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SUMMABILITY OF MULTIPLE TRIGONOMETRIC FOURIER SERIES BY LINEAR METHODS

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Abstract. The paper deals with the problem of estimation of deviations of functions of several variables from linear means of their multiple trigonometric Fourier series. An approach of reducing this problem to the corresponding problem for functions of single variable is developed.

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1. INTRODUCTION

Let R^s $(s \geq 1)$ be the s-dimensional Euclidean space and let N^s be the set of lattice points in R^s. By $x = (x_1, \ldots, x_s)$, $y = (y_1, \ldots, y_s)$ we denote the points of the space R^s and by $m = (m_1, \ldots, m_s)$, $n = (n_1, \ldots, n_s)$ the points of the set N^s. For any $i = 1, \ldots, s$ we set $\bar{i} = \{1, \ldots, i\}$. If B is an arbitrary subset of the set $\bar{s}$, then by x_B we denote the point $(x'_1, \ldots, x'_s)$, where $x'_i = x_i$ when $i \in B$ and $x'_i = 0$ when $i \in \bar{s} \backslash B = B'$. By $|B|$ denote the number of elements of the set B. If $B = \{i_1, \ldots, i_{|B|}\}$, then

$$\sum_{\nu_B = n_B}^{m_B} c_\nu = \sum_{\nu_{i_1} = n_1}^{m_1} \cdots \sum_{\nu_{i_{|B|}} = n_{|B|}}^{m_{|B|}} c_{\nu_{i_1}, \ldots, \nu_{i_{|B|}}}.$$

We will also use the following notation: by Π_s we denote the set of all nonempty subsets of the set $\bar{s}$; $e = (1, \ldots, 1)$; $m \pm n = (m_1 \pm n_1, \ldots, m_s \pm n_s)$; $m \geq n$ denotes $m_i \geq n_i$, $i \in \bar{s}$; $m \to \infty$ denotes $m_i \to \infty$, $i \in \bar{s}$; $dt_B = \prod_{i \in B} dt_i$ and $dt = \prod_{i=1}^{s} dt_i$.

Let X^s be either the Lebesgue space $L_p(T^s)$, $T \in [-\pi, \pi]$, $1 \leq p < \infty$, or the space of continuous functions $C(T^s)$. If $f \in X^s$, by $\|f\|_{X^s}$ we denote the norm of f in the space X^s. Besides, for any $B \in \Pi_s$ we set

$$\|f\|_{X^{|B|}} = \begin{cases} \left((2\pi)^{-|B|} \displaystyle\int_{T^{|B|}} |f(t)|^p dt_B\right)^{\frac{1}{p}}, & X^{|B|} = L_p(T^{|B|}), \quad 1 \le p < \infty, \\ \max\limits_{\substack{t_j \in T \\ j \in B}} |f(t)|, & X^{|B|} = C(T^{|B|}). \end{cases}$$

It is clear that

$$\left\| \|f\|_{X^{|B|}} \right\|_{X^{|B'|}} = \|f\|_{X^s} \ . \tag{1.1}$$

For $f \in X^s$ we set

$$\Delta_{h_{\{j\}}} f(x) = f(x + h_{\{j\}}) - f(x),$$

and define $\Delta_{h_B} f(x)$ to be the repeated applications of the operation h_j when j runs over the set $B \subset \overline{s}$. By the expression

$$\sup_{\substack{|h_j| < \delta_j \\ j \in B}} \|\Delta_{h_B} f(x)\|_{X^s} = \omega_B(f, \delta_B)_{X^s}, \quad \delta_B \ge 0, \tag{1.2}$$

we define the modulus of continuity of a function $f \in X^s$ with respect to those variables whose indices belong to the set $B \subset \overline{s}$.

Let $T_{m_j}^{(j)}(x) \in X^s$ be a trigonometric polynomial of degree m_j ($m_j \ge 0$) with respect to the variable x_j and coefficients depending on the remaining variables x_i, $i \in \{j\}'$. For $m_j < 0$ we assume that $T_{m_j}^{(j)}(x) = 0$. The following sum

$$T_m(x) = \sum_{j \in \overline{s}} T_{m_j}^{(j)}(x) \tag{1.3}$$

is called the trigonometric quasi-polynomial of degree m (see [1]). Denote by P_n the set of trigonometric quasi-polynomials of degree $\le n$, that is,

$$P_n = \{T_m(x) : \ m \le n\}.$$

The quantity

$$\mathcal{H}_n(f)_{X^s} = \inf_{T_m \in P_n} \|f - T_m\|_{X^s} \ , \tag{1.4}$$

introduced by M. K. Potapov [2], is called the best "angular" approximation of a function f (see [2]), or the best approximation of f by the quasi-polynomials of degree $\le n$ (see [1]). In what follows we will use the last term. M. K. Potapov [2] and Yu. A. Brudnyi [1] have found relationships between the best approximation $\mathcal{H}_n(f)_{X^s}$

and the modulus of continuity $\omega(f, n^{-1})$. Specifically, they proved direct and inverse theorems for the approximation of a function $f \in X^s$ by trigonometric polynomials.

Let the multiplicative matrix $\Lambda = \Big(\prod_{j=1}^{s} \lambda^{(j)}_{n_j,k_j} \Big) = (\lambda_{n,k})$ be given. The matrices $\Lambda_j = (\lambda^{(j)}_{n_j,k_j})$, $j = 1, \ldots, s$, are called constituent matrices of the matrix Λ. Let the elements of Λ be such that for any $\varphi \in X^1$ and $f \in X^s$ we have the following relations:

$$\begin{aligned} L_{n_j}(\varphi, x_j) &= \sum_{m_j=-\infty}^{\infty} \widehat{\varphi}_{m_j} \exp(im_j x_j) \lambda^{(j)}_{n_j,|m_j|} \\ &= \frac{1}{2\pi} \int_T \varphi(x_j + t_j) \sum_{m_j=-\infty}^{\infty} \exp(im_j t_j) \lambda^{(j)}_{n_j,|m_j|} dt_j, \quad j \in \{1, \ldots, N\}, \end{aligned} \tag{1.5}$$

$$\begin{aligned} L_n(f, x) &= \sum_{m=-\infty}^{\infty} \widehat{f}_m \prod_{j=1}^{s} \exp(im_j x_j) \lambda^{(j)}_{n_j,|m_j|} \\ &= \frac{1}{(2\pi)^s} \int_{T^N} f(x+t) \prod_{j=1}^{s} \Big(\sum_{m_j=-\infty}^{\infty} \exp(im_j t_j) \lambda^{(j)}_{n_j,|m_j|} \Big) dt, \end{aligned} \tag{1.6}$$

$$\frac{1}{2\pi} \int_T \sum_{m_j=-\infty}^{\infty} \exp(im_j t_j) \lambda^{(j)}_{n_j m |m_j|} dt_j = 1, \quad j \in \{1, \ldots, N\}. \tag{1.7}$$

2. THE MAIN RESULT

Theorem 2.1. *Let for any $j \in \overline{s}$ and $\varphi \in X^1$ the inequalities*

$$\|\varphi - L_{n_j}(\varphi)\|_{X^1} \le \varkappa_{\Lambda_j} \sum_{k_j=0}^{\infty} E_{k_j}(\varphi)_{X^1} \cdot a^{(j)}_{n_j,k_j} \tag{2.1}$$

hold, where (a_{n_j,k_j}), $j \in \overline{s}$, are nonnegative matrices. Then for any $f \in X^s$

$$\|f - L_n(f)\|_{X^s} \le \varkappa_\Lambda \sum_{B \in \Pi_s} \sum_{k_B=0}^{\infty} \mathcal{H}_{k_B - e_{B'}}(f) \prod_{j \in B} a^{(j)}_{n_j,k_j}. \tag{2.2}$$

Proof. First, using the method of induction, we show that for any natural s

$$f(x+t) - f(x) = \sum_{B \in \Pi_s} \Delta_{t_B} f(x). \tag{2.3}$$

Indeed, for $s = 1$ the equality (2.3) is trivial. Assuming that (2.3) is valid for any $s = d$, it is easy to check that for any $x, t \in R^{d+1}$

$$\sum_{B \in \Pi_{d+1}} \Delta_{t_B} f(x) = \Delta_{t_{\{d+1\}}} \Big(\sum_{B \in \Pi_d} \Delta_{t_B} f(x) \Big) + \sum_{B \in \Pi_d} \Delta_{t_B} f(x) + \Delta_{t_{\{d+1\}}} f(x). \tag{2.4}$$

In view of the assumption we have

$$\sum_{B\in\Pi_d} \Delta_{t_B} f(x) = f(x+t_{\overline{d}}) - f(x).$$

From this and (2.4) we get

$$\sum_{B\in\Pi_{d+1}} \Delta_{t_B} f(x) = f(x+t) - f(x+t_{\overline{d}}) - f(x+t_{\{d+1\}}) + f(x)$$
$$+ f(x+t_{\overline{d}}) - f(x) + f(x+t_{\{d+1\}}) - f(x) = f(x+t) - f(x),$$

yielding (2.3) for any natural s.

Further, it follows from (1.6), (1.7) and (2.3) that

$$L_n(f,x) - f(x) = (2\pi)^{-s} \int_T (f(x+t) - f(x)) \prod_{j=1}^{s} \Big(\sum_{m_j=-\infty}^{\infty} \exp(im_jt_j) \lambda^{(j)}_{n_j,|m_j|} \Big) dt$$
$$= \sum_{B\in\Pi_s} (2\pi)^{-s} \int_{T^s} \Delta_{t_B} f(x) \prod_{j=1}^{s} \Big(\sum_{m_j=-\infty}^{\infty} \exp(im_jt_j) \lambda^{(j)}_{n_j,|m_j|} \Big) dt = \sum_{B\in\Pi_s} I(B,f,\Lambda,x).$$

Therefore we have

$$\|f - L_n(f)\|_{X^s} \le \sum_{B\in\Pi_s} \|I(B,f,\Lambda,x)\|_{X^s}\ . \tag{2.5}$$

We now proceed to estimate the norm $\|I(B,f,\Lambda,x)\|_{X^s}$. Note first that $\Delta_{t_B} f(x)$ does not depend on $t_{B'}$. Consequently, in view of (1.7), we get

$$I(B,f,\Lambda_B,x) = (2\pi)^{-s} \int_{T^s} \Delta_{t_B} f(x) \prod_{j=1}^{s} \Big(\sum_{m_j=-\infty}^{\infty} \exp(im_jt_j) \lambda^{(j)}_{n_j,|m_j|} \Big) dt \tag{2.6}$$
$$= (2\pi)^{-|B|} \int_{T^{|B|}} \Delta_{t_B} f(x) \prod_{j\in B} \Big(\sum_{m_j=-\infty}^{\infty} \exp(im_jt_j) \lambda^{(j)}_{n_j,|m_j|} \Big) dt_B.$$

Assuming that for any $B \in \Pi_s$ the inequality

$$\|I(B,f,\Lambda_B,x)\|_{X^{|B|}} \le \varkappa_{\Lambda_B} \sum_{k_B=0}^{\infty} \|f - T_{k_B - e_{B'}}\|_{X^{|B|}} \prod_{j\in B} a^{(j)}_{n_j,k_j} \tag{2.7}$$

is proved, where $T_{k_B-e_{B'}}$ is any trigonometric quasi-polynomial of degree $k_B - e_{B'}$ and $\Lambda_B = \Big(\prod_{j\in B} \lambda^{(j)}_{n_j,k_j}\Big)$, in view of (1.1), we obtain

$$\|I(B,f,\Lambda_B,x)\|_{X^s} \le \varkappa_{\Lambda_B} \sum_{k_B=0}^{\infty} \|f - T_{k_B-e_{B'}}\|_{X^s} \prod_{j\in B} a^{(j)}_{n_j,k_j}\ .$$

From the latter inequality and (2.5), by virtue of arbitrariness of $T_{k_B-e_{B'}}(x)$ the assertion of the theorem follows.

To complete the proof, it remains to prove the inequality (2.7) for any $B \in \Pi_s$. We apply induction on the number of elements in the set B. Consider first the case where $|B| = 1$, that is, B consists of one element.

By (2.1), for any trigonometric polynomial $T^{(j)}_{k_j}(x)$ of degree k_j with respect to the variable x_j with coefficients depending on the remaining variables x_i, $i \in \{j\}'$, we have

$$\|f - L_{n_j}(f)\|_{X^{|\{j\}|}} \le \varkappa_{\Lambda_j} \sum_{k_j=0}^{\infty} \|f - T^{(j)}_{k_j}\|_{X^{|\{j\}|}} a_{n_j,k_j} \; .$$

Let $B = \{j\}$. It follows from (1.5), (1.7) and (2.6) that

$$L_{n_j}(f, x) - f(x) = I(\{j\}, f, \Lambda_{\{j\}}, x).$$

The last two relations imply (2.7) in the case where $B \in \Pi_s$ consists of one element.

Assume now that (2.7) is true for those $B \in \Pi_s$ for which $|B| = d$, and prove it when $|B| = d + 1$.

Let $B_1 = \{j_1, \dots, j_d\}$ and $B_2 = \{j_1, \dots, j_{d+1}\}$. Using (2.6) it is easy to show that

$$I(B_2, f, \Lambda, x) = I(\{j_{d+1}\}, I(B_1, f, \Lambda_{B_1}), \Lambda_{\{d+1\}}, x), \tag{2.8}$$

$$I(B_1, f_1 + f_2, \Lambda_{B_1}, x) = I(B_1, f_1, \Lambda, x) + I(B_1, f_2, \Lambda, x). \tag{2.9}$$

Besides, if $T^{(j_{d+1})}_{k_{j_{d+1}}}$ is any trigonometric polynomial of degree $k_{j_{d+1}}$ with respect to the variable $x_{j_{d+1}}$ with coefficients depending on the remaining variables, then $I(B, T^{(j_{d+1})}_{k_{j_{d+1}}}, \Lambda, x)$ will be a trigonometric polynomial of degree $k_{j_{d+1}}$ with respect to the variable $x_{j_{d+1}}$.

Therefore, by the validity of inequality (2.7) when B consists of one element,for $B = \{j_{d+1}\}$ we have

$$\begin{aligned} &\|I(B_2, f, \Lambda_{B_2}, x)\|_{X^{|\{j_{d+1}\}}} \\ &\le \varkappa_{\Lambda_{j_{d+1}}} \sum_{k_{j_{d+1}}=0}^{\infty} \left\| I(B_1, f, \Lambda_{B_1}, x) - I(B_1, T^{(j_{d+1})}_{k_{j_{d+1}}}, \Lambda_{B_1}, x) \right\|_{X^{|\{j_{d+1}\}|}} a^{(j_{d+1})}_{n_{j_{d+1}}, k_{j_{d+1}}} \; . \end{aligned} \tag{2.10}$$

Assuming (2.7) when $B_1 = \{j_1, \ldots, j_d\}$, in view of (2.9), we have

$$\left\| I(B_1, f, \Lambda_{B_1}, x) - I(B_1, T^{(j_{d+1})}_{k_{j_{d+1}}}, \Lambda_{B_1}, x) \right\|_{X^{|B_1|}} \leq \varkappa_{\Lambda_{B_1}} \sum_{k_{B_1}=0}^{\infty} \left\| f - T^{(j_{d+1})}_{k_{j_{d+1}}} - T_{k_{B_1}-e_{B_1'}} \right\|_{X^{|B_1|}} \prod_{j\in B_1} a^{(j)}_{n_j,k_j}\ , \tag{2.11}$$

where $T_{k_{B_1}-e_{B_1'}}$ is any trigonometric quasi-polynomial of degree $k_{B_1} - e_{B_1'}$. Observe that (see (1.3)) for any preassigned $T_{k_{B_2}-e_{B_2'}}$ (trigonometric quasi-polynomial of degree $k_{B_2} - e_{B_2'}$) one can choose $T^{(j_{d+1})}_{k_{j_{d+1}}}$ and $T_{k_{B_1}-e_{B_1'}}$ to satisfy

$$T_{k_{B_2}-e_{B_2'}}(x) = T^{(j_{d+1})}_{k_{j_{d+1}}}(x) + T_{k_{B_1}-e_{B_1'}}(x).$$

Now, taking $X^{|B_1|}$- norm of both sides of (2.10) and using (1.1) and (2.11), we get

$$\|I(B_2, f, \Lambda_{B_1}, x)\|_{X^{|B_2|}} \leq \varkappa_{\Lambda_{B_2}} \sum_{k_{B_2}=0}^{\infty} \|f - T_{k_{B_2}-e_{B_2'}}\|_{X^{|B_2|}} \prod_{j\in B_2} a^{(j)}_{n_j,k_j}\ ,$$

where $T_{k_{B_2}-e_{B_2'}}$ is any trigonometric quasi-polynomial of degree $k_{B_2} - e_{B_2'}$.

This proves (2.7) when $|B| = d + 1$, and hence for any $B \in \Pi_s$. The proof of Theorem 1.1 is completed. □

Corollary 2.1. *Let the conditions of Theorem 2.1 and*

$$\sum_{k_j=0}^{\infty} a_{n_j,k_j} < \varkappa_{\Lambda_j}\ , \quad j \in \overline{s}, \tag{2.12}$$

be fulfilled. Then

$$\|f - L_n(f)\|_{X^s} \leq \varkappa_\Lambda \sum_{j\in\overline{s}} \sum_{k_j=0}^{\infty} E^{\{j\}}_{k_j}(f)_{X^s} a^{(j)}_{n_j,k_j}\ . \tag{2.13}$$

Proof. According to the definitions of the best approximation by quasi-polynomials (1.4) and partial best approximation by trigonometric polynomials we have

$$H_{n_{\{i\}}-e_{\{i\}'}}(f)_{X^s} = E^{\{j\}}_{n_i}(f)_{X^s}\ , \quad i \in \overline{s}. \tag{2.14}$$

Let us take any $B \in \Pi_s$ and let $i \in B$. Using the monotonicity of $H_n(f)_{X^s}$, and (2.12), (2.14), we obtain

$$\sum_{k_B=0}^{\infty} H_{k_B-e_{B'}}(f) \prod_{j\in B} a^{(j)}_{n_j,k_j} \leq \sum_{k_B=0}^{\infty} E^{\{i\}}_{k_i}(f)_{X^s} \prod_{j\in B} a^{(j)}_{n_j,x_j} \leq \varkappa_\Lambda \sum_{k_i=0}^{\infty} E^{\{i\}}_{k_i}(f)_{X^s} a^{(i)}_{n_i,k_i}\ . \tag{2.15}$$

This and (2.2) imply the assertion of Corollary 2.1. □

Assume that we have a sequence $\{\varepsilon_k\}_{k\geq -e}$ decreasing with respect to each index k_j, $j\in\overline{s}$ and satisfying the condition

$$\lim_{k_j\to\infty}\varepsilon_k=0,\quad j\in\overline{s}.$$

Consider the following classes of functions

$$X^1(\varepsilon^{(j)})=\left\{\varphi:\varphi\in X^1,\ \ E^{\{j\}}_{k_j}(\varphi)_{X^1}\leq\varepsilon_{k_{\{j\}}-e_{\{j\}'}}\right\},\quad j\in\overline{s},$$
$$X^s(\varepsilon)=\{f:f\in X^s,\ \ H_k(f)_{X^s}\leq\varepsilon_k\}.$$

Now we are going to show that under the condition (2.12) the result of Theorem 2.1 is final in the following sense.

Assume that for any $j\in\overline{s}$

$$\sup_{\varphi\in X^1(\varepsilon^{(j)})}\|\varphi-L_{n_j}(\varphi)\|_{X^1}\geq\varkappa_{\Lambda_j}\sum_{k_j=0}^{\infty}\varepsilon_{k_{\{j\}}-e_{\{j\}'}}a^{(j)}_{n_j,k_j}\,. \tag{2.16}$$

Then

$$\sup_{f\in X^s(\varepsilon)}\|f-L_n(f)\|_{X^s}\geq\varkappa_\Lambda\sum_{B\in\Pi}\sum_{k_B=0}^{\infty}\varepsilon_{k_B-e_{B'}}(f)\prod_{j\in B}a^{(j)}_{n_j,k_j}\,. \tag{2.17}$$

In fact, there exists $i\in\overline{s}$ such that

$$\varkappa_{\Lambda_i}\sum_{k_i=0}^{\infty}\varepsilon_{k_{\{i\}}-e_{\{i\}'}}a^{(i)}_{n_i,k_i}\geq\frac{1}{s+1}\sum_{j\in\overline{s}}\varkappa_{\Lambda_j}\sum_{k_j=0}^{\infty}\varepsilon_{k_{\{j\}}-e_{\{i\}'}}a^{(j)}_{n_j,k_j}\,. \tag{2.18}$$

It follows from (2.16) that there exists a function $\psi\in X^1(\varepsilon^{(i)})$ such that

$$\|\psi-L_{n_i}(\psi)\|_{X^1}\geq\varkappa_{\Lambda_i}\sum_{k_i=0}^{\infty}\varepsilon_{k_{\{i\}}-e_{\{i\}'}}a^{(i)}_{n_i,k_i}\,. \tag{2.19}$$

Let us take any $B\in\Pi_s$ and let $j\in B$. Then using the monotonicity of ε_k and (2.12), we obtain

$$\sum_{k_B=0}^{\infty}\varepsilon_{k_B-e_{B'}}\prod_{d\in B}a_{n_d,x_d}\leq\varkappa_\Lambda\sum_{k_j=0}^{\infty}\varepsilon_{k_{\{j\}}-e_{\{j\}'}}a^{(j)}_{n_j,k_j}\,.$$

This inequality yields

$$\sum_{B\in\Pi_s}\sum_{k_B=0}^{\infty}\varepsilon_{k_B-e_{B'}}\prod_{d\in B}a_{n_d,x_d}\leq\varkappa_\Lambda\sum_{j\in\overline{s}}\sum_{k_j=0}^{\infty}\varepsilon_{k_{\{j\}}-e_{\{j\}'}}a^{(j)}_{n_j,k_j}\,. \tag{2.20}$$

Assume now $f(x)=\psi(x_i)$. Then, in view of (1.7), we have

$$L_n(f,x)=L_{n_i}(\psi,x_i).$$

Therefore

$$\|f - L_n(f)\|_{X^s} = \|\psi - L_{n_i}(\psi)\|_{X^1} \ .$$

This and (2.18)–(2.20) imply (2.17).

Consider now the approximative properties of rectangular Fourier sums. This special case of Theorem 2.1 needs special attention, since in the one-dimensional case the properties of partial sums of Fourier series are studied in more detail than other linear means.

Let $f \in L(T^2)$ and let $S_n(f,x)$ be the partial sums of Fourier series of function f. Below Y^s stands for one of the spaces $C(T^s)$ or $L(T^s)$. Observe that for spaces $L_p(T^s)$, $1 < p < \infty$ the results considered below are less significant.

In the one-dimensional case K. I. Oskolkov in [3] and [4] proved that

$$\|f - S_n(f)\|_{Y^1} \le \varkappa \sum_{\nu=0}^{n} \frac{E_{n+\nu}(f)_{Y^1}}{\nu+1} \ .$$

From this and Theorem 2.1 we obtain the following result.

Theorem 2.2. *Let $f \in Y^s$. Then*

$$\|f - S_n(f)\|_{Y^s} \le \varkappa_s \sum_{B \in \Pi_s} \sum_{\nu_B=0}^{n_B} H_{(n+\nu)_B - e_{B'}}(f)_{Y^s} \prod_{j \in B} \frac{1}{\nu_j+1} \ . \tag{2.21}$$

According to (2.14), in the special case of $s = 2$, $Y^2 = C(T^2)$, the inequality (2.21) becomes

$$\|f - S_{n_1 n_2}(f)\|_{C(T^2)} \le \varkappa \Bigg(\sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \frac{H_{n_1+\nu_1, n_2+\nu_2}(f)_{C(T^2)}}{(\nu_1+1)(\nu_2+1)} + \sum_{\nu_1=0}^{n_1} E^{\{1\}}_{n_1+\nu_1}(f)_{C(T^2)} \cdot \frac{1}{\nu_1+1} + \sum_{\nu_2=0}^{n_2} E^{\{2\}}_{n_2+\nu_2}(f)_{C(T^2)} \cdot \frac{1}{\nu_2+1} \ . \tag{2.22}$$

Next, in view of D. Jackson's theorem (see (1.2)) we have

$$E^{(i)}_{n_i+\nu_i}(f)_{C(T^2)} < \varkappa \omega_{\{i\}} \left(f, \frac{1}{n_i+\nu_i} \right)_{C(T^2)} < \varkappa \omega_{\{i\}} \left(f, \frac{1}{n_i+1} \right)_{C(T^2)}, \quad i = 1, 2.$$

Finally, according to Potapov - Brudnyi theorem (see [1], [2]), we have

$$H_{n_1+\nu_1, n_2+\nu_2}(f)_{C(T^2)} \le \varkappa \omega_{\{1,2\}} \left(f, \frac{1}{n_1+1}, \frac{1}{n_2+1} \right)_{C(T^2)} .$$

Putting together the last two inequalities and (2.22) we obtain

$$\begin{aligned}(2.23)\quad & \|f - S_{n_1,n_2}(f)\|_{C(T^2)} \\ & \le \varkappa\Big[\omega_{\{1\}}\Big(f, \frac{1}{n_1+1}\Big)_{C(T^2)} \ln(n_1+2) + \omega_{\{2\}}\Big(f, \frac{1}{n_2+1}\Big)_{C(T^2)} \ln(n_2+2) \\ & \qquad + \omega_{\{1,2\}}\Big(f, \frac{1}{n_1+1}, \frac{1}{n_2+1}\Big)_{C(T^2)} \ln(n_1+2)\ln(n_2+2)\Big].\end{aligned}$$

This unimprovable inequality was obtained by L. V. Zhizhiashvili [5, p. 178]. This inequality, on account of

$$\omega_{\{1,2\}}\Big(f, \frac{1}{n_1+1}, \frac{1}{n_2+1}\Big)_{C(T^2)} \le 2\min\left(\omega_{\{1\}}\Big(f, \frac{1}{n_1+1}\Big)_{C(T^2)}, \omega_{\{2\}}\Big(f, \frac{1}{n_2+1}\Big)_{C(T^2)}\right),$$

yields the following corollary of Theorem 2.2:

$$\begin{aligned}(2.24)\quad & \|f - S_{n_1,n_2}(f)\|_{C(T^2)} \\ & \le \varkappa\Big[\omega_{\{1\}}\Big(f, \frac{1}{n_1+1}\Big)_{C(T^2)} \ln(n_1+2) + \omega_{\{2\}}\Big(f, \frac{1}{n_2+1}\Big)_{C(T^2)} \ln(n_2+2) \\ & + \min\left(\omega_{\{1\}}\Big(f, \frac{1}{n_1+1}\Big)_{C(T^2)}, \omega_{\{2\}}\Big(f, \frac{1}{n_2+1}\Big)_{C(T^2)}\right) \ln(n_1+2)\ln(n_2+2)\Big].\end{aligned}$$

Let now $\omega_1(t)$ and $\omega_2(t)$ be given moduli of continuity, and let

$$H_{\omega_1,\omega_2} = \left\{ f : f \in C(T^2),\ \ \omega_{\{j\}}\Big(f, \frac{1}{n_j+1}\Big)_{C(T^2)} \le \omega_j\Big(\frac{1}{n_j+1}\Big)_{C(T^2)},\ \ j = 1,2 \right\}.$$

From the asymptotic equality, obtained by A. I. Stepanetz (see [6, p. 195]) for the quantity

$$\sup_{f \in H_{\omega_1,\omega_2}} \|f - S_{n_1,n_2}(f)\|_{C(T^2)},$$

we get the unimprovability of the estimate (2.24) in the class of functions H_{ω_1,ω_2}. This estimate, however, becomes crude for those functions of the class H_{ω_1,ω_2} whose best approximations by quasi-polynomials decrease rapidly. For instance, if

$$f(x_1, x_2) = \sum_{k_1=0}^{\infty} \frac{\cos k_1 x_1}{2^{k_1}} \sum_{k_2=0}^{\infty} \frac{\cos k_2 x_2}{2^{k_2}},$$

it can be easily shown that the estimates (2.23) and (2.24) in this case become more crude than estimate (2.22).

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