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SEVAN METHODOLOGIES REVISITED: RANDOM LINE PROCESSES

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Abstract. The paper studies random line processes that are translation invariant in probability distribution, and whose first and second order moment measures possess continuous densities. The purpose is to review the analytical apparatus based on the concept of *horizontal or vertical windows and corresponding Palm-type probability distributions* that are now proved below to exist. That apparatus enables to study the relation between the quantities $p_k(l,\alpha)$ and $\pi_k(l,\alpha)$, where $p_k(l,\alpha)$ = probability to have k hits by the lines from Z on a "test segment"of length l and direction α, while $\pi_k(l,\alpha)$ = conditional probability of the same event, the condition being that the test segment lies on one of the lines from Z. Palm equations for *horizontal windows* have been known since long, but for *vertical windows* they were first put down in the last chapter of the book [4], under stronger condition of Euclidean motions invariance of Z. The paper considers two different *models* that imply Poissonity of the probabilities $p_k(l,\alpha)$. Translation invariant line processes can be viewed as stationary states of random dynamical arrays of countably many particles each moving with constant speed along the *test line*, and these models are of special interest in that context. In a model-free setting, the paper presents a formula for calculation of the conditional intensity $\Lambda(\alpha) = l^{-1}\sum_k k\,\pi_k(l,\alpha)$. That formula includes quantities depending on the distribution of the *typical vertex shape*. "Sevan metodologies" have been the topic of authors plenary report at the Rasht (Iran) meeting in 2011. This usage is motivated in a special historical section below; another section is devoted to detailed description of Sevan methodologies themselves.[1]

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1. Some history

The collection of papers "Stochastic Geometry" [1] edited by E.F. Harding and D.G. Kendall inaugurated in 1974 a new direction in the theory of random point processes: random processes of geometrical elements (lines in the plane, or in space, planes in

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three dimensions etc.) that can be represented as points in appropriate manifolds. Earlier an effort to coordinate and promote that research took place at Oberwolfach meeting on Integral Geometry and Geometric Probability held in June 1969, with organizers D.G.Kendall and Klaus Krickeberg. Remarkably, that Symposium started a rare instance of East - West mathematical cooperation when the materials of the symposium were published in Soviet Armenia in 1970, in a special issue of the Armenian Academy mathematical *Izvestiya* [5]. In 1976, Buffon Needle Bicentenary International Conference was held at Sevan, Armenia [2] jointly sponsored by Royal Society, French and Armenian Academies. The Second Sevan Symposium on Stochastic and Integral Geometry [10], [12] was held in 1985. To commemorate that development, the methodologies presented by the author at the Sevan meetings of 1976 and 1985 we now call "Sevan"

D.G. Kendall who visited the first Sevan Conference, was at that time the President of the London Mathematical Society. In his "Introduction to Stochastic Geometry"in [1] he wrote that "the whole existing literature concerned with stochastic geometry"could be found within the pages of the collection [1]. A prominent place in [1] belongs to the paper by a young Cambridge mathematician Rollo Davidson entitled "Construction of Line-Processes: Second order Properties". That paper by Rollo Davidson was originally published in Soviet Armenia [5]. Quite tragically, Rollo Davidson died (on 29 July 1970 in a mountain-climbing accident) about a month before he could realize a planned visit to Armenian Academy (Yerevan) sponsored by the Royal Society [6]. The collection [1] was a tribute to his memory.

The topic of random line processes dominated the collection [1]. In Kendall's words, there is a sense in which "Stochastic geometry takes its origin in Crofton's famous article in the IXth edition of the *Encyclopedia Britannica*"that contained a study of Euclidean motion invariant Poisson line process. Rollo Davison's work was "an attempt to eschew Crofton's approach": in fact that pioneering work launched a series of studies of general random line processes, like [7]-[10],[12] and Chapter 9 of [4]. The collection [1] contained also several papers devoted to the concept of Palm distribution, an important tool in the present study.

2. Sevan methodologies.

We concentrate on three items that are applied in the present paper.

Combinatorial Integral Geometry We use the notation: $I\!R^2$ = the Euclidean plane, $\mathbf{G}$ = the space of Euclidean lines in $I\!R^2$ with usual Moebius band topology, see [3]. Euclidean motion invariant locally-finite measure dg in the space $\mathbf{G}$ (see (3.1) below) is uniquely defined by the condition

$$\int_{[P_1P_2]} dg = 2\,|P_1, P_2|,$$

where $[P_1, P_2]$ = the set of lines that separate two endpoints of a "needle"$P_1, P_2 \in \mathbf{R}^2$, $|P_i, P_j|$ is the Euclidean distance between P_i and P_j. In 1890 J.J.Sylvester considered the following problem. Let in the plane, n "needles"ν_1,...,ν_n be fixed in general position. The value of that measure on the sets

$$\mathbf{A} = \bigcap_{i=1}^{n} [\nu_i] \quad \text{or} \quad A = \bigcup_{i=1}^{n} [\nu_i]$$

in each case was known to have the representation

$$\int_A dg = \sum_{i<j} u_{ij}(\mathbf{A})\,|P_i, P_j|, \tag{2.1}$$

with some integer coefficients $u_{ij}(\mathbf{A})$. J.J. Sylvester asked for an algorithm of calculation of the integers $u_{ij}(A)$ for each set. Only in 1973 a solution was given in [11], known as the *solution of Buffon–Sylvester problem* [11], [3]. It is as follows.

Assume we have a finite collection of points

$$\{P_i\} = \{P_1, ..., P_N\} \subset I\!R^2.$$

We introduce an equivalence relation: two lines g_1, $g_2 \in \mathbf{G}$ which do not belong to any $[P_i]$ (where $[P]$ = the bundle of lines through P) we call equivalent if they induce the same separation of the set $\{P_i\}$ into two subsets.

An equivalence class Υ (a maximal set of equivalent lines) is always a connected set in the topology of $\mathbf{G}$, but its closure will not be compact if for each line $g \in \Upsilon$ the total $\{P_i\}$ lies in one of the two half-planes separated by g. All other equivalence classes have compact closures: these we call *atoms*. We denote

$r\{P_i\}$ = the minimal ring $r\{P_i\}$ of subsets of $\mathbf{G}$ which contains all atoms,

$[P_i, P_j] = \{g \in \mathbf{G} : g$ separates P_i from $P_j\}$, (the so called *Buffon sets*),

g_{ij} = the line through P_i and P_j.

An element $\mathbf{A} \in r\{P_i\}$ necessarily has the form $\mathbf{A} = \bigcup a_i$, where a_i are some of the atoms of $r\{P_i\}$.

If g_{ij} contains no points from $\{P_i\}$ except P_i and P_j, and the number of points in $\{P_i\}$ exceeds 2, then there exist exactly four different equivalence classes Υ for which we have $g_{ij} \in \partial\Upsilon$. We denote them as $\Upsilon_{ij}(++)$, $\Upsilon_{ij}(--)$, $\Upsilon_{ij}(+-)$ and $\Upsilon_{ij}(-+)$. We make a convention that

every line $g \in \Upsilon_{ij}(++)$ or $g \in \Upsilon_{ij}(--)$ leaves P_i and P_j in one half-plane,

every line $g \in \Upsilon_{ij}(+-)$ or $g \in \Upsilon_{ij}(-+)$ leaves P_i and P_j in different half-planes.

Given $\mathbf{A} \in r\{P_i\}$, the values of the indicator function

$$I_{\mathbf{A}}(g) = \begin{cases} 1, & \text{if } g \in \mathbf{A}, \\ 0, & \text{otherwise} \end{cases}$$

on the lines from the above four sets we denote correspondingly as

$$I_{\mathbf{A}}(i^+, j^-) \equiv I_{\mathbf{A}}(g) \text{ for } g \in \Upsilon_{ij}(+-), \quad I_{\mathbf{A}}(i^-, j^+) \equiv I_{\mathbf{A}}(g) \text{ for } g \in \Upsilon_{ij}(-+),$$

$$I_{\mathbf{A}}(i^+, j^+) \equiv I_{\mathbf{A}}(g) \text{ for } g \in \Upsilon_{ij}(++), \quad I_{\mathbf{A}}(i^-, j^-) \equiv I_{\mathbf{A}}(g) \text{ for } g \in \Upsilon_{ij}(--).$$

The following result was proved in [3] in several different ways. Actually (2.1) is valid for any $A \in r\{P_i\}$. Under the condition that no line g_{ij} contains any points from $\{P_i\}$ other than P_i and P_j, the algorithm of calculation of the coefficients $u_{ij}(A)$ reduces to the *four indicator rule*:

$$u_{ij}(\mathbf{A}) = I_{\mathbf{A}}(i^+, j^-) + I_{\mathbf{A}}(i^-, j^+) - I_{\mathbf{A}}(i^+, j^+) - I_{\mathbf{A}}(i^-, j^-).$$

If the number of points in $\{P_i\}$ equals 2, i.e. $\{P_i\} = \{P_1, P_2\}$ then $r\{P_i\}$ contains only one element $\mathbf{A}$ for which the above remains valid since formally $I_{\mathbf{A}}(i^+, j^+) = I_{\mathbf{A}}(i^-, j^-) = 0$ and we get $u_{12}(\mathbf{A}) = 2$.

For the case where g_{ij} contains points from $\{P_i\}$ other from P_i and P_j (2.1) remains valid for every $A \in r\{P_i\}$, while the algorithm requires modification.

The class (+); A 2-set P_i, P_j belongs to the class $(+)$ if the interior of the linear segment P_i, P_j does not contain any points from $\{P_i\}$. For every $P_i, P_j \in (+)$, the equivalence classes $\Upsilon_{ij}(+-)$ and $\Upsilon_{ij}(-+)$ are uniquely defined (necessarily both are atoms).

The class (-): A 2-set P_i, P_j belongs to the class (-) if the interior of the complement (within g_{ij}) of the linear segment P_i, P_j does not contain points from $\{P_i\}$. For $P_i, P_j \in$ (-), the equivalence classes $\Upsilon_{ij}(++)$ and $\Upsilon_{ij}(--)$ are also uniquely defined (one of the two can fail to be an atom).

We write

$$u'_{ij}(\mathbf{A}) = I_{\mathbf{A}}(i^+, j^-) + I_{\mathbf{A}}(i^-, j^+), \quad \text{well defined for } P_i, P_j \text{ from the class } (+),$$

$$u''_{ij}(\mathbf{A}) = I_{\mathbf{A}}(i^+, j^+) + I_{\mathbf{A}}(i^-, j^-), \quad \text{well defined for } P_i, P_j \text{ from the class } (-),$$

General algorithm. For any finite set of points $\{P_i\} \subset I\!R^2$ with number of points greater then 2, and every $\mathbf{A} \in r\{P_i\}$

$$\int I_{\mathbf{A}}(g)\, dg = \sum_{(+)} u'_{ij}(\mathbf{A})\, |P_i P_j| - \sum_{(-)} u''_{ij}(\mathbf{A})\, |P_i P_j|. \tag{2.2}$$

The book [3] contains numerous corollaries and generalizations of (2.2), while Chapter 10 of [4] contains the first case of calculation based on the coefficients $u'_{ij}(\mathbf{A})$ and $u''_{ij}(\mathbf{A})$.

Translational analysis of realizations. Let Z be a realization of a random line processes in $\mathbf{R}^2$,

$\mathbf{T}_2$ = the group of parallel translations of $\mathbf{R}^2$,

$\mathbf{P}$ = probability distribution of Z assumed to be invariant with respect to $\mathbf{T}_2$,

dt = Haar measure on the group $\mathbf{T}_2$ (corresponds to Lebesgue measure in $\mathbf{R}^2$),

tZ = translation of Z by $t \in \mathbf{T}_2$.

The method is based on the study of integrals

$$\int_{\mathbf{b}} f(tZ)\, dt,$$

where $\mathbf{b}$ corresponds to some disc in $\mathbf{R}^2$, while $f(Z)$ is some function defined in the space of realizations Z. In the cases of interest $f(Z) = f_\varepsilon(Z)$ also depends on some small parameter ε, and it is possible to find the limit

$$\lim_{\varepsilon \to 0} \int_{\mathbf{b}} f_\varepsilon(tZ)\, dt = x(Z). \tag{2.3}$$

By the invariance assumption concerning $\mathbf{P}$ we have the identity

$$\int f_\varepsilon(tZ)\, d\mathbf{P} \;=\; \int f_\varepsilon(Z)\, d\mathbf{P}.$$

In case there exists a function $y(Z)$ with finite integral (expectation) with respect to $\mathbf{P}$ such that uniformly in ε

$$\int_{\mathbf{b}} f_\varepsilon(tZ)\, dt \;\leq\; y(Z), \tag{2.4}$$

then by Lebesgue bounded convergence theorem and Fubini theorem, integration in $d\mathbf{P}$ would imply

$$\lim_{\varepsilon\to 0} \int f_\varepsilon(Z)\, d\mathbf{P} \;=\; ||\mathbf{b}||^{-1} \int x(Z)\, d\mathbf{P}, \tag{2.5}$$

where $||\mathbf{b}||$ stands for the value of dt-measure (area) of $\mathbf{b}$. In the lemmas of Section 4 we take $||\mathbf{b}|| = 1$.

The simplest illustration of above general idea can be found in [4], pages 137, 193, where the method leads to the well known in the theory of translation invariant random point processes concept of “Palm Distribution”. (Even earlier case is [14], where one of the sections is entitled “The move-and-average method”.) The same method for line processes Z that are distribution invariant with respect to the Euclidean group was used in [4], Chapter 10. This led to the concept of "Palm Distribution"for the corresponding class of line processes. In Section 4 below we apply that methodology to line processes and the group $\mathbf{T}_2$, and so demonstrate the existence of "Palm-type"probabilities Π_v, and Π_{vv} (v stands for "vertical"windows). The same approach with minimal changes applies to "Palm-type"probabilities Π_h and Π_{hh} (h stands for "horizontal"windows).

Factorization of invariant measures. A considerable part of the book [4] is devoted to measures in the products of the spaces of geometrical elements that are invariant with respect to groups acting in the carrier spaces. Normally such measures split into two factors, one of the two being the Haar measure on the group. The problem then is to find the other measure factor. This is the "factorization"in the book’s title. In [4] $\mathbf{T}_2$-invariant measures in the space $\mathbf{G}$ of lines and in the space $\mathbf{G} \times \mathbf{G}$ are treated on the basis of that factorization principle. In the present paper such measures come in as the first M_1 and the second M_2 moment measures of $\mathbf{T}_2$-invariant line processes. We will base on the following.

If a locally-finite $\mathbf{T}_2$-invariant measure M_1 possesses a density, then it necessarily has the form

$$\rho_1(\phi)\, dg$$

where ϕ is the direction of $g \in \mathbf{G}$, dg is the Euclidean motions invariant measure on $\mathbf{G}$ and $\rho(\phi)$ is a summable function defined on

$$(0, \pi) = \text{the space of planar directions.}$$

If we assume that the measure M_2 beyond the "diagonal"$g_1 = g_2$ possess a density, then necessarily it has the form

$$\rho_2(\phi_1, \phi_2)\, dg_1\, dg_2,$$

where $\rho_2(\phi_1, \phi_2)$ is some summable function defined on the product space $(0, \pi) \times (0, \pi)$. We will use the Jacobian result (see [4], p. 37)

$$\rho_2(\phi_1, \phi_2)\, dg_1\, dg_2 \;=\; \sin\tau\, \rho_2(\phi_1, \phi_2)\, d\phi_1\, d\phi_2\, dQ, \tag{2.5}$$

where Q is the point where the lines g_1 and g_2 intersect, dQ is the planar Lebesgue, $d\phi_i$ are usual angular measures in the space of directions $(0, \pi)$.

3. Random line processes

The purpose of the present section is to presents necessary basic concepts from the line processes theory together with much of the notation used in the paper.

The space of sensed lines in the plane $\mathbf{R}^2$ can be represented by a cylindrical surface

$$[0, 2\pi) \times (-\infty, +\infty),$$

where $[0, 2\pi)$ stands for the circle of unit radius. Each sensed line then receives natural coordinates (ϕ, p), where $\phi =$ direction of the line, $p =$ the signed distance of the line from the origin on the plane. A non-sensed line we denote by g. The space $\mathbf{G}$ of (non-sensed) lines in the plane, $g \in \mathbf{G}$, is obtained from the above cylinder by identifying pairs of lines that coincide except for directions. In this way $\mathbf{G}$ receives topology of the Möbius Band.

A random collection Z of lines in the plane is called a random line process in case Z corresponds to some random point process in $\mathbf{G}$. (Sometimes, as appropriate, below we use Z to denote a fixed realization of a random line process.) Let Z be a random line process such that Z with probability 1 has no lines parallel to $g_0 =$ a "test line"on the plane. Then Z possesses random marked point process $\{x_i, \psi_i\}$ representation, where each $x_i \in g_0$ is the point where a line from Z intersects g_0, and ψ_i is the corresponding intersection angle. Conversely, a marked point processes $\{x_i, \psi_i\}$ generates via given g_0 a random line processes Z_{g_0} due to the map

$$\{x_i, \psi_i\} \to \{g_i\} \, = \, Z_{g_0}$$

where g_i is the line that hits $x_i \in g_0$ under angle ψ_i.

In case $\{x_i, \psi_i\}$ is invariant in distribution with respect to g_0-preserving translations of the plane, the corresponding Z_{g_0} in general is not translation invariant. Let $\{x_i, \psi_i\}_d$ be the marked point process induced by Z_{g_0} on the line g_d parallel to and distance d from g_0. The problem of existence of limiting distribution for $\{x_i, \psi_i\}_d$ as $d \to \infty$ attracted much attention some decades ago in the case where on g_0 the sequences $\{x_i\}$ and $\{\psi_i\}$ are assumed independent and $\{\psi_i\}$ is a sequence of independent angles, see [15], [16].

The probability distribution of Z we denote as $\mathbf{P}$: it is a probability measure that lives in the space of realizations of line processes, or, more properly, on the sigma-algebra ∇ defined to be the image of the sigma-algebra well known in the theory of random point processes. Elements of ∇ are called events. A classical example due to Crofton is the Poisson line process governed by the standard Euclidean motion invariant measure

$$dg \, = \, \sin \psi \, d\psi \, dl, \tag{3.1}$$

where

$l =$ the usual one dimensional coordinate of the point $g \cap \gamma$ on some reference line γ,

$\psi =$ the angle between the reference line and g.

By definition, it corresponds to the Poisson point process on $\mathbf{G}$ governed by dg.

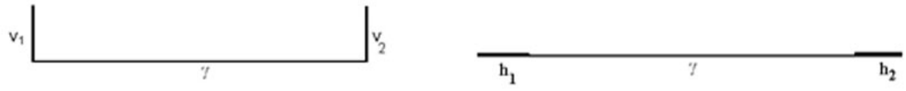

Рис. 1. Two pairs of “windows” at the endpoints of γ

We recall that a Poisson point process on $\mathbf{G}$ governed by a measure m that lives on $\mathbf{G}$ puts k points in a Borel set $B \subset \mathbf{G}$ with probability

$$\frac{[m(B)]^k}{k!} \exp^{-m(B)},$$

while the numbers k for disjoint sets B are independent.

Poisson line processes governed by measures of the form $\rho(\phi)\, dg$, where $\rho_1(\phi)$ is some density defined on $[0, \pi)$ are all $\mathbf{T}_2$ - invariant in distribution.

Basic Events. Let α be some direction in the plane to be called "horizontal γ be a "test"line segment in the plane of length l and planar direction α. So γ lies on a horizontal "test"line, see Fig.1, v_1 and v_2 = vertical windows, both of length ε, and

h_1 and h_2 = horizontal windows, both of length ε. We write

$\binom{u}{k}$ = the segment $u \subset \mathbf{R}^2$ is hit by exactly k lines from Z

(we say that $g \in \mathbf{G}$ *hits* u if g separates the endpoints of u). Given several test segments u_1, u_2,... u_m and nonnegative integers k_1, k_2,..., k_m we consider the events

$$\binom{u_1}{k_1} \cap \binom{u_2}{k_2} \cap ... \cap \binom{u_m}{k_m}.$$

The events of the above type are said to belong to the class ∇_0 if the endpoints of γ are not among the endpoints of the segments $u_1, u_2, ..., u_m$. (The class ∇_0 serves measure continuation purposes in the Lemmas 1.2 below.) We write $\begin{pmatrix} u_1 & u_2 \\ k_1 & k_2 \end{pmatrix}$ for the intersection of $\binom{u_1}{k_1}$ and $\binom{u_2}{k_2}$. For the probabilities of such events we use notation like $\mathbf{P}\begin{pmatrix} u_1 & u_2 \\ k_1 & k_2 \end{pmatrix}$.

For a line segment u we define the event $\binom{v}{1u} \subset \binom{v}{1}$ as

$$\binom{v}{1u} = \{Z : \text{the unique line from } Z \text{ that hits } v \text{ hits the segment } u\}.$$

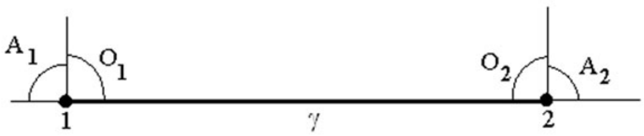

Рис. 2

Assume a probability measure Π is concentrated on realizations Z that with probability $\Pi = 1$ possess a line g_0 that contains one of the endpoints of γ. Then $\Pi(\Theta)$ will stand for the Π-probability of the *event*, that the random direction of g_0 belongs to Θ, where Θ is some arc of planar directions. In case with probability $\Pi = 1$ there are two lines in Z through each endpoint of γ, we will use the notation $\Pi\,[\Theta_1 \cap \Theta_2]$ for the probability of the corresponding intersection event.

In the definition of Acute-Obtuse factorization model given in Section 6 we choose $\Theta_i = A_i$ or $\Theta_i = O_i$, the arcs A_1, O_1, A_2, O_2 are shown on Fig.2 (A stands for *Acute* and O for *Obtuse*). In the proof of Lemma 5.1 we will use the event relations valid for each endpoint of γ, i.e. for $i = 1, 2$:

$$\text{(3.2)} \qquad \lim_{\varepsilon\to 0} \partial \begin{pmatrix} \gamma & v_i \\ k & 1\gamma \end{pmatrix} = \begin{pmatrix} \gamma \\ k-1 \end{pmatrix} \cap A_i, \qquad \lim_{\varepsilon\to 0} \partial \begin{pmatrix} \gamma & v_i \\ k & 1 d_i \end{pmatrix} = \begin{pmatrix} \gamma \\ k \end{pmatrix} \cap O_i,$$

where d_i is the hypotenuse spanning γ and v_i, while ∂ stands for the boundary of a set.

Densities of Moment Measures. The present paper considers line processes that are invariant in distribution with respect to the group $\mathbf{T}_2$. Within that class we specify the subclasses

$D1$ = line processes with first moment measure possessing a continuous density

$$\rho_1(\phi)\, dg, \quad \text{where } \phi \text{ is the direction of the line } g,$$

$D2$ = line processes with second moment measure possessing a continuous density

$$\rho_2(\phi_1, \phi_2)\, dg_1\, dg_2, \quad \text{where } \phi_1, \phi_2 \text{ are the directions of the lines } g_1, g_2.$$

The assumption $Z \in D1$ implies

$$\lambda(\alpha) \,=\, |\gamma|^{-1} \sum k \, \mathbf{P} \binom{\gamma}{k} \,<\, \infty$$

and hence as the length $|\gamma|$ tends to zero

$$\mathbf{P} \binom{\gamma}{1} \,=\, \lambda(\alpha) \, |\gamma| \,+\, o(|\gamma|) \quad \text{and} \quad \mathbf{P} \binom{\gamma}{k} \,=\, o(|\gamma|) \;\; \text{for} \;\; k > 1,$$

i.e. the set of hits on any test line is *orderly* in the usual random point process sense. From (3.1)

$$\lambda(\alpha) \,=\, \int \rho_1(\phi) \, \sin(\alpha, \phi) \, d\phi, \quad \lambda(v) \,=\, \int \rho_1(\phi) \, |\cos(\alpha, \phi)| \, d\phi,$$

where v stands for the vertical (perpendicular to α) direction,

$(\alpha, \phi) = \psi =$ the angle between directions α and ϕ,

$d\phi =$ the usual rotation invariant measure in the space of planar directions.

It will become clear (Lemma 4.2) that for the events

$$V_2 \,=\, \begin{pmatrix} v_1 & v_2 \\ 1 & 1 \end{pmatrix} \cap (2), \quad \text{and} \quad H \,=\, \begin{pmatrix} h_1 & h_2 \\ 1 & 1 \end{pmatrix}$$

where (2) is an event defined as

$(2) = \{Z :$ the windows v_1, v_2 are hit by two different lines from $Z\}$,

the assumption $Z \in D2$ implies

$$\mathbf{P}(V_2) \,=\, S_{vv}(\alpha) \, \varepsilon^2 \,+\, o(\varepsilon^2) \qquad \text{and} \qquad \mathbf{P}(H) \,=\, S_{hh}(\alpha) \, \varepsilon^2 \,+\, o(\varepsilon^2),$$

where

$$S_{hh}(\alpha) \,=\, \int_0^\pi \int_0^\pi \rho_2(\phi_1, \phi_2) \, \sin(\alpha, \phi_1) \sin(\alpha, \phi_2) \, d\phi_1 \, d\phi_2,$$

$$S_{vv}(\alpha) \,=\, \int_0^\pi \int_0^\pi \rho_2(\phi_1, \phi_2) \, | \cos(\alpha, \phi_1) \; \cos(\alpha, \phi_2) \, | \, d\phi_1 \, d\phi_2. \tag{3.3}$$

As for the event

$$V_1 \,=\, \begin{pmatrix} v_1 & v_2 \\ 1 & 1 \end{pmatrix} \cap (1)$$

where $(1) = \{Z :$ the windows v_1, v_2 are hit by the same line from $Z\}$, the relation

$$\mathbf{P}(V_1) \,=\, \rho_1(\alpha) \, \frac{\varepsilon^2}{l} \,+\, o(\varepsilon^2) \tag{3.4}$$

does not seem to be automatically valid, hence the definition: a line process $Z \in D_1$ is called orderly if it satisfies (3.4). We note that due to

$$\int_{[v_1] \cap [v_2]} dg \,=\, \frac{\varepsilon^2}{l} \,+\, o(\varepsilon^2)$$

the right-hand side of (3.4) is the asymptotical expression of the value of the first moment measure of $Z \in D1$ on the set $[v_1] \cap [v_2]$ = lines that hit both v_1 and v_2.

The Vertex Process. Let $\{Q_i\}$ be the set of vertices of $Z \in D2$: each vertex Q_i is a point where some two lines from Z intersect. We postulate that with probability $\mathbf{P} = 1$ no triads of lines from Z meet at a point. (In a broader framework of random segment processes related questions were considered in [13] and [14].) To each vertex Q_i correspond the marks (dependence on i is suppressed):

(ϕ_1, ϕ_2) = the directions of the two lines g_1, $g_2 \in Z$ that meet at Q_i and

τ = the angle between the directions ϕ_1 and ϕ_2,

(ϕ_1, ϕ_2) is the translational and τ is the Euclidean "shape"of Q_i.
By (2.6) $\{Q_i\}$ happens to be a point process of finite intensity λ_Q:

$$\lambda_Q = \int\int \sin\tau\, \rho_2(\phi_1, \phi_2)\, d\phi_1\, d\phi_2.$$

According to [4], the probability density defined on the product $(0, \pi) \times (0, \pi)$ as

$$\frac{1}{\lambda_Q} \sin\tau\, \rho_2(\phi_1, \phi_2)\, d\phi_1\, d\phi_2 \tag{3.5}$$

describes the *translational random shape of a typical vertex* in $\{Q_i\}$. The corresponding expectation we denote as $\mathbf{E}_Q$. It follows that

$$\mathbf{E}_Q \frac{1}{\sin\tau} < \infty.$$

The results of Section 7 below (now published for the first time) permit to express the "conditional"intensity

$$\Lambda(\alpha) = \sum k\, \pi_k(l, \alpha)$$

via the intensities $\lambda(\alpha) = \sum k\, p_k(l, \alpha)$, λ_Q and some averages E_Q of certain parameters depending on random shape of a typical vertex in $\{Q_i\}$. We will often use the well-known *Wilhelm Blaschke relation* (rediscovered in [4])

$$2\,\rho_1(\alpha) = \lambda(\alpha) + \lambda''(\alpha). \tag{3.6}$$

4. Palm type probabilities

In Lemma 4.1 we can choose v, A, O to be either v_1, A_1, O_1 or v_2, A_2, O_2 as on Figs. 1,2. We note that

$$\mathbf{P}\binom{v}{1} = \lambda(v)\,\varepsilon + O(\varepsilon), \qquad \mathbf{P}\binom{v}{k} = o(\varepsilon) \quad \text{for } k = 2, 3, ...$$

are well known facts of the theory of stationary point processes on a line ($\lambda(v)$ is the intensity of the intersections point process on lines of vertical direction).

Lemma 4.1. *For every line process $Z \in D1$ and every event $C \in \nabla_0$ there exists a limit*

$$\Pi_v(C) = \lim_{\varepsilon \to 0} \frac{\mathbf{P}\left[C \cap \binom{v}{1}\right]}{\mathbf{P}\binom{v}{1}} \tag{4.1}$$

that defines, by means of probability continuation, a probability measure Π_v on ∇. This Π_v is concentrated on realizations Z that possess a line through an end-point of γ. In particular, the probabilities $\Pi_v\left[\binom{\gamma}{k} \cap A\right]$ and $\Pi_v\left[\binom{\gamma}{k} \cap O\right]$ are well defined.

Proof. We apply the notation of the Translational analysis subsection of Section 3. Let $\mathbf{b} \subset \mathbf{T}_2$ correspond to the disc $b \subset \mathbf{R}^2$ i.e. $b = \{tO, t \in \mathbf{B}\}$, O is the origin. We put

$$f_\varepsilon(Z) = \varepsilon^{-1}\, I_{\binom{v}{1}}(Z)\, I_C(Z)$$

(product of two indicator functions). For realizations Z from the set that has probability $\mathbf{P} = 1$ we easily establish

$$x(Z) = \sum_{\chi_i} |\cos(\chi_i, \gamma)| \int_{u \in \chi_i} I_C(t_u Z)\, du, \tag{4.2}$$

where

$t_u =$ shift that takes the point $u \in \mathbf{R}^2$ to the common endpoint of v and γ,

$\chi_i =$ the chord $b \cap g_i$, $g_i \in Z$,

$(\chi_i, \gamma) =$ the angle between χ_i and γ,

$du =$ the length measure on the chord χ_i. Clearly

$$\int_{tO \in b} f_\varepsilon(tZ)\, dt \le c\, y(Z),$$

where $c =$ diameter of b, while $y(Z) =$ the number of lines from Z that hit b. If Z is random $Z \in D1$ with probability distribution $\mathbf{P}$, then

$$\int y(Z)\, d\mathbf{P} \;=\; \int_{[b]} \rho_1(\phi)\, dg < \infty,$$

where $[b] \subset \mathbf{G}$ is the set of lines that hit b. This proves the existence of

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\, \mathbf{P}\left[C \cap \binom{v}{1}\right].$$

In particular, replacing C by the total space of realizations we get the existence of

$$\lambda(v) \;=\; \lim_{\varepsilon\to 0} \varepsilon^{-1}\, \mathbf{P}\binom{v}{1}.$$

The two limits together yield (4.1). From (4.2) follows the probability continuation formula valid at least for every $C \in \nabla$:

$$\Pi_v(C) \;=\; \int d\mathbf{P} \sum_{\chi_i} |\cos(\chi_i, \gamma)| \int_{\chi_i} I_C(t_u Z)\, du. \tag{4.3}$$

Lemma 4.1 is proved. $\square$

In the next Lemma 4.2 the event V_2 is as defined in Section 3 and we again apply Translational analysis as in Section 2. The quantities S_{hh} and S_{vv} are given by (3.3).

Lemma 4.2. *For every line process $Z \in D2$*

$$\mathbf{P}(V_2) \;=\; S_{vv}(\alpha)\, \varepsilon^2 \;+\; o(\varepsilon^2). \tag{4.4}$$

For every $C \in \nabla_0$ there exists the limit

$$\Pi_{vv}(C) \;=\; \lim_{\varepsilon\to 0} \frac{\mathbf{P}(C \cap V_2)}{\mathbf{P}(V_2)}$$

that extends to a probability measure Π_{vv} on ∇. This Π_{vv} is concentrated on realizations Z that possess lines through each end-point of γ. In particular, the probabilities like $\Pi_{vv}\left[\binom{\gamma}{k} \cap A_1 \cap O_2\right]$ are well defined.

Proof. Let $g_1^{(i)}$, $g_2^{(i)}$ be two lines from Z that meet at a vertex Q_i, and τ_i be the angle between the two. By elementary calculations, the dt-measure of the set $\{t \in \mathbf{T}_2 : tv_1 \text{ hits } g_1^{(i)} \text{ and } tv_2 \text{ hits } g_2^{(i)}\}$ equals

$$\frac{|\cos(g_1^{(i)}, \gamma)|\,|\cos(g_2^{(i)}, \gamma)|}{\sin \tau_i}\, \varepsilon^2. \tag{4.5}$$

Now using the notation of the Translational analysis subsection of Section 3 we put

$$f_\varepsilon(Z) \;=\; \varepsilon^{-2} I_{V_2}(Z)\, I_C(Z).$$

With probability 1 the limiting function $x(Z)$ exists; we write down, its explicit expression under a simplifying assumption about realization Z. Let $\{t_i\} \in \mathbf{b}$ be determined by the condition that for each t_i the set $t_i Z$ contains both endpoints of γ. If we assume, that Z has the property that for no pairs (i,m) the points $t_i Q_m$ coincide with endpoints of γ, then (4.5) implies

$$x(Z) \;=\; \sum_{t_i\in\mathbf{b}} \frac{|\cos(g_1^{(i)},\gamma)|\,|\cos(g_2^{(i)},\gamma)|}{\sin\tau_i}\, I_C(t_i Z). \tag{4.6}$$

It is easy to find an explicit expression for $x(Z)$ in case of general Z. We do not put it down because the present proof needs only existence of the limit that defines $x(Z)$. What the proof needs is the inequality

$$f_\varepsilon(Z) \;\le\; \sum_{t_i O\in b} \frac{1}{\sin\tau_i} \;=\; y(Z)$$

valid for every realization Z; it directly follows from (4.5). By(3.5), the function $y(z)$ is summable, hence

$$\Pi_{vv}(C) \;=\; \int x(Z)\, d\mathbf{P}. \tag{4.7}$$

As a by-product we get (4.4), and the proof ends in the same way as in Lemma 4.1. □

Further Remarks. First we briefly present the results for the case of horizontal windows (see Fig.1) that can be easily proved by the Translational analysis method of Section 2 above.

There is a counterpart of Lemma 4.1 that states the existence of the limit

$$\Pi_h(C) \;=\; \lim_{\varepsilon\to 0} \frac{\mathbf{P}\left[C\cap \binom{h}{1}\right]}{\mathbf{P}\binom{h}{1}}$$

for every line process $Z \in D1$ and every $C \in \nabla_0$, where h is one of the two horizontal windows h_1, h_2. The analog of (4.3) happens to be

$$\Pi_h(C) \;=\; \int d\mathbf{P} \sum_{\chi_i} \sin(\chi_i,\gamma) \int_{\chi_i} I_C(t_u Z)\, du. \tag{4.8}$$

From (4.3) and (4.8) we conclude that

$$\Pi_v(C) \;=\; [\lambda(v)]^{-1}\,\mathbf{E}_h\,|\cot\psi|\,I_C(Z), \tag{4.9}$$

where $\mathbf{E}_h$ stands for the expectation with respect to measure Π_h, while $I_C(Z)$ is the usual indicator function of the event $C \in \nabla$, and ψ is the direction of the line through the endpoint of γ that exists with Π_h-probability 1.

The counterpart of Lemma 4.2 states that for $Z \in D2$

$$\mathbf{P}(H) \;=\; S_{hh}(\alpha)\,\varepsilon^2 \;+\; o(\varepsilon^2),$$

where $H = \begin{pmatrix} h_1 & h_2 \\ 1 & 1 \end{pmatrix}$, and the existence of the limit

$$\Pi_{hh}(C) \;=\; \lim_{\varepsilon\to 0}\frac{\mathbf{P}(C\cap H)}{\mathbf{P}(H)}.$$

The analogs of (4.3) and (4.8) for $\Pi_{hh}(C)$ and $\Pi_{vv}(C)$ we write down under an additional assumption that (4.6) holds with probability one.

Given a test interval γ, let a line process $Z \in D2$ satisfy (4.6) with probability 1. Then

$$\Pi_{hh}(C) \;=\; \int d\mathbf{P}\sum_{t_i\in\mathbf{b}}\frac{\sin(g_1^{(i)},\gamma)\,\sin(g_2^{(i)},\gamma)|}{\sin\tau_i}\,I_C(t_iZ),$$

$$\Pi_{vv}(C) \;=\; \int d\mathbf{P}\sum_{t_i\in\mathbf{b}}\frac{|\cos(g_1^{(i)},\gamma)|\,|\cos(g_2^{(i)},\gamma)|}{\sin\tau_i}\,I_C(t_iZ),$$

implying

$$\Pi_{vv}(C) \;=\; [S_{vv}(\alpha)]^{-1}\,\mathbf{E}_{hh}|\cot\psi_1|\,|\cot\psi_2|\;I_C(Z), \tag{4.10}$$

where S_{vv} is given by (3.3), $\mathbf{E}_{hh}$ stands for the expectation with respect to Π_{hh}, and ψ_1, ψ_2 are the directions of the two lines through the two endpoints of γ that exist with Π_{hh}-probability 1.

5. "Palm equations" for vertical and horizontal windows

For the probability distribution $\mathbf{P}$ of a translation invariant Z, we reasonably write

$$p_k(l,\alpha) \;=\; \mathbf{P}\begin{pmatrix}\gamma\\ k\end{pmatrix},$$

where l is the length and α is the (horizontal) direction of γ. The Lemmas 3,4 refer to the following differential operators acting on $p_k(l,\alpha)$.

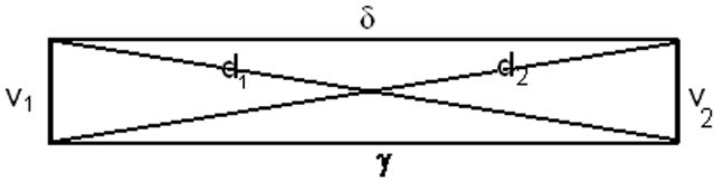

Рис. 3

Let the segments γ, σ, d_1 and d_2 be as on Fig. 3. After appropriate choice of positive rotation in the space of planar directions α we have

$$\lim_{l\to 0}(\lambda(v)\,|v|)^{-1}\left[\mathbf{P}\binom{d}{k} - \mathbf{P}\binom{\gamma}{k}\right] = \frac{\partial p_k(l,\alpha)}{\partial \alpha}.$$

By formal Taylor expansion we find ($\delta =$ the angle between γ and d_2 is $\varepsilon\, l^{-1} + o(\varepsilon)$)

$$\lim_{\varepsilon\to 0}\varepsilon^{-2}\left[\mathbf{P}\binom{d_1}{k} - \mathbf{P}\binom{\gamma}{k} - \mathbf{P}\binom{\sigma}{k} + \mathbf{P}\binom{d_2}{k}\right]$$

$$= \lim_{\varepsilon\to 0}\frac{p_k(\sqrt{l^2+\varepsilon^2}, \alpha+\delta) - 2p_k(l,\alpha) + p_k(\sqrt{l^2+\varepsilon^2}, \alpha-\delta)}{\varepsilon^2}$$

$$= l^{-1}\frac{\partial p_k(l,\alpha)}{\partial l} + l^{-2}\frac{\partial^2 p_k(l,\alpha)}{\partial \alpha^2}. \tag{5.1}$$

Generally speaking, both identities are valid under certain smoothness conditions imposed on the function $p_k(l,\alpha)$. However the first identity always holds for $Z \in D1$, while the condition $Z \in D2$ does not guarantee (5.1). So for $Z \in D2$ we speak about additional *smoothness condition* (5.1).

In Lemma 5.3 that follows we can choose d, v, A, O to be either d_1, v_1, A_1, O_1 or d_2, v_2, A_2, O_2 as on Figs. 2, 3.

Lemma 5.1. *If $Z \in D1$, then the first order vertical window Palm equation is valid:*

$$[\lambda(v)\,l]^{-1}\frac{\partial p_k(l,\alpha)}{\partial \alpha} =$$

$$\Pi_v\left[\binom{\gamma}{k-1}\cap A\right] + \Pi_v\left[\binom{\gamma}{k}\cap O\right] - \Pi_v\left[\binom{\gamma}{k}\cap A\right] - \Pi_v\left[\binom{\gamma}{k-1}\cap O\right]. \tag{5.2}$$

Proof. We represent $\binom{\gamma}{k}$ as a union of mutually exclusive events

$$\binom{\gamma}{k} = \bigcup_{j\geq 0} \begin{pmatrix} \gamma & v \\ k & j \end{pmatrix}.$$

By (3.2), when $|v| \to 0$

$$\sum_{j>2} \mathbf{P} \begin{pmatrix} \gamma & v \\ k & j \end{pmatrix} = o(|v|),$$

and therefore

$$\mathbf{P} \binom{\gamma}{k} = \mathbf{P} \begin{pmatrix} \gamma & v \\ k & 0 \end{pmatrix} + \mathbf{P} \begin{pmatrix} \gamma & v \\ k & 1 \end{pmatrix} + o(|v|).$$

Similarly

$$\mathbf{P} \binom{d}{k} = \mathbf{P} \begin{pmatrix} d & v \\ k & 0 \end{pmatrix} + \mathbf{P} \begin{pmatrix} d & v \\ k & 1 \end{pmatrix} + o(|v|).$$

Because γ, d and v are sides of a triangle, a set equality

$$\begin{pmatrix} \gamma & v \\ k & 0 \end{pmatrix} = \begin{pmatrix} d & v \\ k & 0 \end{pmatrix}$$

follows. By subtraction

$$\mathbf{P} \binom{d}{k} - \mathbf{P} \binom{\gamma}{k} = \mathbf{P} \begin{pmatrix} d & v \\ k & 1 \end{pmatrix} - \mathbf{P} \begin{pmatrix} \gamma & v \\ k & 1 \end{pmatrix} + o(|v|). \tag{5.3}$$

By (3.2) we have

$$\mathbf{P} \begin{pmatrix} \gamma & v \\ k & 1 \end{pmatrix} = \mathbf{P} \begin{pmatrix} \gamma & v \\ k-1 & 1\gamma \end{pmatrix} + \mathbf{P} \begin{pmatrix} \gamma & v \\ k & 1d \end{pmatrix} =$$

$$= \lambda(v)\,|v|\,\Pi_v \left[\binom{\gamma}{k-1} \cap A \right] + \lambda(v)\,|v|\,\Pi_v \left[\binom{\gamma}{k} \cap O \right] + o(|v|),$$

and similarly

$$\mathbf{P} \begin{pmatrix} d & v \\ k & 1 \end{pmatrix} = \mathbf{P} \begin{pmatrix} d & v \\ k-1 & 1\gamma \end{pmatrix} + \mathbf{P} \begin{pmatrix} d & v \\ k & 1d \end{pmatrix} + o(|v|) =$$

$$= \lambda(v)\,|v|\,\Pi_v \left[\binom{\gamma}{k} \cap O \right] + \lambda(v)\,|v|\,\Pi_v \left[\binom{\gamma}{k-1} \cap A \right] + o(|v|).$$

It remains to substitute this into (5.3), divide the result by $|v|$ and calculate the limits. This proves Lemma 5.3. $\square$

In the next lemma we use conditioning by the event V_1 defined in Section 3. Intuitively, $\pi_k(l, \alpha)$ is the conditional probability of $\binom{\gamma}{k}$, conditional upon the event "γ belongs to a line from Z". We say that a line process $Z \in D2$ is orderly if it satisfies (3.4).

Lemma 5.2. *If $Z \in D2$ is orderly and the smoothness condition (5.1) is satisfied, then the limits*

$$\pi_k(l,\alpha) \;=\; \lim_{|h|\to 0} \frac{\mathbf{P}\left[\binom{\gamma}{k} \cap V_1\right]}{\mathbf{P}(V_1)}.$$

exist for $k = 0, 1, 2, ...$ and satisfy the second order vertical window Palm equation

(5.4)
$$\frac{\partial p_k(l,\alpha)}{\partial l} + l^{-1}\frac{\partial^2 p_k(l,\alpha)}{\partial \alpha^2} \;=\; 2\,\rho_1(\alpha)\;[\pi_{k-1}(l,\alpha) \;-\; \pi_k(l,\alpha)] + S_{vv}(\alpha)\,l\;[y_k - 2y_{k-1} + y_{k-2}]\,.$$

where $\pi_{-1}(l,\alpha) \;=\; 0$, while for $k = 0, 1, 2, ...$

$$y_k \;=\; \Pi_{vv}\left[\binom{\gamma}{k} \cap A_1 \cap O_2\right] + \Pi_{vv}\left[\binom{\gamma}{k} \cap O_1 \cap A_2\right] -$$
$$- \,\Pi_{vv}\left[\binom{\gamma}{k} \cap A_1 \cap A_2\right] - \Pi_{vv}\left[\binom{\gamma}{k} \cap O_1 \cap O_2\right],$$

with $y_{-1} = y_{-2} = 0$.

Proof. On the set $[v_1] \cap [v_2]$= lines that hit both v_1 and v_2 we define

$\chi = \chi(g) =$ the segment cut from $g \in \mathbf{G}$ by v_1 and v_2,

$\binom{\chi}{k}_*$ = the lines $g \in [v_1] \cap [v_2]$ for which $\chi(g)$ is hit by k lines from realization Z,

$I_{\binom{\chi}{k}_*}(Z,g)$ = a usual indictor function defined in the (Z,g)-space, $g \in [v_1] \cap [v_2]$.

The function

$$I_k(Z,g) \;=\; I_{[v_1]\cap[v_2]}(g)\, I_{\binom{\chi}{k}_*}(Z,g)$$

we integrate first with respect to dg and then with respect to probability distribution $\mathbf{P}$ of Z. By interchange of the integration order

$$\int d\mathbf{P} \int I_k(Z,g)\,dg \;=\; \int_{[v_1]\cap[v_2]} \mathbf{P}\binom{\chi}{k}\,dg.$$

Both v_i being vertical, as ε tends to 0 we have $\chi \;=\; l + O(\varepsilon^2)$. Therefore (see around (3.4))

$$\int_{[v_1]\cap[v_2]} \mathbf{P}\binom{\chi}{k}\,dg \;=\; \mathbf{P}\binom{\gamma}{k}\,\frac{\varepsilon^2}{l} \;+\; o(\varepsilon^2). \tag{5.5}$$

On the other hand, the set $[v_1] \cap [v_2] \cap \binom{\chi}{k}_* \subset \mathbf{G}$ belongs to the ring $r\{P_i\}$, where we take

$\{P_i\} = \{q_i\} \cup \{p_i\}$ with $\{q_i\} =$ the four endpoints of v_1 and v_2 and $\{p_i\} =$ points where the lines from Z hit v_1, v_2, γ_1 or σ_2. Therefore the integral $\int I_k(Z,g)\,dg$ can be decomposed according to (2.2). So we need the combinatorial coefficients u'_{ij} and u''_{ij} for the set $[v_1] \cap [v_2] \cap \binom{\chi}{k}_*$. As explained in Section 2 above, these coefficients have purely combinatorial nature; they have been put down in [4], pages 262-267 in the section "Averaging a combinatorial decomposition"of Chapter 10.

For instance for the 2-sets $\{P_i, P_j\}$ that belong to the closure of γ (or σ) we have

$u'_{ij} = 0$ if at least one point from $\{P_i, P_j\}$ belongs to the interior of γ (or σ) and

$u''_{ij} = -I_{\binom{\gamma}{k}_*}(Z,\gamma)$ if $\{P_i, P_j\}$ are the two endpoints of γ (the same for σ).

Also, for the 2-sets $\{P_i, P_j\}$ that belong to the closure of d_1 (or d_2) we have

$u'_{ij} = 0$ if at least one point from $\{P_i, P_j\}$ belongs to the interior of d_1 (or d_2) and

$u'_{ij} = I_{\binom{d_1}{k}_*}(Z, d_1)$ if $\{P_i, P_j\}$ are the two endpoints of d_1 (the same for d_2).

Thus the joint contribution of the mentioned 2-sets after averaging (i.e. after integration with respect to $\mathbf{P}$) happens to be, compare with (5.1)

$$\sqrt{l^2+\varepsilon^2}\,\mathbf{P}\binom{d_1}{k} + \sqrt{l^2+\varepsilon^2}\,\mathbf{P}\binom{d_2}{k} - l\,\mathbf{P}\binom{\gamma}{k} - l\,\mathbf{P}\binom{\sigma}{k} =$$

$$\left[l^{-1}p_k(l,\alpha) + \frac{dp_k(l,\alpha)}{dl} + l^{-1}\frac{\partial^2 p_k(l,\alpha)}{\partial\alpha^2}\right]\varepsilon^2 + o(\varepsilon^2)$$

Due to (5.5), after dividing by ε^2 and calculating the limit we get the left-hand side of the equation (5.4). The sum of the remaining members is responsible for the right-hand side of (5.4). A detailed derivation contained in [4], pages 262-267 leads to (5.5). We note that the Euclidean motions invariance of $\mathbf{P}$ assumed in [4] automatically guarantees "orderly"behavior of $\mathbf{P}$ in the sense of (3.4) and yields

$$\frac{\partial^2 p_k(l,\alpha)}{\partial\alpha^2} = 0.$$

The proof is complete. □

We remark that an alternative proof of Lemma 5.2 can be found in complete detail in [9]. It uses representations of the events $\binom{d_1}{k}$, $\binom{\gamma}{k}$, $\binom{\sigma}{k}$ and $\binom{d_2}{k}$ as unions of the events of the type $\begin{pmatrix} U & v_1 & v_2 \\ k & k_1 & k_2 \end{pmatrix}$ and an analysis in the style of the proof of Lemma 5.1.

The similar results for $\binom{\gamma}{k}$ in case instead of vertical windows we use horizontal h_1 and h_2 are as follows. If $Z \in D1$, then

$$\frac{d\,p_k(l,\alpha)}{d\,l} = \lim_{l\to 0} \varepsilon^{-1}\left[\mathbf{P}\binom{\gamma\cup h}{k} - \mathbf{P}\binom{\gamma}{k}\right] = \lambda(\alpha)\left[\Pi_h\binom{\gamma}{k-1} - \Pi_h\binom{\gamma}{k}\right]. \tag{5.6}$$

If $Z \in D2$, then with h_1 and h_1 as on Fig. 1

$$\frac{\partial^2 p_k(l,\alpha)}{\partial l^2} = \lim_{l\to 0} \varepsilon^{-2}\left[\mathbf{P}\binom{\gamma\cup h_1\cup h_2}{k} - \mathbf{P}\binom{\gamma\cup h_1}{k} - \mathbf{P}\binom{\gamma\cup h_2}{k} + \mathbf{P}\binom{\gamma}{k}\right] =$$

$$S_{hh}(\alpha)\left[\Pi_{hh}\binom{\gamma}{k-2} - 2\,\Pi_{hh}\binom{\gamma}{k-1} + \Pi_{hh}\binom{\gamma}{k}\right]. \tag{5.6}$$

By summation, (5.4) implies $2\,\rho_1(\alpha) = \lambda + \lambda''$; hence (5.4) can be called a *decomposition of W.Blaschke's relation* (3.6).

6. Factorization models

The product events $\binom{\gamma}{k} \cap A_2$ and $\binom{\gamma}{k} \cap A_1 \cap A_2$ that appear in Lemmas 3,4 suggest the question: what happens in the simplest case, where on these events the probabilities Π_v and Π_{vv} factorize?

Definition 6.1. *We say that $Z \in D2$ satisfying conditions of Lemma 5.2 belongs to Acute-Obtuse Independence class (or is an AOI model) on (any) test line g_0 of direction α if for every segment $\gamma \subset g_0$*

$$\Pi_v\left[\binom{\gamma}{k}\cap A_2\right] = \Pi_v\binom{\gamma}{k}\,\Pi_v(A_2), \quad \Pi_v\left[\binom{\gamma}{k}\cap O_2\right] = \Pi_v\binom{\gamma}{k}\,\Pi_v(O_2),$$

$$\Pi_{vv}\left[\binom{\gamma}{k}\cap A_1\cap A_2\right] = \Pi_{vv}\binom{\gamma}{k}\,\Pi_{vv}(A_1)\,\Pi_{vv}(A_2),$$

$$\Pi_{vv}\left[\binom{\gamma}{k}\cap O_1\cap O_2\right] = \Pi_{vv}\binom{\gamma}{k}\,\Pi_{vv}(O_1)\,\Pi_{vv}(O_2),$$

$$\Pi_{vv}\left[\binom{\gamma}{k}\cap A_1\cap O_2\right] = \Pi_{vv}\binom{\gamma}{k}\,\Pi_{vv}(A_1)\Pi_{vv}(O_2),$$

$$\Pi_{vv}\left[\binom{\gamma}{k}\cap O_1\cap A_2\right] = \Pi_{vv}\binom{\gamma}{k}\,\Pi_{vv}(O_1)\Pi_{vv}(A_2).$$

All Poisson $Z \in D2$ and their mixtures are in fact AOI models. Given stationary marked point process $\{x_i, \psi_i\}$ on a test line g_0 (see Section 3), to each x_i we apply the transformation $\psi_i \to \pi - \psi_i$ or keep ψ_i intact, according to independent tosses of a coin. In case the limit of $\{x_i, \psi_i\}_d$, as $d \to \infty$ happens to be a line process $Z \in D_2$, then the letter will necessarily be an AOI model. Unsolved problem: can this approach produce AOI models beyond the class of Poisson mixtures, see [15]?

A model $Z \in$AOI is Symmetrical if always

$$\Pi_v(A) = \frac{1}{2}, \quad \Pi_v(O) = \frac{1}{2},$$

$$\Pi_{vv}(A_1) = \Pi_{vv}(A_2) = \frac{1}{2}, \quad \Pi_{vv}(O_1) = \Pi_{vv}(O_2) = \frac{1}{2}.$$

In the Symmetrical AOI case (5.2) writes

$$\frac{\partial p_k(l,\alpha)}{\partial \alpha} = 0, \tag{6.1}$$

while (5.4) reduces to

$$\frac{\partial p_k(l,\alpha)}{\partial l} + l^{-1}\frac{\partial^2 p_k(l,\alpha)}{\partial \alpha^2} = 2\,\rho_1(\alpha)\,\left[\pi_{k-1}(l,\alpha) - \pi_k(l,\alpha))\right]. \tag{6.2}$$

It turns out that if additionally, the model Z is directionally stable at α, i.e. if

$$\frac{\partial^2 p_k(l,\alpha)}{\partial \alpha^2} = 0 \quad \text{for} \quad k =, 1, 2, ..., \tag{6.3}$$

then the probabilities $p_k(l,\alpha)$ and $\pi_k(l,\alpha)$ satisfy the equations system

$$\frac{\partial p_k(l,\alpha)}{\partial l} = 2\,\rho_1(\alpha)\,\left[\pi_{k-1}(l,\alpha) - \pi_k(l,\alpha)\right].$$

The additional condition

$$p_k(l,\alpha) = \pi_k(l,\alpha) \;\text{ and }\; \frac{\partial^2 p_k(l,\alpha)}{\partial \alpha^2} = 0, \quad k = 0,1,2,..., \quad l \in (0,\infty) \tag{6.4}$$

then implies Poissonity of the probabilities $p_k(l,\alpha)$ (with parameter $\lambda(\alpha)l$, see (3.6)). Another model that implies Poissonity of the probabilities $p_k(l,\alpha)$ was considered in [9]; it is defined by the *factorization assumptions* F1,F2 and F3 as below based on above formulae for horizontal windows.

By a remarkable interplay of signs, (4.9) and (5.2) yield

$$\Pi_v\left(\binom{\gamma}{k} \cap A\right) - \Pi_v\left(\binom{\gamma}{k} \cap O\right) = \frac{1}{\lambda(v)}\mathbf{E}_h\, I\binom{\gamma}{k}\cot\psi, \tag{6.5}$$

where I stands for the indicator function of the corresponding event (dependence on Z suppressed). By (4.10) and a similar signs interplay, for the quantities y_k in Lemma 4 we get

$$y_k = [S_{vv}(\alpha)]^{-1}\mathbf{E}_{hh} I\binom{\gamma}{k}\cot\psi_1\cot\psi_2. \tag{6.6}$$

Assumption F1: $\cot\psi$ and I_C as in (4.9), for $C = \binom{\gamma}{k}$ are Π_h-uncorrelated, i.e.

$$\mathbf{E}_h I \binom{\gamma}{k} \cot\psi = \Pi_h \binom{\gamma}{k} \mathbf{E}_h \cot\psi.$$

We have

$$\mathbf{E}_h \cot\psi = (\lambda(\alpha))^{-1} \int \cos\psi f_1(\phi) d\psi = (\lambda(\alpha))^{-1}\lambda'(\alpha)$$

with $\lambda'(\alpha)$ denoting the first derivative in α. By (5.6)

$$\lambda(\alpha)\left[\Pi_h \binom{\gamma}{k-1} - \Pi_h \binom{\gamma}{k}\right] = \frac{\partial p_k(t,\alpha)}{\partial t},$$

so F1 implies the differential equation

$$\frac{\partial p_k(l,\alpha)}{\partial \alpha} = t \cdot (\lambda(\alpha))^{-1}\lambda'(\alpha)\frac{\partial p_k(l,\alpha)}{\partial l}.$$

By standard method of characteristics, its general solution has the form

$$p_k(l,\alpha) = q_k(\lambda(\alpha)l), \tag{6.7}$$

where $q_k(\cdot)$ is some function of one argument.

Assumption F2: the random variables $\cot\psi_1$ $\cot\psi_2$ and $I\binom{\gamma}{k}$ as in (4.10) are Π_h-uncorrelated, i.e.

$$\mathbf{E}_{hh} I \binom{\gamma}{k} \cot\psi_1 \cot\psi_2 = \Pi_{hh} \binom{\gamma}{k} \mathbf{E}_{hh} \cot\psi_1 \cot\psi_2.$$

Due to (5.7) and (4.10), this brings (5.4) to the form

$$l\,\frac{\partial p_k(l,\alpha)}{\partial l} + \frac{\partial^2 p_k(l,\alpha)}{\partial \alpha^2} = 2\,\rho_1(\alpha)\left[\pi_{k-1}(l,\alpha) - \pi_k(l,\alpha)\right] + l^2\frac{\partial^2 p_k(l,\alpha)}{\partial l^2}\ \mathbf{E}_{hh}\cot\psi_1\cot\psi_2.$$

By a direct substitution of (6.7) and (3.6) we get, that under F1 and F2

$$(\lambda + \lambda'')\, q_k' + l[(\lambda')^2 - \lambda^2\ \mathbf{E}_{hh}\cot\psi_1\cot\psi_2]\, q_k'' = (\lambda + \lambda'')\,[\pi_{k-1}(l,\alpha) - \pi_k(l,\alpha)]. \tag{6.8}$$

Assumption F3:

$$\mathbf{E}_{hh}\cot\psi_1\cot\psi_2 = \mathbf{E}_h\cot\psi_1 \mathbf{E}_h \cot\psi_2 = [\lambda'(\alpha)]^2[\lambda(\alpha)]^{-2}.$$

Under F3 the equation (6.8) transforms to

$$q_k' = \pi_{k-1}(l,\alpha) - \pi_k(l,\alpha). \tag{6.9}$$

This infinite system of equations is easily solved if we assume (6.4)(which in [9] was called *sufficient mixing* condition). Under (6.4), the solution of (6.9) satisfying natural initial conditions $q_0(0) = 1$ and $q_k(0) = 0$ for $k > 0$ yields Poisson probabilities with unit parameter $q_k(t) = \frac{t^k}{k!}\, e^{-t}$. We summarize: *if the three factorization properties F1, F2 and F3 are valid for any direction α and length t, then the property of sufficient mixing (6.4) implies that $p_k(t,\alpha)$ are Poisson probabilities with parameter $\lambda(\alpha)t$.*

7. A model-free result

We start with definition of the quantities $C(a)$ and $S(a)$ posing in the theorem that follows. We put

$$C(\alpha) \;=\; \Pi_{vv}\,[A_1 \cap O_2] + \Pi_{vv}\,[O_1 \cap A_2] \;-\; \Pi_{vv}\,[A_1 \cap A_2] - \Pi_{vv}\,[O_1 \cap O_2]\,,$$

and express $C(\alpha)$ as a double integral over $(0,\pi)\times(0,\pi)$. We identify the direction ϕ_1 with the angle ψ_1 and direction ϕ_2 with the angle ψ_2, assuming that the angle ψ_1 is measured from α (= direction of γ) in the *clockwise* direction while the angle ψ_2 is measured from α in the *anticlockwise* direction. Under this convention for both $i = 1, 2$ we get (see Fig. 2)

$$\{\psi_i \in A_i\} = (0, \frac{\pi}{2}) \quad \text{and} \quad \{\psi_i \in O_i\} = (\frac{\pi}{2}, \pi).$$

This yields

$$C(\alpha) \;=\; \int_0^\pi \int_0^\pi \rho_2(\phi_1,\phi_2)\, \cos\psi_1\, \cos\psi_2\, d\phi_1\, d\phi_2.$$

As for $S(\alpha)$, we put $S(\alpha) = S_{hh}(\alpha)$, see (3.3), i.e.

$$S(\alpha) \;=\; \int_0^\pi \int_0^\pi \rho_2(\phi_1,\phi_2)\, \sin\psi_1\, \sin\psi_2\, d\phi_1\, d\phi_2.$$

If we additionally assume rotation invariance, then $S = S(\alpha)$, $C = C(\alpha)$, $\rho_1 = \rho_1(\alpha)$, $\Lambda = \Lambda(\alpha)$ are constants and by (3.6) $2\,\rho_1 = \lambda$, where λ is the intensity of hits on test lines of any direction. By τ we denote the angle between the directions ϕ_1 and ϕ_2.

Theorem 7.1. *For every orderly-(3.4) line process $Z \in D2$ that satisfies the smoothness condition (5.1) we have*

$$4\,\rho_1(\alpha)\,\Lambda(\alpha) \;=\; 2\,S(\alpha) + \frac{\partial^2 S(\alpha)}{\partial\alpha^2} \;-\; 2\,C(\alpha). \tag{7.1}$$

If $Z \in D2$ happens to be rotation invariant then $S(\alpha) = S$ and $C(\alpha) = C$ are constants:

$$S \;=\; \int_0^\pi \rho_2(\tau)\,(\pi-\tau)\,\cos\tau\,d\tau \;+\; \int_0^\pi \rho_2(\tau)\,\sin\tau\,d\tau, \tag{7.2}$$

$$C = \int_0^\pi \rho_2(\tau)\,(\pi-\tau)\,\cos\tau\,d\tau - \int_0^\pi \rho_2(\tau)\,\sin\tau\,d\tau, \tag{7.3}$$

and the product of the constant values $\Lambda = \Lambda(\alpha)$ *and* $\lambda = \lambda(\alpha)$ *equals*

$$\lambda\,\Lambda = \frac{2}{\pi}\,\lambda_Q. \tag{7.4}$$

Proof. For every line process $Z \in D2$ the second moment

$$\sum k^2\,p_k(l,\alpha)$$

can be found by integration in the space $\mathbf{G}\times\mathbf{G}$:

$$\sum k^2\,p_k(l,\alpha) = \int_{[\gamma]}\int_{[\gamma]} \rho_2(\phi_1,\phi_2)\,dg_1dg_2 + \lambda(\alpha)\,l = S(\alpha)\,l^2 + \lambda(\alpha)\,l, \tag{7.5}$$

where $[\gamma] = \{g \in \mathbf{G} : g \text{ hits } \gamma\}$ while $\lambda(\alpha)\,l$ is the contribution of the diagonal set $\{g_1 = g_2\}$. We have the identities

$$\sum_{k=0}^{\infty} k^2\,[\pi_{k-1}(l,\alpha) - \pi_k(l,\alpha)] = 2\sum_{k=1}^{\infty} k\,\pi_k(l,\alpha) + \sum_{k=0}^{\infty}\pi_k(l,\alpha) = 2\,\Lambda(\alpha)\,l + 1,$$

and

$$\sum_{k=0}^{\infty} k^2\,[y_k - 2y_{k-1} + y_{k-2}] = 2\sum_{k=0}^{\infty} y_k =$$
$$\Pi_{vv}\,[A_1\cap A_2] + \Pi_{vv}\,[O_1\cap O_2] - \Pi_{vv}\,[A_1\cap O_2] - \Pi_{vv}\,[O_1\cap A_2] =$$
$$= 2\,(S_{vv})^{-1}\,C(\alpha),$$

From (7.5) we get

$$2\,S(\alpha) + \frac{\lambda(\alpha)}{l} + \frac{\partial^2 S(\alpha)}{\partial\alpha^2} + \frac{1}{l}\frac{\partial^2\lambda(\alpha)}{\partial\alpha^2} =$$
$$2\,\frac{\rho_1(\alpha)}{l}\,[2\,\Lambda(\alpha)\,l + 1] + 2\,C(\alpha).$$

Because of the identity (3.6) the above is equivalent to (7.1).

The Case of Rotation Invariant Z. In that case (7.1) reduces to

$$S = \lambda\,\Lambda + C. \tag{7.6}$$

For rotation-invariant Z the calculation of S and C can be reduced to one-dimensional integration, for in that case

$$\rho_2(\phi_1,\phi_2) = \rho_2(\tau),$$

where $\tau > 0$ is the angle between directions ϕ_1 and ϕ_2. The calculation for S runs as follows.

We consider two cases $\psi_2 = \psi_1 + \tau$ or $\psi_2 = \psi_1 - \tau$, so

$$S \;=\; S_1 \;+\; S_2,$$

where

$$S_1 = \int_0^\pi \sin\psi\, d\psi \int_0^{\pi-\psi} \sin(\psi+\tau)\,\rho_2(\tau)\,d\tau \;=\; S_{11} \;+\; S_{12},$$

$$S_2 = \int_0^\pi \sin\psi\, d\psi \int_0^{\psi} \sin(\psi-\tau)\,\rho_2(\tau)\,d\tau \;=\; S_{21} \;-\; S_{22},$$

where in turn

$$S_{11} \;=\; \int_0^\pi \sin^2\psi\, d\psi \int_0^{\pi-\psi} \cos\tau\,\rho_2(\tau)\,d\tau \;=\; \int_0^\pi \cos\tau\,\rho_2(\tau)\,d\tau \int_0^{\pi-\tau} \sin^2\psi\, d\psi,$$

$$S_{12} \;=\; \int_0^\pi \sin\psi\,\cos\psi\, d\psi \int_0^{\pi-\psi} \sin\tau\,\rho_2(\tau)\,d\tau \;=\; \int_0^\pi \sin\tau\,\rho_2(\tau)\,d\tau \int_0^{\pi-\tau} \sin\psi\,\cos\psi\, d\psi,$$

$$S_{21} \;=\; \int_0^\pi \sin^2\psi\, d\psi \int_0^{\psi} \cos\tau\,\rho_2(\tau)\,d\tau \;=\; \int_0^\pi \cos\tau\,\rho_2(\tau)\,d\tau \int_\tau^{\pi} \sin^2\psi\, d\psi,$$

$$S_{22} \;=\; \int_0^\pi \sin\psi\,\cos\psi\, d\psi \int_0^{\psi} \sin\tau\,\rho_2(\tau)\,d\tau \;=\; \int_0^\pi \sin\tau\,\rho_2(\tau)\,d\tau \int_\tau^{\pi} \sin\psi\,\cos\psi\, d\psi.$$

The interior integrals are easily calculated, so we get

$$S_{11} \;=\; S_{21} \;=\; \int_0^\pi \;\rho_2(\tau)\;\cos\tau\,\left[\frac{\pi-\tau}{2} \;+\; \frac{\sin 2\tau}{4}\right]\,d\tau,$$

$$S_{12} \;=\; \frac{1}{2}\int_0^\pi \;\rho_2(\tau)\;\sin^3\tau\,d\tau,$$

$$S_{22} \;=\; -\,\frac{1}{2}\int_0^\pi \;\rho_2(\tau)\;\sin^3\tau\,d\tau.$$

Since $S \;=\; S_{11} \;+\; S_{12} \;+\; S_{21} \;-\; S_{22}$, we get

$$S \;=\; \int_0^\pi \;\rho_2(\tau)\;\cos\tau\,\left[\pi-\tau \;+\; \frac{\sin 2\tau}{2}\right]\,d\tau \;+\; \int_0^\pi \;\rho_2(\tau)\;\sin^3\tau\,d\tau,$$

which is equivalent to (7.2).

Similarly, the term C writes as

$$C \;=\; C_1 \;+\; C_2,$$

where

$$C_1 \;=\; \int_0^\pi \cos\psi\, d\psi \int_0^{\pi-\psi} \cos(\psi+\tau)\,\rho_2(\tau)\,d\tau \;=\; C_{11} \;-\; C_{12},$$

$$C_2 \;=\; \int_0^\pi \cos\psi\, d\psi \int_0^{\psi} \cos(\psi-\tau)\,\rho_2(\tau)\,d\tau \;=\; C_{21} \;+\; C_{22}.$$

We have

$$C_{11} \;=\; \int_0^\pi \cos^2\psi\, d\psi \int_0^{\pi-\psi} \cos\tau\,\rho_2(\tau)\,d\tau \;=\; \int_0^\pi \cos\tau\,\rho_2(\tau)\,d\tau \int_0^{\pi-\tau} \cos^2\psi\, d\psi,$$

$$C_{12} = \int_0^\pi \sin\psi\,\cos\psi\,d\psi \int_0^{\pi-\psi} \sin\tau\,\rho_2(\tau)\,d\tau = \int_0^\pi \sin\tau\,\rho_2(\tau)\,d\tau \int_0^{\pi-\tau} \sin\psi\,\cos\psi\,d\psi,$$

$$C_{21} = \int_0^\pi \cos^2\psi\,d\psi \int_0^{\psi} \cos\tau\,\rho_2(\tau)\,d\tau = \int_0^\pi \cos\tau\,\rho_2(\tau)\,d\tau \int_\tau^{\pi} \cos^2\psi\,d\psi,$$

$$C_{22} = \int_0^\pi \sin\psi\,\cos\psi\,d\psi \int_0^{\psi} \sin\tau\,\rho_2(\tau)\,d\tau = \int_0^\pi \sin\tau\,\rho_2(\tau)\,d\tau \int_\tau^{\pi} \sin\psi\,\cos\psi\,d\psi.$$

We find

$$C_{11} = C_{21} = \int_0^\pi \rho_2(\tau)\,\cos\tau \left[\frac{\pi-\tau}{2} - \frac{\sin 2\tau}{4}\right] d\tau,$$

$$C_{12} = \frac{1}{2}\int_0^\pi \rho_2(\tau)\,\sin^3\tau\,d\tau,$$

$$C_{22} = -\frac{1}{2}\int_0^\pi \rho_2(\tau)\,\sin^3\tau\,d\tau,$$

and since $C = C_{11} - C_{12} + C_{21} + C_{22}$, finally

$$C = \int_0^\pi \rho_2(\tau)\,\cos\tau \left[\pi - \tau - \frac{\sin 2\tau}{2}\right] d\tau - \int_0^\pi \rho_2(\tau)\,\sin^3\tau\,d\tau,$$

which is equivalent to (7.3).

The last assertion of the theorem 7.1 follows from

$$\lambda\,\Lambda = S - C = 2\int_0^\pi \rho_2(\tau)\,\sin\tau\,d\tau = \frac{2}{\pi}\,\lambda_Q, \tag{7.7}$$

where λ_Q is the intensity of the vertex process $\{Q_i\}$. The proof of the theorem 7.1 is complete.

Let us consider the random vertex process $\{Q_i\}$ we discussed in Section 3. We define a vertex shape as an ordered pair (ϕ_1, ϕ_2) of planar directions to the two lines from Z that meet at a vertex Q_i. Given some direction α, with the typical vertex in $\{Q_i\}$ we associate two random variables

$$s(\alpha) = \frac{\sin(\alpha,\phi_1)\,\sin(\alpha,\phi_2)}{\sin\tau} \quad \text{and} \quad c(\alpha) = \frac{\cos(\alpha,\phi_1)\,\cos(\alpha,\phi_2)}{\sin\tau},$$

where the angles we measure in a way to have $(\alpha,\phi_1) = \psi_1$ and $(\alpha,\phi_2) = \psi_2$, see above, while τ is the angle between directions ϕ_1 and ϕ_2. The following Corollary follows directly from (3.6) and the Theorem 7.1 just proved.

Corollary 7.1. *The conditional intensity $\Lambda(\alpha)$ depends on the translational shape of the typical vertex in $\{Q_i\}$. In fact for every line process $Z \in D2$ that satisfies the conditions of the above theorem*

$$\Lambda(\alpha) = \frac{\lambda_Q}{\lambda(\alpha) + \lambda''(\alpha)} \left[\mathbf{E}_Q\,s(\alpha) + \frac{\partial^2}{\partial\alpha^2}\mathbf{E}_Q\,s(\alpha) - \mathbf{E}_Q\,c(\alpha)\right].$$

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