper points at the possibility that in certain asymptotical sense X-rays in only three different directions can be enough for approximate reconstruction of centrally symmetric convex domains. The accuracy of reconstruction would tend to become perfect in the limit, as the directions of the three X-rays change, all three converging to some given direction. The analysis leading to that conclusion is based on two lemmas of Section 1 and *Pleijel type identity for parallel X-rays* derived in Sections 2 and 3. These tools together supply a system of two differential equations with respect to two unknown functions that describe the two *branches* of the domain boundary **D**. The system is easily resolved. The solution intended to provide a complete tomography reconstruction of **D**, happens however to depend on a two dimensional parameter, whose "real value"remains unknown. So tomography reconstruction of **D** becomes possible if a satisfactory approximation to that unknown “real value” can be found. In the last section a test procedure for the individual candidates for “approximate real value” of the parameter is described. A uniqueness theorem concerning tomography of circular discs is proved.

MSC2010 numbers: 52A22; 53C65; 60D05;
Keywords: X-ray; tomography; convex planar domain; reconstruction.

1. Two lemmas

In the present paper, $\mathcal{D}$ denotes the class of bounded convex domains **D** with continuously differentiable boundary $\partial\mathbf{D}$ that possesses no linear segments. The space $\mathcal{C}$ of planar directions we identify with $(0, 2\pi)$ converted to a circle, and let $\alpha \in \mathcal{C}$ be a reference direction. In $\mathcal{C}$ we consider the usual angular coordinate $\varepsilon \in (0, 2\pi)$ assuming that ε is measured clockwise from direction α ($\varepsilon = 0$ coincides with direction α).

Let X_ε be the axis of direction ε that contains the origin O in the plane. For each direction $\varepsilon \in \mathcal{C}$ we consider the right system of Cartesian coordinates $(x_\varepsilon, y_\varepsilon)$ with x-axis coinciding with X_ε. We will use both notation: $X_\alpha = X_0$.

By $[\mathbf{D}]_\alpha$ we denote the space of chords of $\mathbf{D} \in \mathcal{D}$ that are perpendicular to the direction α. The point $u \in X_\alpha$ where the linear continuation of the chord $\chi \in [\mathbf{D}]_\alpha$ hits X_α we call the base of $\chi \in [\mathbf{D}]_\alpha$. We denote

$\chi_u =$ the chord from $[\mathbf{D}_\alpha]$ whose base is $u \in X_\alpha$, $|\chi_u| =$ the length of χ_u,

$p_\alpha =$ (open) perpendicular projection of $\mathbf{D}$ on X_α (the range of u).

$\chi_M =$ the longest chord in $[\mathbf{D}_\alpha]$,

$u_M =$ the base of χ_M, $u_M \in X_\alpha$,

$L =$ the left part of p_α separated by u_M,

$R =$ the right part of p_α separated by u_M.

Lemma 1.1. *For any $\mathbf{D} \in \mathcal{D}$ and any choice of α, the longest chord χ_M is unique in $[\mathbf{D}]_\alpha$, while the length function $|\chi_u|$ is strictly monotone increasing on L and strictly monotone decreasing on R.*

A satisfactory proof of Lemma 1.1 can be obtained easily by considering the graphs of the continuous functions $t_1(u)$ and $t_2(u)$ each defined in the interior of p_α:

$t_i(u) =$ tan of the angle between α and the lines tangent to the boundary $\partial\mathbf{D}$ at the upper $(i = 2)$ or lower $(i = 1)$ endpoints of $\chi(u)$. The two graphs can have only one intersection point, whose projection on X_a happens to be u_M.

Without loss of generality, we assume that the convex domain $\mathbf{D}$ lies totally in the half-plane $y_\alpha > 0$, i.e. in the left half-plane bounded by X_α. Thus we can speak about upper and lower endpoint of χ_u for every $u \in p_\alpha$. *Elevation* of a chord $\chi \in [\mathbf{D}]_\alpha$ is defined to be the y_α-value of the lower endpoint of χ.

By Lemma 1.1, the following two "elevation functions"are well defined on the interval $(0, M)$:

$U_L(l) =$ elevation of the chord χ_u that satisfies $|\chi_u| = l$ and $u \in L$,

$U_R(l) =$ the same for $u \in R$.

Lemma 1.2. *For every $\mathbf{D} \in \mathcal{D}$ the identity*

$$U_L(l) - U_R(l) = \frac{d}{d\alpha} \int_0^l \rho_\alpha(\tau)\, d\tau$$

is valid for every $l \in (0, M)$, where $\rho_\alpha(\tau)$ is the distance between two chords from $[\mathbf{D}_\alpha]$ of common length τ.

The remaining part of the section contains a proof of Lemma 1.2.
Let $f(T)$ be a sufficiently smooth function defined for $T > 0$ which vanishes identically in some neighborhood of $T = 0$. We consider the integral

$$J(\varepsilon) = \int_{[\mathbf{D}]_\varepsilon} f(|\chi|)\, dx_\varepsilon = \int_{p_\varepsilon} f(T(x_\varepsilon))\, dx_\varepsilon, \tag{1.1}$$

where

$p_\varepsilon =$ the perpendicular projection of $\mathbf{D}$ on X_ε,

$dx_\varepsilon =$ the Lebesgue measure on X_ε (equivalently, on $[\mathbf{D}]_\varepsilon$)),

$T(x_\varepsilon) := |\chi| =$ the length of the chord $\chi \in [\mathbf{D}]_\varepsilon$ whose base is $x_\varepsilon \in X_\varepsilon$.

The purpose is to calculate the first derivative of $J(\varepsilon)$ at $\varepsilon = 0$.
For given ε let us consider the map that sends X_ε into X_0:

$$x_\varepsilon \to u, \quad u \in X_0, \quad \text{with Jacobian } \frac{dx_\varepsilon}{dx_0} = \cos\varepsilon,$$

where u denotes the point where the line perpendicular to ε and containing x_ε hits X_0. To change the integration variable in (1.1) from x_ε to u we write

$T_\varepsilon(u) =$ the length of the chord $\chi \in [\mathbf{D}]_\varepsilon$ whose base $x_\varepsilon \in X_\varepsilon$ maps into $u \in X_0$,
in particular

$T_0(u) =$ the length of the chord $[\mathbf{D}]_0$ whose base is $u \in X_0$, $[\mathbf{D}]_0 = [\mathbf{D}]_\alpha$.

For every ε from sufficiently small neighborhood of direction 0 the assumption as regards $f(u)$ allows to replace p_ε in (1.1) by $p_0 \subset X_0$, therefore for $\varepsilon \to 0$

$$J(\varepsilon) = \cos\varepsilon \int_{p_0} f(T_\varepsilon(u))\, du = \int_{p_0} f(T_\varepsilon(u))\, du + o(\varepsilon^2) \tag{1.2}$$

(for simplicity we write du instead of dx_0). In (1.2) the integration domain does not depend on ε, so the derivative of $J(\varepsilon)$ at $\varepsilon = 0$ happens to be

$$\frac{d}{d\varepsilon} J(0) = \int_{p_0} \frac{df(T_0(u))}{d\varepsilon}\, du. \tag{1.3}$$

Let $g_\varepsilon(u)$ be the axis perpendicular to direction ε that contains $u \in p_0$. In the Cartesian system that correspond to $\varepsilon = 0$, the coordinates of the two ($i = 1, 2$) points where $g_\varepsilon(u)$ meets $\partial\mathbf{D}$ let be

$$x_{0i} = x_{0i}(\varepsilon, u), \quad y_{0i} = y_{0i}(\varepsilon, u), \quad \text{and}$$

$$r_i = r_i(u, \varepsilon) = \text{ the distance from } u \text{ to } (x_{0i}, y_{0i}),\ r_2 \geq r_1 \geq 0.$$

By $T_\varepsilon(u) = r_2(u, \varepsilon) - r_1(u, \varepsilon)$ we have

$$\frac{dT_\varepsilon(u)}{d\varepsilon} = \frac{d(r_2 - r_1)}{d\varepsilon} = \frac{\partial r_2}{dx_{02}} \frac{dx_{02}}{d\varepsilon} - \frac{\partial r_1}{dx_{01}} \frac{dx_1}{d\varepsilon}.$$

We note that for $\varepsilon = 0$ and $i = 1, 2$

$$\frac{\partial r_i(u,0)}{dx_{i0}} = \frac{\partial Y_i(u)}{du}, \qquad \frac{dx_{0i}(0,u)}{d\varepsilon} = Y_i(u),$$

where by definition

$$Y_1(u) = r_1(u,0) \quad \text{and} \quad Y_2(u) = r_2(u,0).$$

It is important, that the graphs of the functions $Y_1(u)$ and $Y_2(u)$, both defined on p_0 and satisfying $Y_1(u) \leq Y_2(u)$, represent the two *branches* of $\partial\mathbf{D}$ that project on p_0. Now (1.3) takes the form

$$\frac{d}{d\varepsilon} J(0) = \int_{p_0} f'(T_0(u))\, W(u)\, du, \tag{1.4}$$

where

$$W(u) = Y_2(u)\, \frac{dY_2(u)}{du} - Y_1(u)\, \frac{dY_1(u)}{du}.$$

A standard δ - formalism permits to calculate the integral in (1.4) if for some $\tau > 0$ we choose

$$f(z) = h_\tau(z) \text{ where } h_\tau(z) = 0 \text{ for } z < \tau, \text{ and } h_\tau(z) = 0 \text{ otherwise,} \tag{1.5}$$

in which case $h'_\tau(z) = \delta_\tau(z)$. For that choice

$$J(\varepsilon) = \rho_\alpha(\tau) = \text{the distance between } \chi_1 \text{ and } \chi_2,$$

where χ_1 and χ_2 are the two parallel chords of $\mathbf{D}$, both of length τ and perpendicular to direction ε. We have

$$\frac{\partial}{\partial\varepsilon} \rho_\alpha(\tau) = \int_{p_0} \delta_\tau(T_0(u))\, W(u)\, du. \tag{1.6}$$

Therefore

$$\int_{p_0} \delta_\tau(T_0(u))\, W(u)\, du = \int_L \delta_\tau(T_0(u))\, W(u)\, du + \int_R \delta_\tau(T_0(u))\, W(u)\, du =$$

$$\int_0^M \delta_\tau(T)\, W(x)\, \frac{du_L(T)}{dT}\, dT + \int_M^0 \delta_\tau(T)\, W(x)\, \frac{du_R(T)}{dT}\, dT,$$

where for $T > 0$

$u_L(T) =$ the point from L for which $T_0(u_L(T)) = T$,

$u_R(T) =$ the point from R for which $T_0(u_R(T)) = T$.

Thus

$$\int_{p_0} \delta_\tau(T_0(u))\, W(u)\, du = W(u_L(\tau))\, \frac{du_L(\tau)}{d\tau} - W(u_R(\tau))\, \frac{du_R(\tau)}{d\tau}. \tag{1.7}$$

From (1.4) we find

$$\frac{dY_i(u_L(\tau))}{du}\, \frac{du_L(\tau)}{d\tau} = \frac{dU_{iL}(\tau)}{d\tau},$$

$$\frac{dY_i(u_R(\tau))}{du}\,\frac{du_R(\tau)}{d\tau} \;=\; \frac{dU_{iR}(\tau)}{d\tau},$$

where $i = 1, 2$ while

$$U_{iL}(\tau) \;=\; Y_i(u_L(\tau)) \quad \text{and} \quad U_{iR}(\tau) \;=\; Y_i(u_R(\tau)). \tag{1.8}$$

Taken together with (1.6) this implies

$$\frac{d}{d\varepsilon}\rho_\alpha(\tau) \;=\; \left[U_{2L}(\tau)\,\frac{dU_{2L}(\tau)}{d\tau} \;-\; U_{1L}(\tau)\,\frac{dU_{1L}(\tau)}{d\tau}\right] -$$

$$-\left[U_{2R}(\tau)\,\frac{dU_{2R}(\tau)}{d\tau} \;-\; U_{1R}(\tau)\,\frac{dU_{1R}(\tau)}{d\tau}\right], \tag{1.9}$$

We integrate (1.9) in $d\tau$ from 0 to some $T > 0$. Using formulae like

$$\int_0^T U_{2L}(\tau)\,\frac{dU_{2L}(\tau)}{d\tau}\,d\tau \;=\; \int_0^{U_{2L}(T)} y\,dy \;=\; \frac{1}{2}\,U_{2L}^2(T),$$

we get

$$2\int_0^T \frac{d}{d\varepsilon}\rho_\alpha(\tau)\,d\tau \;=\; U_{2L}^2(T) \;-\; U_{1L}^2(T) \;-\; U_{2R}^2(T) \;+\; U_{1R}^2(T).$$

On the other hand

$$U_{2L}(T) \;=\; U_{1L}(T) \;+\; T, \qquad U_{2R}(T) \;=\; U_{1R}(T) \;+\; T,$$

yielding

$$2\int_0^T \frac{d}{d\varepsilon}\rho_\alpha(\tau)\,d\tau \;=\; [U_{1L}(T) \;+\; T)]^2 - U_{1L}^2(T) - \;[U_{1R}(T) \;+\; T]^2 \;+\; U_{1R}^2(T) \;=$$

$$2U_{1L}(T)\,T \;-\; 2U_{1R}(T)\,T \;=\; 2T\,[U_{1L}(T) \;-\; U_{1R}(T)],$$

or finally

$$U_{1L}(T) \;-\; U_{1R}(T) \;=\; \frac{1}{T}\int_0^T \frac{d}{d\varepsilon}\rho_\alpha(\tau)\,d\tau. \tag{1.10}$$

which coincides with the assertion of the Lemma 1.2.

2. Totally disintegrated Pleijel identity

We will be considering subsets of the space

$\mathbf{G}$ = the space of lines in the plane, the lines $g \in \mathbf{G}$ carry no orientation. We prefer to use the "translational"(ϕ, t) parametrization of lines $g \in \mathbf{G}$:

ϕ = direction of g, $\phi \in (0, \pi)$ converted to a circle and

t = signed distance of the line g from the origin ($t \in (-\infty, \infty)$ can be identified with a translation of g in the direction perpendicular to ϕ). In $\mathbf{G}$ there exists unique (up to a constant factor) measure dg invariant with respect to the Euclidean motions. In the (ϕ, t) parametrization

$$dg = d\phi\, dt, \tag{2.1}$$

where $d\phi$ is the uniform measure in the space of planar directions (identified as usual with the Lebesgue measure on the interval $(0, \pi)$), dt is the Lebesgue measure on $(-\infty, \infty)$.

Let $\mathbf{D}$ be a *strictly convex* bounded domain in the plane $\mathbb{R}^2$ with piecewise-smooth boundary $\partial\mathbf{D}$. From now on

$[\mathbf{D}]$ = the space of linear chords of $\mathbf{D}$, $[\mathbf{D}] \subset \mathbf{G}$,

$g \in [\mathbf{D}]$ = a linear chord of $\mathbf{D}$,

dg = restriction of the measure (2.1) to $[\mathbf{D}]$.

χ = length of the chord g: $\chi = \chi(g) = |g|$) (notation preferred in [1] and [2]).

Clearly the linear chords $g \in [\mathbf{D}]$ inherit the (ϕ, t) parametrization from lines in $\mathbb{G}$. We also denote

$[\mathbf{D}]*$ = the space of oriented linear chords of $\mathbf{D}$,

ν = an element of $[\mathbf{D}]*$, we can speak about the *start* endpoint of ν on $\partial\mathbf{D}$,

$[\nu]$, $[g]$ = the set of chords that hit ν or g (so $[\nu], [g] \subset [\mathbf{D}]$),

$[\nu]*$, $[g]*$ = the set of oriented chords that hit ν or g (i.e. $[\nu]*, [g]* \subset [\mathbf{D}]*$).

The identity (2.2) below essentially presents the combinatorial solution of the Buffon–Sylvester problem for n needles (see [1] and [2], pages 107-109).

We set

$\nu_1, ..., \nu_n$ = a sequence of elements of $[\mathbf{D}]*$, we assume that $n > 1$,

$I_m(\nu) = 1$ if ν is hit by m chords from the collection $\nu_1, ..., \nu_n$, otherwise $I_m(\nu) = 0$ (we say that ν hits a chord g if ν contains exactly one point from the interior of g).

$$A = \bigcap_1^n [\nu_i] \subset [\mathbf{D}]$$

and choose some chord $g_0 \in [\mathbf{D}]$. In the space $[\mathbf{D}]$ we consider the delta–measure δ_0 concentrated on g_0. Assuming that no three endpoints of the chords $\nu_1, ..., \nu_n$ lie on a line and that g_0 avoids all these endpoints, the "four indicator formula" yields

$$2\delta_0(A) = 2\sum_{i=1}^n I_{n-1}(\nu_i)\, \delta_0([\nu_i]) + \sum I_{n-2}(d_i)\, \delta_0([d_i]) - \sum I_{n-2}(s_i)\, \delta_0([s_i]). \tag{2.2}$$

Here each d_i or s_i is a segment joining a pair of endpoints of two different chords, say ν_r and ν_l from the collection $\nu_1, ..., \nu_n$. By definition

d_i if ν_r and ν_l lie in different half-planes with respect to continuation of d_i,

s_i if ν_r and ν_l lie in one half-plane with respect to continuation of s_i.

On the space $([\mathbf{D}]*)^n$ of sequences $\nu_1, ..., \nu_n$, we consider the product measure

$$d\nu_1 ... d\nu_n,$$

where each measure $d\nu_i$ (a measure on $[\mathbf{D}]*$) locally coincides with dg (a measure on $[\mathbf{D}]$).

We integrate (2.2) over $([\mathbf{D}]*)^n$ with respect to that product measure. First

$$2 \int ... \int \delta_0(A)\, d\nu_1 ... d\nu_n = 2 \prod_1^n \int_{[g_0]*} d\nu_i = 2\,(4\chi_0)^n, \tag{2.3}$$

where $\chi_0 = |g_0|$ is the length of the chord g_0. Using symmetry we find

$$2 \int ... \int \sum_{i=1}^n I_{n-1}(\nu_i)\, \delta_0([\nu_i])\, d\nu_1 ... d\nu_n = 2n \int_{[g_0]*} d\nu_1 \int ... \int I_{n-1}(\nu_1)\, d\nu_2 ... d\nu_n =$$

$$= 4n \int_{[g_0]} (4\chi)^{n-1}\, dg. \tag{2.4}$$

In the next integral calculation we use the expression of $d\nu$ in the coordinates

$l =$ the start endpoint of ν, $l \in \partial\mathbf{D}$,

$\psi =$ the angle between ν and the line tangent to $\partial\mathbf{D}$ at l, that is

$$dg = \sin\psi\, d\psi\, dl,$$

where dl is the length measure on $\partial\mathbf{D}$. We find (a similar calculation is contained in [1] as well as in [2], page 155)

$$\int \left[\sum I_{n-2}(d_i)\, \delta_0([d_i]) - \sum I_{n-2}(s_i)\, \delta_0([s_i]) \right] d\nu_1 ... d\nu_n =$$

$$= -8n(n-1) \left[\int_{L_1} \int_{L_2} + \int_{L_2} \int_{L_1} \right] (4\chi)^{n-2} \cos\beta_1 \cos\beta_2\, dl_1\, dl_2 =$$

$$= -4n(n-1) \int_{[g_0]} (4\chi)^{n-1} \cot\beta_1 \cot\beta_2\, dg, \tag{2.5}$$

where in the second line

$\chi =$ the length of linear chord between l_1 and l_2 (the chord l_1, l_2),

$L_1, L_2 =$ the parts of $\partial\mathbf{D}$ separated by the endpoints of g_0,

$dl_i =$ length measures on L_i, $i = 1, 2$,

$\beta_1, \beta_2 =$ the angles between the chord l_1, l_2 and $\partial\mathbf{D}$ at its endpoints l_1 and l_2, both taken to lie within $\partial\mathbf{D}$, in the same half-plane with respect to (continuation of) l_1, l_2.

In the third line of (2.5) we used the Jacobian relation (see [2], page 50)

$$dg = \frac{\sin\beta_1 \sin\beta_2}{\chi}\, dl_1 dl_2.$$

Putting together (2.3), (2.4) and (2.5), after deleting the common factor 4^n we obtain what we now call the *totally disintegrated Pleijel identity*

$$\chi_0^n = \frac{n}{2} \int_{[g_0]} \chi^{n-1}\, dg \;-\; \frac{1}{2}\, n\,(n-1) \int_{[g_0]} \chi^{n-1}\, \cot\beta_1\, \cot\beta_2\, dg. \tag{2.6}$$

This identity was used for derivation of "disintegrated iso-perimetric inequality" in [3], in the context of point X-ray theory; as for parallel X-rays, they require one more integration.

3. Integration over a bundle of parallel chords

Given a planar direction $\alpha \in (0, \pi)$, we consider the following subset of $[\mathbf{D}]$:

$[\mathbf{D}]_\alpha$ = the family of parallel chords of $\mathbf{D}$ that have direction perpendicular to α.

The chords from $[\mathbf{D}]_\alpha$ are parameterized solely by the translational parameter t already mentioned above. For simplicity we suppress the explicit mentioning of α in the notation:

dt = one dimensional Lebesgue measure on $[\mathbf{D}]_\alpha$,

g_t = the chord from $[\mathbf{D}]_\alpha$ that corresponds to t,

χ_t = the length of g_t.

We put down (2.6) for a chord $g_t \in [\mathbf{D}]_\alpha$:

$$(\chi_t)^n = \frac{n}{2} \int_{[g_t]} \chi^{n-1}\, dg \;-\; \frac{1}{2}\, n\,(n-1) \int_{[g_t]} \chi^{n-1}\, \cot\beta_1\, \cot\beta_2\, dg. \tag{3.1}$$

This being an identity valid for every t, we integrate it by the measure dt. First we mention that for any $g \in [\mathbf{D}]$ integration of the indicator function $I_{[g_t]}(g)$ yields

$$\int_{[\mathbf{D}]_\alpha} I_{[g_t]}(g)\, dt \;=\; \chi \sin\widehat{\alpha\phi},$$

where $\widehat{\alpha\phi}$ is the angle between the direction α and the direction ϕ of the chord g, while $\chi = |g|$. Hence by an interchange of integration order, for any function $f(g)$ defined on $[\mathbf{D}]$ we get

$$\int_{[\mathbf{D}]_\alpha} dt \int_{[g_t]} f(g)\, dg \;=\; \int_{[\mathbf{D}]} f(g)\, dg \int_{[\mathbf{D}]_\alpha} I_{[g_t]}(g)\, dt \;=\; \int_{[\mathbf{D}]} f(g)\, \chi\, \sin\widehat{\alpha\phi}\, dg.$$

Therefore integrating (3.1) yields

(3.2)
$$\int_{[\mathbf{D}]_\alpha} (\chi_t)^n\, dt \;=\; \frac{n}{2} \int_{[\mathbf{D}]} \chi^n\, \sin\widehat{\alpha\phi}\; dg \;-\; \frac{n(n-1)}{2} \int_{[\mathbf{D}]} \chi^n\, \cot\beta_1\, \cot\beta_2\, \sin\widehat{\alpha\phi}\; dg.$$

By (1.1) the identity (3.2) can be written as

$$\int_{[\mathbf{D}]_\alpha} (\chi_t)^n\, dt \;=\; \frac{n}{2} \int_0^\pi \sin\widehat{\alpha\phi}\; d\phi \int_{[\mathbf{D}]_\phi} (\chi_t)^n\, dt \,- \tag{3.3}$$

$$- \frac{n(n-1)}{2} \int_0^\pi \sin \widehat{\alpha\phi}\, d\phi \int_{[\mathbf{D}]_\phi} (\chi_t)^n \cot\beta_1 \cot\beta_2 \, dt.$$

Clearly the relation (3.3) has a specific form

$$Z(\alpha) = \int_0^\pi z(\phi) \sin \widehat{\alpha\phi}\, d\phi,$$

where $Z(\alpha)$ and $z(\phi)$ are some functions defined on $(0, \pi)$. From the last equation $z(\phi)$ can be found in terms of $Z(\alpha)$ (see [2], pages 30-31): in operator notation,

$$z(\phi) = \mathcal{A}\{Z(\alpha)\},$$

the operator $\mathcal{A}$ being

$$\mathcal{A}\{Z(\alpha)\} = \frac{1}{2}\left[Z(\phi) + \frac{d^2}{d\phi^2} Z(\phi)\right],$$

with the second derivative of $Z(\alpha)$ defined on the basis of the $(0, \pi)$ model of the space of planar directions. In the case (3.3) we have

$$Z(\alpha) = \int_{[\mathbf{D}]_\alpha} (\chi_t)^n \, dt$$

and

$$z(\phi) = \frac{n}{2} \int_{[\mathbf{D}]_\phi} (\chi_t)^n \, dt - \frac{n(n-1)}{2} \int_{[\mathbf{D}]_\phi} (\chi_t)^n.$$

If the boundary $\partial\mathbf{D}$ is sufficiently smooth, then the operator $\mathcal{A}$ is well defined: a sufficient condition is that the tangent direction should change continuously along $\partial\mathbf{D}$, and this property we will assume below. So we get

$$\int_{[\mathbf{D}]_\phi} (\chi_t)^n \, dt + \frac{d^2}{d\phi^2} \int_{[\mathbf{D}]_\phi} (\chi_t)^n \, dt =$$

$$n \int_{[\mathbf{D}]_\phi} (\chi_t)^n \, dt - n(n-1) \int_{[\mathbf{D}]_\phi} (\chi_t)^n \cot\beta_1 \cot\beta_2 \, dt.$$

Thus for every integer $n > 1$ and every direction α (to replace ϕ) we come to what we call the *Pleijel type identity for parallel X-rays*

$$\int_{[\mathbf{D}]_\alpha} (\chi_t)^n \, dt - \frac{1}{n-1} \frac{d^2}{d\alpha^2} \int_{[\mathbf{D}]_\alpha} (\chi_t)^n \, dt = n \int_{[\mathbf{D}]_\alpha} (\chi_t)^n \cot\beta_1 \cot\beta_2 \, dt.$$

For sufficiently broad class of functions $f(x)$ it implies

$$\int_{[\mathbf{D}]_\alpha} f(\chi_t)\, dt - \frac{d^2}{d\alpha^2} \int_{[\mathbf{D}]_\alpha} F(\chi_t)\, dt = \int_{[\mathbf{D}]_\alpha} f'(\chi_t)\, \chi_t \cot\beta_1 \cot\beta_2 \, dt, \tag{3.4}$$

where

$$F(x) = x \int_0^x \frac{f(u)}{u^2} \, du.$$

The identity (3.4) remains valid for certain "generalized functions"as well. First we choose the function $f(u)$ in (3.4) to be

$$f(u) = h_T(u), \tag{3.5}$$

where for some $T > 0$

$$h_T(u) = 0 \quad \text{if} \quad u < T, \quad \text{and} \quad h_T(u) = 1 \quad \text{if} \quad u > T.$$

Then by standard formalism

$$f'(u) = \delta_T(u) = \text{ the usual delta-function concentrated on } T$$

and

$$F(x) = x \int_0^x \frac{h_T(u)\, du}{u^2} = x\, h_T(x) \int_T^x \frac{du}{u^2},$$

i.e.

$$F(x) = \frac{x - T}{T} h_T(x).$$

Hence in case of (3.5) the identity (3.4) takes the form

(3.6)

$$\int_{[\mathbf{D}]_\alpha} h_T(\chi_t)\, dt - \frac{1}{T} \frac{d^2}{d\alpha^2} \int_{[\mathbf{D}]_\alpha} (\chi_t - T)\, h_T(\chi_t)\, dt = \int_{[\mathbf{D}]_\alpha} \delta_T(\chi_t)\, \chi_t \cot\beta_1 \cot\beta_2\, dt.$$

We consider the functions

$\rho_\alpha(T) =$ the distance between the two parallel chords from $[\mathbf{D}]_\alpha$ both of length T,

$H_\alpha(T) =$ the area of the part of $\mathbf{D}$ between the two parallel chords of length T as above minus the rectangular area $T\, \rho_\alpha(T)$. We call $H_\alpha(T)$ the "area function". Clearly

$$\int_{[\mathbf{D}]_\alpha} h_T(\chi_t)\, dt = \rho_\alpha(T) \quad \text{and} \quad \int_{[\mathbf{D}]_\alpha} (\chi_t - T)\, h_T(\chi_t)\, dt = H_\alpha(T),$$

so (3.6) rewrites as

$$\rho_\alpha(T) - \frac{1}{T} \frac{d^2}{d\alpha^2} H_\alpha(T) = \int_{[\mathbf{D}]_\alpha} \delta_T(\chi_t)\, \chi_t \cot\beta_1 \cot\beta_2\, dt. \tag{3.7}$$

With this Pleijel-type identity we work in the next section.

4. The differential equations

After writing the right-hand side of (3.7) as

$$\int_L \delta_T(|\chi|)\, |\chi| \cot\beta_1 \cot\beta_2\, dt + \int_R \delta_T(|\chi|)\, |\chi| \cot\beta_1 \cot\beta_2\, dt,$$

where L and R are as in Lemma 1 above, in each of the two integrals we make an integration variable change, choosing the chord length $|\chi|$ to replace t. So by the usual δ -formalism (compare with (1.7))

$$\int_{[\mathbf{D}]_\alpha} \delta_T(|\chi|)\,|\chi|\,\cot\beta_1\,\cot\beta_2\,du\;=$$

$$T\,\cot\beta_1^{(L)}\,\cot\beta_2^{(L)}\,\frac{d\,u_L(T)}{dT}\;-\;T\,\cot\beta_1^{(R)}\,\cot\beta_2^{(R)}\,\frac{d\,u_R(T)}{dT}, \tag{4.1}$$

where $u_L(T)$ and $u_R(T)$ are as in Section 1, while the angles $\beta_i^{(L)}, \beta_i^{(R)}$, $i = 1, 2$, correspond to the (unique) chord of length T based in L or R correspondingly.

Preparing the definition below, we make a convention that the vertices of the angles $\beta_i^{(L)}$ and $\beta_i^{(R)}$, $i = 1, 2$, lie on the graph of the corresponding function $Y_i(u)$. While $Y_1(u)$ and $Y_2(u)$ are the two *branches* of $\mathbf{D}$ defined in Section 1, we use the standard writing

$$\frac{d\,Y_i(u_L(T))}{du}\;=\;\text{the value of }\frac{d\,Y_i(u))}{du}\text{ at the point }u\;=\;u_L(T)$$

Another additional convention is that both $\beta_1^{(L)}$ and $\beta_2^{(R)}$ lie to the left of the corresponding chord of length T.

Definition 4.1. *For $i = 1, 2$ we define the functions $s_i(u)$ by the relations*

$$\cot\beta_i^{(L)}\;=\;s_i(u_L(T))\,\frac{d\,Y_i(u_L(T))}{du}\quad\text{and}\quad\cot\beta_i^{(R)}\;=\;s_i(u_R(T))\,\frac{d\,Y_i(u_R(T))}{du}$$

and put

$$\sigma(u)\;=\;s_1(u)\,s_2(u).$$

The standard geometrical interpretation of a derivative implies

$$\sigma(u)\textit{ attains only values }+1\textit{ or }-1.$$

From the definition of the class $\mathcal{D}$ we conclude that for values of u sufficiently close to the ends of the corresponding p_α necessarily

$$\sigma(u)\;=\;-1.$$

The jumps of $\sigma(u)$ occur exactly at two points u_1 and u_2 uniquely determined by the equations

$$\frac{d\,Y_i(u_i)}{du}\;=\;0,\quad i\;=\;1,2$$

(this corresponds to the idea of exactly two lines of direction α tangent to $\mathbf{D} \in \mathcal{D}$). So the following lemma is valid.

Lemma 4.1. *For any* $\mathbf{D} \in \mathcal{D}$ *and any direction* α, *the function* $\sigma(u)$ *attains the value* $+1$ *within the interval* $(u_1, u_2) \subset p_\alpha$ *and the value* -1 *in the interior of the remaining part of* p_α.

Writing as appropriate

$$u_L = u_L(T) \quad \text{and} \quad u_R = u_R(T),$$

as well as for every $T \in (0, M)$

$$\Psi_L = \frac{d\,Y_1(u_L)}{du} \quad \text{and} \quad \Psi_R = \frac{d\,Y_1(u_R)}{du},$$

we put (4.1) as

$$\rho_\alpha(T) - \frac{1}{T}\frac{d^2}{d\alpha^2}H_\alpha(T) =$$

$$\sigma(u_L)\,T\,\Psi_L\,\frac{d\,Y_2(u_L)}{du}\,\frac{d\,u_L(T)}{dT} - \sigma(u_R)\,T\,\Psi_R\,\frac{d\,Y_2(u_R)}{du}\,\frac{d\,u_R(T)}{dT}.$$

We also have additional equations valid for every $T \in (0, M)$:

$$\frac{d\,Y_2(u_L)}{du} = \Psi_L + \frac{d\,T_0(u_L)}{du} \quad \text{and} \quad \frac{d\,Y_2(u_R)}{du} = \Psi_R + \frac{d\,T_0(u_R)}{du}, \tag{4.2}$$

that follow from $Y_2(u) = Y_1(u) + T_0(u)$. This reduces the previous equation to

$$\rho_\alpha(T) - \frac{1}{T}\frac{d^2}{d\alpha^2}H_\alpha(T) = \sigma(u_L)\,T\,\frac{d\,u_L(T)}{dT}\left[\Psi_L^2 + \Psi_L\,\frac{d\,T_0(u_L)}{du}\right] - \tag{4.3}$$

$$- \sigma(u_R)\,T\,\frac{d\,u_R(T)}{dT}\left[\Psi_R^2\,\frac{d\,u_R(T)}{dT} + \Psi_R\,\frac{d\,T_0(u_R)}{du}\right].$$

Our purpose is to find Ψ_L and Ψ_R. This will be possible if to (4.3) we add an additional equation for Ψ_L and Ψ_R. We turn to the equation (1.10) and differentiate:

$$\frac{d}{dT}U_{1L}(T) = \frac{d}{dT}U_{1R}(T) + Q(T) \quad \text{with} \quad Q(T) = \frac{d}{dT}\left[\frac{1}{T}\int_0^T \frac{d}{d\varepsilon}\rho_\alpha(\tau)\,d\tau\right]. \tag{4.4}$$

Since $U_{1L}(T) = Y_1(u_L)$ identically (compare with (1.8)), we find

$$\frac{dU_{1L}(T)}{dT} = \frac{dY_1(u_L)}{du}\,\frac{du_L}{dT} = \Psi_L\,\frac{du_L}{dT}.$$

This relation remains valid if we replace L by R. Thus the additional equation happens to be

$$\Psi_L\,\frac{du_L}{dT} = \Psi_R\,\frac{du_R}{dT} + Q(T) \quad \text{or} \quad \Psi_L = \Psi_R\,\frac{du_R}{du_L} + Q(T)\,\frac{dT}{du_L}. \tag{4.5}$$

Substitution of (4.5) into (4.3) yields the following lemma.

Lemma 4.2. *Given* $\mathbf{D} \in \mathcal{D}$, *for* $T \in (0, M)$ *the function*

$$\frac{d\,Y_1(u_R)}{du} = \Psi_R$$

satisfies the quadratic equation

$$A\,\Psi_R^2 + B\,\Psi_R + C = 0, \tag{4.6}$$

with coefficients

$$A = \sigma(u_L)\,T\left[\frac{du_R}{du_L}\right]^2 \frac{d\,u_L(T)}{dT} - \sigma(u_R)\,T\,\frac{d\,u_R(T)}{dT},$$

$$B = \sigma(u_L)\,T\,\frac{du_R}{du_L}\,[2\,Q(T) + 1] - \sigma(u_R)\,T,$$

$$C = \sigma(u_L)\,T\,Q^2(T)\,\frac{dT}{du_L} - \left[\rho_\alpha(T) - \frac{1}{T}\frac{d^2}{d\alpha^2}H_\alpha(T)\right],$$

where

$$u_L = u_L(T),\ u_R = u_R(T),\quad \frac{du_R}{du_L}$$

have been defined as functions of T. *After* Ψ_R *is found from (4.6), we can find*

$$\frac{d\,Y_1(u_L)}{du} = \Psi_R\,\frac{du_R}{du_L} + Q(T)\,\frac{dT}{du_L},$$

where

$$Q(T) = \frac{d}{dT}\left[\frac{1}{T}\int_0^T \frac{d}{d\varepsilon}\rho_\alpha(\tau)\,d\tau\right].$$

5. The problem of tomographic reconstruction of D

In mathematical tomography [4] the function $T_\alpha(u) = T_0(u)$ is called an X-ray of $\mathbf{D}$ perpendicular to the direction α. The quantities that can be calculated on the basis of a given X-ray $T_\alpha(u)$ for some single direction α can be called *single ray*. Thus the functions

$$u_L(T),\ u(T),\quad \frac{d\,u_L(T)}{dT},\quad \frac{d\,u_R(T)}{dT},\quad \frac{du_R}{du_L},\quad \frac{dT(u_L)}{du},\quad \frac{dT(u_R)}{du},\quad \rho_\alpha(T),\quad H_\alpha(T) \tag{5.1}$$

that appear in the expressions for A, B, C in Lemma 4 are single ray. On the other hand, the functions

$$Q(T) \quad \text{and} \quad \frac{d^2}{d\alpha^2}H_\alpha(T) \tag{5.2}$$

found in the same expressions are not single ray. From the presence of the directional derivative in the expression for $Q(T)$, we conclude that two X-rays $T_\phi(u)$ in directions close to α would be enough for approximate evaluation of that quantity. That evaluation would tend to become exact in the limit, as the directions of the two X-rays change, both converging to direction α. By the presence of the second directional derivative, evaluation of the second quantity in (5.2) would require knowledge of $T_\phi(u)$ for three values ϕ close enough to α. This evaluation too is of the asymptotically exact nature, similar to the mentioned property of the evaluation of $Q(T)$. These remarks suggest an approach to tomographic reconstruction of $\mathbf{D} \in \mathcal{D}$ postulating that the quantities in (5.1) and (5.2) are "known". However, beyond (5.1)–(5.2) there are discontinuous quantities

$$\nu_L(T) = \sigma(u_L) \quad \text{and} \quad \nu_L(T) = \sigma(u_R) \tag{5.3}$$

on which A, B, C depend. There is another discontinuous function $S(T)$ that also attain only values $+1$ and -1: it appears in the expression for Ψ_R found from (2.8) as

$$\Psi_R(A, B, C, S(T)) = \frac{-B + S(T)\sqrt{B^2 - 4AC}}{2A}. \tag{5.4}$$

What about them?

By Lemma 4.1, both $\nu_L(T)$ and $\nu_R(T)$ can have exactly 1 jump in the interval $(0, M)$ (for smaller values of T both equal -1). As for $S(T)$, this function may have jumps only at values of T, for which the square root in (5.4) vanishes. This essentially reduces the problem to finding the jump locations for $\nu_L(T)$ and $\nu_R(T)$.

The class of functions $\nu(T)$ defined on the interval $(0, M)$ that attain only values -1 (for smaller values of T) and 1 and possess exactly one points of discontinuity we identify with the interval $(0, M)$. The space of pairs $[\nu_1(T), \nu_2(T)]$ we identify with the square $(0, M) \times (0, M)$. Lemma 4.2 offers an algorithm of *search* within $(0, M) \times (0, M)$, that would permit to find, basing on the quantities (5.1)–(5.2) the pair $[\nu_L(T), \nu_R(T)]$ that approximates

$[\mu_L(T), \mu_R(T)] =$ the element from $(0, M) \times (0, M)$ that really corresponds to $\mathbf{D} \in \mathcal{D}$ under study.

Given a pair $[\nu_L(T), \nu_R(T)]$, we choose an $S(T)$ according to the above remark and calculate $\Psi_R(A, B, C, S(T))$ as in (5.4). By Lemma 4.2, and (4.2) we obtain both functions

$$\frac{dY_1(u)}{du} \quad \text{and} \quad \frac{dY_2(u)}{du}.$$

and with them the two functions $Y_1(u)$ and $Y_2(u)$ (each determined up to a shift perpendicular to direction α). In this way we obtain a map

$$[\nu_L(T), \nu_R(T)] \Rightarrow [Y_1(u),\ Y_2(u)]. \tag{5.5}$$

It is clear, that if $[\nu_L(T), \nu_R(T)] = [\mu_L(T), \mu_R(T)]$, then $Y_1(u)$ would become the lower and $Y_2(u)$ the upper branch of $\partial\mathbf{D}$.
The map (5.5) can be used in a search for a satisfactory approximation for $\sigma_0(u), S_0(T)$. Here some continuous functional $\mathcal{F}$ defined in the space of pairs $[\,Y_1(u),\ Y_2(u)\,]$ can help:

$\mathcal{F}[\,Y_1(u),\ Y_2(u)\,]$ should play the role of distance from the pair $[\,Y_1(u),\ Y_2(u)\,]$ to the set $\mathcal{D}$,

$\mathcal{F}[\,Y_1(u),\ Y_2(u)\,] = 0$ if Y_1 and Y_2 are the upper and lower branches of boundary of some $\mathbf{D} \in \mathcal{D}$.

The search for $[\nu_L(T), \nu_R(T)]$ approximating $[\mu_L(T), \mu_R(T)]$ based on some concrete $\mathcal{F}$ can be as follows. It is necessary to choose some (sufficiently small) $\varepsilon > 0$ and have a source of test candidate pairs $[\nu_L(T), \nu_R(T)]$. The candidates can be supplied say, by Monte Carlo method as random points in $(0, M) \times (0, M)$, or come from some lattice in that square. Given a candidate pair, $Y_1(u)$ and $Y_2(u)$ are calculated on the basis of (3.1)-(3.2); if $\mathcal{F}[\,Y_1(u),\ Y_2(u)\,] > \varepsilon$, then $[\nu_L(T), \nu_R(T)]$ is rejected, otherwise it is accepted as an approximation for $[\mu_L(T), \mu_R(T)]$. The convex hull of the figure obtained by appropriately shifted graphs of $Y_1(u)$ and $Y_2(u)$ is then accepted as an (approximate) reconstruction of $\mathbf{D}$. This would yield asymptotically exact reconstruction of $\mathbf{D} \in \mathcal{D}$ by means of three parallel X-rays.

Now we give an example where complete tomographic reconstruction of a $\mathbf{D} \in \mathcal{D}$ can be done without described search procedure.

Let for some direction α, some $r > 0$ and every $u \in (-r, r)$

$$T_\alpha(u) = 2\sqrt{r^2 - u^2}, \tag{5.6}$$

and the directional derivatives at α let be identical zero, i.e. for every $T \in (0, 2r)$

$$Q(T) = 0 \quad \text{and} \quad \frac{d^2}{d\alpha^2} H_\alpha(T) = 0 \tag{5.7}$$

Of course, in case of $\mathbf{D} =$ a circular disc $\mathbf{D}$ of radius r with abscissa of the center at $u = 0$ we have those values, but is the contrary assertion valid?

Without difficulty we get

$$U_{1L}(T) = U_{1R}(T), \quad \text{which is the same as} \quad Y_1(u_L) = Y_1(u_R),$$
$$\frac{du_R}{du_L} = -1, \qquad \frac{d\,u_L(T)}{dT} = +\frac{T}{2\,\rho(T)} \qquad \frac{d\,u_R(T)}{dT} = -\frac{T}{2\,\rho(T)},$$

so the functions in Lemma 3 are found to be

$$A = \frac{T^2}{2\,\rho(T)}\,[\,\sigma(u_L) + \sigma(u_R)\,], \quad B = -T\,[\,\sigma(u_L) + \sigma(u_R)\,], \quad C = -\rho(T),$$

implying

$$B^2 - 4AC = 2T^2 [1 + \sigma(u_L)\sigma(u_R) + \sigma(u_L) + \sigma(u_R)] = 2T^2 c,$$

where c can attain only values 0 or 4. To keep $\Psi_R(A, B, C, S(T))$ continuous and by Lemma 3, we necessarily have to assume that $c = 0$. Hence

$$\Psi_R(A, B, C, S(T)) = \frac{-B}{2A} = \frac{\rho(T)}{T} = \frac{u}{\sqrt{r^2 - u^2}}.$$

Integrating this Ψ_R we get familiar expressions for Y_1 and Y_2 that correspond to a circular disc of radius r. We conclude that the following uniqueness result holds:

Theorem 5.1. *Let for some* $\mathbf{D} \in \mathcal{D}$ *the* X*-ray for some direction* α *be given by (5.6). If for the same* α *holds (5.7), then necessarily* $\mathbf{D}$ *is a circular disc of radius* r.

To end the paper, we point at a corollary, probably of interest in convexity theory.

Corollary 5.1. *For every* $\mathbf{D} \in \mathcal{D}$ *the discriminant* $B^2 - 4AC$ *is continuous and nonnegative; at the discontinuity points of* $S_0(T)$ *necessarily* $B^2 - 4AC = 0$.

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Поступила 4 марта 2012