Известия НАН Армении. Математика, том 47, н. 6, 2012, стр. 3-18.

In memory of my teacher in Math, Nikolaos Stamatas.

ON A CHARACTERIZATION OF ARAKELIAN SETS

G. FOURNODAVLOS

Department of Mathematics, University of Toronto, Canada

E-mail: *grifour@math.toronto.edu*

Abstract. Let K be a compact set in the complex plane, such that its complement in the Riemann sphere, $(\mathbb{C} \cup \{\infty\}) \setminus K$, is connected. Also, let $U \subset \mathbb{C}$ be an open set which contains K. Then there exists a simply connected open set $V \subset \mathbb{C}$ such that $K \subset V \subset U$. We show that if K is replaced by a closed set $F \subset \mathbb{C}$, then the preceding result is equivalent to the fact that F is an Arakelian set in $\mathbb{C}$. This holds in more general case when $\mathbb{C}$ is replaced by any simply connected open set $\Omega \subset \mathbb{C}$. In the case of an arbitrary open set $\Omega \subset \mathbb{C}$, the above extends to the one point compactification of Ω. Furthermore, if we do not require $(\mathbb{C} \cup \{\infty\}) \setminus K$ to be connected, we can demand that each component of $(\mathbb{C} \cup \{\infty\}) \setminus V$ intersects a prescribed set A containing one point in each component of $(\mathbb{C} \cup \{\infty\}) \setminus K$. Using the previous result, we prove that again if we replace K by a closed set F, the latter is equivalent to the fact that F is a set of uniform meromorphic approximation with poles lying entirely in A.

MSC2010 numbers: 30E10.

Keywords: uniform approximation in the complex domain; Arakelian set; simply connected open set.

1. Introduction

When one starts doing approximations in complex analysis, he quickly realizes that topological lemmas play a key role to his work. This may seem strange to someone not familiar with the subject, but actually it is quite natural. A simple explanation is that the main tools for this kind of research, such as Runge's theorem [22] and Mergelyan's theorem [17], contain themselves topological conditions. Our work in the present paper rotates around the following standard topological lemma (see Proposition 3.10.6 in [18]).

Lemma 1.1. *Let $K \subset \mathbb{C}$ be a compact set with $(\mathbb{C} \cup \{\infty\}) \setminus K$ connected. If U is an open set in $\mathbb{C}$ containing K, then there exists a simply connected open set V such that $K \subset V \subset U$.*

We note that throughout this article, when we say that an open subset of $\mathbb{C}$ is simply connected, it may not necessarily be path connected (as it is often the case). An equivalent statement is that the complement of this open set in the Riemann sphere, $\mathbb{C} \cup \{\infty\}$, is connected. The above lemma has given several applications. For instance, in [4] and [16] it was used by the authors to obtain certain universal approximations. More recently, we used it in [8] to show the density of polynomials in a subspace of $A^{\infty}(\Omega)$, the space of holomorphic functions in a (simply connected) open set $\Omega \subset \mathbb{C}$, smooth on the boundary. When I first encountered this lemma, I was interested in knowing if its conclusion still holds when the compact set K is replaced by a closed subset F of $\mathbb{C}$. Eventually, I realized that the answer, in general, is negative and a counterexample is the following:

$$F = \bigcup_{n=1}^{\infty} \Big[\big(\{\sum_{i=1}^{n} \frac{1}{2^i}\} \times [0,n]\big) \cup \big([\sum_{i=1}^{n} \frac{1}{2^i}, \sum_{i=1}^{n+1} \frac{1}{2^i}] \times \{n\}\big) \Big] \cup \big(\{1\} \times [0,\infty)\big).$$

This set F relates to the well-known Arakelian Approximation Theorem [1].

Theorem 1.1. *Let F be a closed set in the complex plane $\mathbb{C}$. Then every function $f : F \to \mathbb{C}$ continuous on F and holomorphic in F^0 ($f \in A(F)$) can be uniformly approximated on F by entire functions, $g \in H(\mathbb{C})$, if and only if the following hold:*

(i) $(\mathbb{C} \cup \{\infty\}) \setminus F$ *is connected and*

(ii) $(\mathbb{C} \cup \{\infty\}) \setminus F$ *is locally connected at* ∞.

Yet, in [19] one finds another proof of Theorem 1.1, based on Mergelyan's theorem, where conditions (i) and (ii) are replaced by the following equivalent condition:

(iii) $\mathbb{C} \setminus F$ has no bounded components and for every closed disk D in $\mathbb{C}$, the union of all bounded components of $\mathbb{C} \setminus (F \cup D)$ is a bounded set. Such a closed set, as in Theorem 1.1, is called an Arakelian set in $\mathbb{C}$. It is easy to see that in our counterexample F does not satisfy condition (ii). This is in accordance with [14], where it was shown that Lemma 1.1 is valid for every Arakelian set in $\mathbb{C}$. In the present article we start by proving that the converse is also true. More precisely, the following theorem holds.

Theorem 1.2. *Let F be a closed subset of $\mathbb{C}$. Then the following are equivalent:*

(1) for every open set $U \subset \mathbb{C}$, which contains F, there exists a simply connected open set V such that $F \subset V \subset U$;

(2) F is an Arakelian set in $\mathbb{C}$.

More generally, Arakelian sets may be defined for an arbitrary open set $\Omega \subset \mathbb{C}$. The question is for a relatively closed set F in Ω, whether every function $f \in A(F)$ can be uniformly approximated on F by holomorphic functions $g \in H(\Omega)$. This was completely settled by Arakelian in [2], where he extended his Theorem 1.1. In this version one considers the one point compactification of Ω, $\Omega \cup \{\alpha\}$. The relatively closed set $F \subset \Omega$, for which the approximation is possible, is called an Arakelian set in Ω and again a purely topological description can be provided.

We extend Theorem 1.2 in this case by replacing the complex plane with a simply connected open set $\Omega \subset \mathbb{C}$. This means that the open set V is still simply connected; that is $(\mathbb{C} \cup \{\infty\}) \setminus V$ is connected. Also, we can further extend Theorem 1.2 in the general case for any open set $\Omega \subset \mathbb{C}$. In this case V is not necessarily simply connected, but its complement $(\Omega \cup \{\alpha\}) \setminus V$ in the one point compactification of Ω has to be connected.

Next, we give two applications of our results. One of these is a simple proof of the fact that if Ω is a simply connected open subset of $\mathbb{C}$, then the union of a locally finite family of pairwise disjoint Arakelian sets in Ω is also an Arakelian set in Ω. We notice that when Ω is not simply connected the latter fails.

N. Tsirivas in his master thesis [25] used a variation of Lemma 1.1, without the assumption that $(\mathbb{C} \cup \{\infty\}) \setminus K$ is connected (see Lemma 2.2 in [6]). In this case one considers a set A which contains one point in each component of $(\mathbb{C} \cup \{\infty\}) \setminus K$ and requires that every component of $(\mathbb{C} \cup \{\infty\}) \setminus V$ intersects A. We consider the same problem, as before, by replacing the compact set K with a closed set F and we manage to fully characterize the closed sets that satisfy the previous conclusion. Indeed, we realize that they are exactly these for which every function $f \in H(F)$, holomorphic in some neighborhood of F, can be uniformly approximated on F by meromorphic functions whose poles lie in A. Finally, we investigate the latter type of approximation for the larger space $A(F)$ and give a characterization relating to the preceding result. We note that this kind of approximation is different and more restrictive than, namely, uniform meromorphic approximation (see Roth's theorem [20]). The former approximation problem (in a much more general context though) was solved by Scheinberg in [24], where a topological description of such closed sets is given.

We remark that the above approximation problems have also been studied in the case of open Riemann surfaces. For example, in [10], [23], [24] the authors obtained their results for arbitrary open Riemann surfaces Ω and closed subsets F of essentially finite genus. We believe that our results can be extended in this case, but this is up to investigation. In the present article we will only consider planar open sets $\Omega \subset \mathbb{C}$ as domains of definition. P. M. Gauthier suggested that some alternative proofs could relate to Runge pairs and harmonic approximation (see [3], [11], [12]). A preliminary version of this article can be found in [7].

2. The main result and applications

In [2] N. U. Arakelian proved the following theorem.

Theorem 2.1. *Let $\Omega \subset \mathbb{C}$ be an open set and F be a relatively closed subset of Ω. Then every function $f \in A(F)$ can be uniformly approximated on F by functions $g \in H(\Omega)$ holomorphic in Ω, if and only if the following hold:*

(i) $(\Omega \cup \{\alpha\}) \setminus F$ is connected and

(ii) $(\Omega \cup \{\alpha\}) \setminus F$ is locally connected at α, where $\Omega \cup \{\alpha\}$ is the one point compactification of Ω.

Such a set F is also called an Arakelian set in Ω.

It is easy to see that condition (ii) is equivalent to the seemingly stronger one: "$(\Omega \cup \{\alpha\}) \setminus F$ is locally connected". From now on, we shall say that B is a "hole of F"in Ω, iff B is a component of $\Omega \setminus F$ which is contained in some compact subset of Ω. Note that $(\Omega \cup \{\alpha\}) \setminus F$ is connected, iff F has no holes in Ω.

Proposition 2.1. *A closed set F in Ω, without holes, is an Arakelian set in Ω, if and only if for every compact set $K \subset \Omega$, the union of all holes of $F \cup K$ in Ω is contained in a compact subset of Ω.*

For the case $\Omega = \mathbb{C}$ see also [19]. We include a proof of the general case, for the sake of completeness.

Proof. ($\Rightarrow$) Suppose that there is a compact set $K \subset \Omega$, such that the union of all holes in Ω of $F \cup K$ is not contained in any compact subset of Ω. The complement $(\Omega \cup \{\alpha\}) \setminus K$ is a neighborhood of α in $\Omega \cup \{\alpha\}$. Hence, there exists a neighborhood

$W \subset (\Omega \cup \{\alpha\}) \smallsetminus K$ of α in $\Omega \cup \{\alpha\}$, such that $W \cap \big[(\Omega \cup \{\alpha\}) \smallsetminus F\big]$ is connected. Since $(\Omega \cup \{\alpha\}) \smallsetminus W$ is contained in a compact subset of Ω, there is a hole B of $F \cup K$ in Ω, such that $B \cap W \neq \emptyset$. Observe that $\partial B \subset F \cup K$. Since $W \cap \big[(\Omega \cup \{\alpha\}) \smallsetminus F\big]$ and $F \cup K$ are disjoint, it follows that $W \cap \big[(\Omega \cup \{\alpha\}) \smallsetminus F\big] \subset B \cup \big[(\Omega \cup \{\alpha\}) \smallsetminus \overline{B}\big]$, which is a contradiction.

($\Leftarrow$) Let $U \subset \Omega \cup \{\alpha\}$ be an open neighborhood of α in $\Omega \cup \{\alpha\}$. The set $K = (\Omega \cup \{\alpha\}) \smallsetminus U \subset \Omega$ is compact. Therefore, the union of all holes in Ω of $F \cup K$ is contained in a compact subset of Ω. Let $B_1, B_2 \ldots$ be those holes. Also, let $W = (\Omega \cup \{\alpha\}) \smallsetminus (K \cup B_1 \cup B_2 \cup \cdots)$. Obviously, W is a neighborhood of α and $W \subset U$. We notice that $W \cap \big[(\Omega \cup \{\alpha\}) \smallsetminus F\big]$ is the union of $\{\alpha\}$ and all the components of $\Omega \smallsetminus (F \cup K)$, which are either unbounded or have zero distance from the boundary of Ω. Thus, $W \cap \big[(\Omega \cup \{\alpha\}) \smallsetminus F\big]$ is connected and the proof is complete. $\square$

Remark 2.1. *In order to determine whether a relatively closed set F, without holes, is an Arakelian set in Ω, it suffices to check the condition of Proposition 2.1 only for an exhausting sequence, $(K_n)_{n\in\mathbb{N}}$, of compact subsets of Ω. Such a sequence can be chosen so that $K_n \subset K^0_{n+1}$, $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} K^0_n = \Omega$ and every component of $(\mathbb{C} \cup \{\infty\}) \smallsetminus K_n$ contains a component of $(\mathbb{C} \cup \{\infty\}) \smallsetminus \Omega$, for all $n \in \mathbb{N}$ (see* [21], *p. 267). The latter is equivalent to the fact that K_n, $n \in \mathbb{N}$, has no holes in Ω.*

Also, assuming that a closed set F, without holes, is not an Arakelian set in Ω, by Proposition 2.1 there exists a compact set K such that the union of all holes of $F \cup K$ is not contained in any compact subset of Ω. If we consider a larger compact set $K \subset \widetilde{K} \subset \Omega$, then the same holds for the union of all holes of $F \cup \widetilde{K}$ in Ω. Thus, we can consider $\widetilde{K}$ to be a finite union of squares in a grid, whose sides are parallel to the coordinate axes and of the same length $\delta > 0$.

Proposition 2.2. *If F is an Arakelian set in Ω, then for every open set $U \subset \Omega$, which contains F, there exists an open set $V \subset \Omega$ such that $F \subset V \subset U$ and $(\Omega \cup \{\alpha\}) \smallsetminus V$ is connected.*

Proof. Let F be an Arakelian set in Ω and $U \subset \Omega$ be an open set such that $F \subset U$. For every $x \in \Omega \smallsetminus U$, we define $d_x = \min\{\frac{dist(x,F)}{2}, \frac{dist(x,\mathbb{C}\smallsetminus\Omega)}{2}, 1\} > 0$. Also, let $(K_n)_{n\in\mathbb{N}}$ be an exhausting sequence of compact subsets of Ω, as in Remark 2.1.

- Observe that the cover $\{D(x,d_x) \mid x \in \Omega \smallsetminus U\}$ of the relatively closed set $\Omega \smallsetminus U$, has a locally finite subcover in Ω, which we denote by $\big\{D(x_i,d_{x_i})\big\}_{i=1}^{\infty}$. Indeed, it suffices to choose a finite cover of disks $D(x,d_x)$, $x \in \Omega \smallsetminus U$, for each of the compact sets $(K_n \smallsetminus K_{n-1}^0) \cap (\Omega \smallsetminus U)$, $n \in \mathbb{N}$, $K_0 = \emptyset$.

This implies that every compact subset of Ω intersects a finite number of disks from $\big\{D(x_i,d_{x_i})\big\}_{i=1}^{\infty}$. Observe that $\bigcup\limits_{i=1}^{\infty} \overline{D(x_i,d_{x_i})} \subset \Omega$ is relatively closed and the set $\{x_i \mid i = 1,2,\dots\}$ has no accumulation points in Ω. Hence, $U_1 = \Omega \smallsetminus \big(\bigcup\limits_{i=1}^{\infty} \overline{D(x_i,d_{x_i})}\big)$ is open and $F \subset U_1 \subset U$.

- We say that a point $x \in \Omega$ is joined with α by a curve Γ in $E \subset \Omega$, if $\Gamma : [0,+\infty) \to E$ is continuous and $\Gamma(0) = x$, $\lim\limits_{t\to+\infty} \Gamma(t) = \alpha$. The image of such a curve, $\Gamma\big([0,+\infty)\big)$, is closed in Ω.

Each x_i can be joined with α by a curve Γ_i in $\Omega \smallsetminus F$, $i = 1,2,\dots$, such that for every $n \in \mathbb{N}$ only finitely many curves intersect the compact set K_n. Indeed, for every $n \in \mathbb{N}$, by Proposition 2.1 there are finitely many x_i's contained in the union of K_n and all the holes of $F \cup K_n$ in Ω. The points that we have not already joined with α (induction hypothesis), are contained in components of $\Omega \smallsetminus (F \cup K_{n-1})$, which are either unbounded or have zero distance from the boundary of Ω. Let x_i be such a point and E be the component of $\Omega \smallsetminus (F \cup K_{n-1})$ which contains it.

- For each $s \geq n$, by Proposition 2.1, E contains a component E_s of $\Omega \smallsetminus (F \cup K_s)$, which is either unbounded or has zero distance from $\partial\Omega$. Further, we can assume that $E_{s-1} \supset E_s$, $s \geq n$, where $E_{n-1} = E$. Let $x_{is} \in E_s$ and let Γ_{is} be a curve in E_{s-1}, which joins x_{is-1} with x_{is} , $s \geq n$, $x_{in-1} = x_i$ (such a curve exists, since E_{s-1} is open and connected). The desired curve, Γ_i, consists of all Γ_{is}, $s \geq n$.

Thus, the family of the curves Γ_i is locally finite in Ω and in particular the union $\bigcup\limits_{i=1}^{\infty} \Gamma_i$ is relatively closed. We can easily see now that the open set $V = U_1 \smallsetminus (\bigcup\limits_{i=1}^{\infty} \Gamma_i)$ has all the desired properties. Since the curves Γ_i do not intersect F, we have $F \subset V \subset U_1 \subset U$ and of course $(\Omega \cup \{\alpha\}) \smallsetminus V = \bigcup\limits_{i=1}^{\infty} \big(\overline{D(x_i,d_{x_i})} \cup \Gamma_i\big) \cup \{\alpha\}$ is connected, which completes the proof. $\square$

Remark 2.2. *The method of proof of Proposition 2.2 implies that V can be chosen so that the collection of components of $\Omega \setminus V$ is locally finite in Ω.*

We note that the previous proposition, in the case $\Omega = \mathbb{C}$, is known (see [14]).

Proposition 2.3. *If F is a closed subset of Ω such that for every open set $U \subset \Omega$, which contains F, there exists an open set $V \subset \Omega$ with $F \subset V \subset U$ and $(\Omega \cup \{\alpha\}) \setminus V$ connected, then F is an Arakelian set in Ω.*

Proof. First, we notice that F has no holes in Ω. Indeed, if B is a hole of F in Ω and $x \in B$, then the open set $U = \Omega \setminus \{x\}$ contains F. Hence, there exists an open set $V \subset \Omega$ such that $F \subset V \subset U$ and $(\Omega \cup \{\alpha\}) \setminus V$ is connected. Evidently, we have $(\Omega \cup \{\alpha\}) \setminus F \supset (\Omega \cup \{\alpha\}) \setminus V$. Therefore, the latter is contained in the component of $(\Omega \cup \{\alpha\}) \setminus F$ that contains α. However, $x \in \big[(\Omega \cup \{\alpha\}) \setminus V\big] \cap B \neq \emptyset$, which is a contradiction, because B is a component of $(\Omega \cup \{\alpha\}) \setminus F$ not containing α.

Suppose now that F is not an Arakelian set in Ω. By Proposition 2.1 there exists a compact set $K \subset \Omega$, such that α is an accumulation point of the union of all holes of $F \cup K$ in Ω. Moreover, Remark 2.1 enables us to assume that K is a finite union of closed squares in a grid, whose sides are parallel to the coordinate axes and of some length $\delta > 0$.

Let $B_1, B_2, \ldots$ be a sequence of holes of $F \cup K$ in Ω and let $x_n \in B_n$, $n \in \mathbb{N}$ be such that $x_n \to \alpha$, as $n \to +\infty$. The open set $U = \Omega \setminus \{x_1, x_2, \ldots\}$ contains F. Thus, there exists an open set V with $F \subset V \subset U$ and $(\Omega \cup \{\alpha\}) \setminus V$ connected. Observe that $(\Omega \cup \{\alpha\}) \setminus V$ intersects B_n and $(\Omega \cup \{\alpha\}) \setminus \overline{B_n}$ for all $n \in \mathbb{N}$. This implies that there exists $y_n \in (\Omega \setminus V) \cap \partial B_n \cap \partial K$, $n \in \mathbb{N}$. Since ∂K is compact, $(y_n)_{n \in \mathbb{N}}$ has a limit point $y \in \partial K$. Also, $y \in \Omega \setminus V$, because $\Omega \setminus V$ is closed in Ω and $y \in \Omega$. We claim that $y \in F \subset V$, which is obviously a contradiction. Indeed, if $y \notin F$, then there exists $\varepsilon > 0$ such that the disk $\overline{D(y, \varepsilon)} \subset \Omega$ does not intersect F. In addition, we can choose $\varepsilon > 0$ small enough, depending on the place of y in the grid, so that $D(y, \varepsilon) \setminus K$ has at most two components. This is a contradiction, since $D(y, \varepsilon) \setminus K \subset \Omega \setminus (F \cup K)$ intersects infinitely many holes of $F \cup K$ in Ω. The proof is complete. $\square$

According to Propositions 2.2 and 2.3, we have the following characterization of Arakelian sets.

Theorem 2.2. *Let $\Omega \subset \mathbb{C}$ be an open set and $\Omega \cup \{\alpha\}$ be its one point compactification. If F is a closed set in Ω, then the following are equivalent:*

(1) for every open set $U \subset \Omega$, which contains F, there exists an open set $V \subset \Omega$ such that $F \subset V \subset U$ and $(\Omega \cup \{\alpha\}) \setminus V$ is connected;

(2) F is an Arakelian set in Ω.

Our achievement so far was, in fact, to show the equivalence of two topological descriptions regarding relatively closed sets in open subsets of $\mathbb{C}$. In other words, Theorem 2.2 takes the following equivalent form for planar open sets Ω.

Theorem 2.3. *Let Ω be an open set in $\mathbb{C}$ and F be a relatively closed subset of Ω. Then the following are equivalent:*

(1) for every open set $U \subset \Omega$ with $F \subset U$ there exists an open set $V \subset \Omega$ such that $F \subset V \subset U$ and $(\Omega \cup \{\alpha\}) \setminus V$ is connected;

(2) $(\Omega \cup \{\alpha\}) \setminus V$ is connected and locally connected.

The advantage of this formulation, as P. M. Gauthier suggested, is that it can be examined if it is true for arbitrary open Riemann surfaces Ω, where the analogue of Theorem 2.1 does not hold in full generality (see [13], [23]).

Lemma 2.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected open set. A set $G \subset \Omega$ has connected complement in $\Omega \cup \{\alpha\}$ if and only if its complement in the Riemann sphere, $(\mathbb{C} \cup \{\infty\}) \setminus G$, is connected.*

Proof. ($\Rightarrow$) Let $G \subset \Omega$ with $(\Omega \cup \{\alpha\}) \setminus G$ connected. Assume that $(\mathbb{C} \cup \{\infty\}) \setminus G$ is not connected. Thus, there are two open sets U_1, U_2 in $\mathbb{C} \cup \{\infty\}$, such that $U_i \cap \big[(\mathbb{C} \cup \{\infty\}) \setminus G\big] \neq \emptyset$, $i = 1, 2$, $(\mathbb{C} \cup \{\infty\}) \setminus G \subset U_1 \cup U_2$ and $U_1 \cap U_2 \cap \big[(\mathbb{C} \cup \{\infty\}) \setminus G\big] = \emptyset$. Since $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ is connected and it is contained in $(\mathbb{C} \cup \{\infty\}) \setminus G$, it follows that $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ is contained in exactly one of the sets U_1, U_2. Without loss of generality, we assume that $(\mathbb{C} \cup \{\infty\}) \setminus \Omega \subset U_1$. Observe that $(\mathbb{C} \cup \{\infty\}) \setminus U_1$ is a compact subset of Ω. This implies that $V_1 = \big[U_1 \cap (\Omega \setminus G)\big] \cup \{\alpha\}$ and the set $V_2 = U_2 \cap (\Omega \setminus G) \subset (\mathbb{C} \cup \{\infty\}) \setminus U_1 \subset \Omega$ are two nonempty disjoint relatively open sets in $(\Omega \cup \{\alpha\}) \setminus G$. Furthermore, it is immediate that $(\Omega \cup \{\alpha\}) \setminus G = V_1 \cup V_2$ and hence we have a contradiction.

($\Leftarrow$) Let $G \subset \Omega$ with $(\mathbb{C} \cup \{\infty\}) \setminus G$ connected. We define the map $\phi : \mathbb{C} \cup \{\infty\} \to \Omega \cup \{\alpha\}$:

$$\phi(x) = \begin{cases} x, & x \in \Omega \\ \alpha, & x \notin \Omega. \end{cases}$$

Obviously, ϕ is continuous and so $\phi\big((\mathbb{C} \cup \{\infty\}) \setminus G\big) = (\Omega \cup \{\alpha\}) \setminus G$ is connected. The proof is complete. $\square$

Combining Theorem 2.2 and Lemma 2.1, we obtain the following theorem.

Theorem 2.4. *Let $\Omega \subset \mathbb{C}$ be a simply connected open set and let $F \subset \Omega$ be a relatively closed set. Then the following are equivalent:*

(1) for every open set $U \subset \Omega$, which contains F, there exists a simply connected open set $V \subset \Omega$ such that $F \subset V \subset U$;

(2) F is an Arakelian set in Ω.

The next corollary is an immediate application of Theorem 2.4.

Corollary 2.1. *Let $\Omega \subset \mathbb{C}$ be a simply connected open set. If $\{F_n\}_{n=1}^{\infty}$ is a locally finite family of pairwise disjoint Arakelian sets in Ω, then the union $\bigcup_{n=1}^{\infty} F_n$ is also an Arakelian set in Ω.*

Proof. Let $U \subset \Omega$ be an open set, which contains the union $\bigcup_{n=1}^{\infty} F_n$. Since $\{F_n\}_{n=1}^{\infty}$ is a locally finite family of pairwise disjoint closed sets in Ω, there exist pairwise disjoint open sets $G_n \subset \Omega$, $n = 1, 2, \ldots$, such that $F_n \subset G_n$ for all n. By Theorem 2.4 there are simply connected open sets V_n with $F_n \subset V_n \subset G_n \cap U$ for every $n \in \mathbb{N}$. Obviously, it holds $\bigcup_{n=1}^{\infty} F_n \subset \bigcup_{n=1}^{\infty} V_n \subset U$ and V_n, $n = 1, 2, \ldots$, are also pairwise disjoint. The latter implies that every component of $\bigcup_{n=1}^{\infty} V_n$ is simply connected and thus $\bigcup_{n=1}^{\infty} V_n$ is a simply connected open set. According to Theorem 2.4, the relatively closed set $\bigcup_{n=1}^{\infty} F_n$ is an Arakelian set in Ω, which completes the proof. $\square$

An alternative proof of the previous result in the case $\Omega = \mathbb{C}$, using Proposition 2.1, could be derived from [5]. In the same article, the authors use their proposition in order to solve a problem of uniform entire approximation. It is worth mentioning that if the conclusion of the preceding corollary is true for some family (not necessarily countable) of pairwise disjoint (connected) Arakelian sets, then there is a method (see

[15]) to join them all, retaining the key properties of an Arakelian set. The following example shows that Corollary 2.1 does not hold when Ω is not simply connected.

Example 2.1. *Let $\Omega = \mathbb{C} \setminus \{0\}$, $F_1 = C(0, r_1)$ and $F_2 = C(0, r_2)$, where $0 < r_1 < r_2$. Observe that F_1, F_2 are two disjoint compact subsets of Ω with connected complements in the one point compactification of Ω. Hence, both sets are Arakelian in Ω. Nevertheless, the union $F_1 \cup F_2$ is not an Arakelian set in Ω, since $(\Omega \cup \{\alpha\}) \setminus (F_1 \cup F_2)$ is not connected.*

The next example shows that even if $\Omega \subset \mathbb{C}$ is a simply connected open set, it is not true that the infinite denumerable union of pairwise disjoint Arakelian sets in Ω is also an Arakelian set in Ω.

Example 2.2. *Let $\Omega = \mathbb{C}$ and let $F_0 = \{2\} \times [0, +\infty)$, $F_n = \big(\{\sum_{i=0}^{n-1} \frac{1}{2^i} - \frac{1}{2^n}, \sum_{i=0}^{n-1} \frac{1}{2^i}\} \times [0, n]\big) \cup \big([\sum_{i=0}^{n-1} \frac{1}{2^i} - \frac{1}{2^n}, \sum_{i=0}^{n-1} \frac{1}{2^i}] \times \{n\}\big)$, $n \geq 1$. It is easy to see that each F_n, $n = 0, 1, \ldots$, is an Arakelian set in $\mathbb{C}$. However, $F = \bigcup_{n=0}^{\infty} F_n$ is not an Arakelian set in $\mathbb{C}$, because despite the fact that F is closed and $(\mathbb{C} \cup \{\infty\}) \setminus F$ is connected, the union of all holes of $F \cup \overline{D(0, r)}$ in $\mathbb{C}$, $r \geq 2$, is unbounded.*

Finally, we present another application of our characterization.

Corollary 2.2. *Let $\Omega \subset \mathbb{C}$ be a simply connected open set and $F \subset \Omega$ be an Arakelian set in Ω. Also, let $f \in A(F)$ be such that $f(z) \neq 0$ for all $z \in F$. Then there exists a function $g \in A(F)$ such that $f = e^g$.*

Proof. According to Tietze's extension theorem, there exists a continuous extension of f on Ω, which we denote by $\widetilde{f} : \Omega \to \mathbb{C}$. Our assumption yields that the open set $U = \Omega \setminus \widetilde{f}^{-1}(\{0\})$ contains F. Thus, by Theorem 2.4 there is a simply connected open set V with $F \subset V \subset U$. If we consider the covering map $exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, then the latter implies that $\widetilde{f}_{|V} : V \to \mathbb{C} \setminus \{0\}$ can be lifted to a continuous function $\mathrm{tg} : V \to \mathbb{C}$, such that $\widetilde{f}_{|V} = e^{\mathrm{tg}}$. The function $g = \mathrm{tg}_{|F}$ is obviously continuous on F and $f = e^g$. Since $f_{|F^0}$ is holomorphic, g is also holomorphic in F^0. Thus, $g \in A(F)$ and the proof is complete. $\square$

We note that for $\Omega = \mathbb{C}$, Corollary 2.2 is known (see [14] for an application).

3. An extension of the main result

As we stated in Lemma 1.1, $(\mathbb{C} \cup \{\infty\}) \smallsetminus K$ must be connected in order for V to be simply connected. However, if we do not require $(\mathbb{C} \cup \{\infty\}) \smallsetminus K$ being disconnected (e.g., when we do approximations with rational functions), then we can obtain the following variation of Lemma 1.1 (see [6], [25]).

Lemma 3.1. *Let $K \subset \mathbb{C}$ be a compact set and $A \subset \mathbb{C} \cup \{\infty\}$ be a set containing one point from each component of $(\mathbb{C} \cup \{\infty\}) \smallsetminus K$. Then for every open set $U \subset \mathbb{C}$ with $K \subset U$, there exists an open set $V \subset \mathbb{C}$ such that $K \subset V \subset U$, every component of $(\mathbb{C} \cup \{\infty\}) \smallsetminus V$ intersects A and $\mathbb{C} \smallsetminus V$ has finitely many components.*

Remark 3.1. *If $(\mathbb{C} \cup \{\infty\}) \smallsetminus K$ is connected and $A = \{\infty\}$, then Lemma 3.1 is actually Lemma 1.1.*

In Section 2 we investigated the relation of Lemma 1.1 with Arakelian's theorem [1], [2], i.e., uniform holomorphic approximation on closed sets. Our efforts in obtaining similar results for Lemma 3.1 indicated that in this case the corresponding extension is closely related to uniform approximation on closed sets by meromorphic functions having prescribed poles. This kind of approximation has been studied in the past and in a very general setting also (see the work due to S. Scheinberg [24]).

Let $M(\Omega)$ denote the set of meromorphic functions in Ω, $H(F)$ the set of holomorphic functions in a (varying) neighborhood of F and $R(G)$ the uniform limits, on G, of rational functions (without poles in G).

Theorem 3.1. *Let $\Omega \subset \mathbb{C}$ be an arbitrary open set and F be a relatively closed subset of Ω. Also, let $B_1, B_2, \dots$ be the holes of F in Ω and let $A \subset \Omega \cup \{\alpha\}$ be a set containing one point in each component of $(\Omega \cup \{\alpha\}) \smallsetminus F$. Then the following are equivalent:*

1. *For every $f \in H(F)$ and $\varepsilon > 0$ there exists $g \in M(\Omega)$, all of whose poles lie in $A \cap \Omega$, such that $\|g - f\|_F < \varepsilon$.*
2. *For every compact set $K \subset \Omega$ the union of the holes of $F \cup K$ in Ω, which do not intersect A, is contained in a compact subset of Ω.*
3. *For every open set $U \subset \Omega$, which contains F, there exists an open set $V \subset \Omega$ such that $F \subset V \subset U$, every component of $(\Omega \cup \{\alpha\}) \smallsetminus V$ intersects A and the collection of all components of $\Omega \smallsetminus V$ is locally finite in Ω.*

Proof. The proof of 1. $\Leftrightarrow$ 2. is given in [24], Theorem 1. So we prove 2. $\Leftrightarrow$ 3. First we show 2. $\Rightarrow$ 3. Let $U \subset \Omega$ be an open set with $F \subset U$ and let $A = \{b_1, b_2, \dots\} \cup \{\zeta\}$, where $b_n \in B_n$, $n \in \mathbb{N}$ and ζ belongs to the component of $(\Omega \cup \{\alpha\}) \setminus F$ containing α. According to our assumption and Proposition 2.1, $\widetilde{F} = F \cup (\bigcup_{n=1}^{\infty} B_n)$ is an Arakelian set in Ω, because for any compact set $K \subset \Omega$ there is at most one hole of $\widetilde{F} \cup K$ in Ω, which intersects A and each hole of $\widetilde{F} \cup K$ is also a hole of $F \cup K$. Hence, by Theorem 2.2 there exists an open set $\widetilde{V} \subset \Omega$ such that $\widetilde{F} \subset \widetilde{V} \subset U \cup (\bigcup_{n=1}^{\infty} B_n)$ and $(\Omega \cup \{\alpha\}) \setminus \widetilde{V}$ is connected.

Under the notation used in the proof of Proposition 2.2, we may assume that $\Omega \setminus U = \bigcup_{i=1}^{\infty} \overline{D(x_i, d_{x_i})}$, where $\{\overline{D(x_i, d_{x_i})} \mid i = 1, 2, \dots\}$ is a locally finite family of closed disks in Ω. Let $k_1, k_2, \dots$ be the indices for which $A_n = \{x_1, x_2, \dots\} \cap B_{k_n} \neq \emptyset$, $n \in \mathbb{N}$. Also, let $\{K_m\}_{m \in \mathbb{N}}$ be an exhausting sequence of compact subsets of Ω and let H_m, $m \in \mathbb{N}$, be the union of the holes of $F \cup K_m$ in Ω which do not intersect A. We can join inductively the x_i's contained in each B_{k_n} with b_{k_n} by curves $\Gamma_{in} \subset B_{k_n}$, $n \in \mathbb{N}$, such that every K_m intersects at most finitely many of all these curves.

- For $m \in \mathbb{N}$, our assumption and the fact that the set $\{x_1, x_2, \dots\}$ has no accumulation points in Ω, imply that there are finitely many x_i's contained in $K_m \cup H_m$. If such a point x_i is also contained in some B_{k_n} and we have not already joined it with b_{k_n} (induction hypothesis), then it must be contained in a hole $B \subset B_{k_n}$ of $F \cup K_{m-1}$ in Ω, $K_0 = \emptyset$, which contains $b_{k_n} \in A$. Whenever the case, we join x_i with b_{k_n} by a curve $\Gamma_{in} \subset B \subset (\Omega \setminus K_{m-1}) \cap B_{k_n}$.

Thus, the family of the curves Γ_{in} is locally finite in Ω and more particularly the union $\bigcup_{n} \bigcup_{x_i \in A_n} \Gamma_{in}$ is closed in Ω. Lastly, ζ can be joined with α by a curve Γ in $\Omega \setminus \widetilde{F} \subset \Omega \setminus F$, since $\widetilde{F}$ is an Arakelian set in Ω (see the proof of Proposition 2.2).

We define V to satisfy

$$(\Omega \cup \{\alpha\}) \setminus V = \big[(\Omega \cup \{\alpha\}) \setminus \widetilde{V}\big] \cup \Gamma \cup \bigcup_{n} \Big[\bigcup_{x_i \in A_n} \big(\overline{D(x_i, d_{x_i})} \cup \Gamma_{in}\big)\Big].$$

It is evident that V is open, $F \subset V \subset U$ and every component of $(\Omega \cup \{\alpha\}) \setminus V$ intersects A. Finally, by Remark 2.2 (applied to $\widetilde{V}$) and the construction of the curves Γ_{in}, $x_i \in A_n$, $n \in \mathbb{N}$, it follows that the collection of all components of

$\Omega \smallsetminus V = (\Omega \smallsetminus \widetilde{V}) \cup \Gamma \cup \bigcup_n \big[\bigcup_{x_i \in A_n} \big(\overline{D(x_i, d_{x_i})} \cup \Gamma_{in}\big)\big]$ is locally finite in Ω, yielding the implication 2. $\Rightarrow$ 3.

To prove 3. $\Rightarrow$ 2., let $\widetilde{F} = F \cup (\bigcup_{n=1}^{\infty} B_n)$, $A = \{b_1, b_2, \ldots\} \cup \{\zeta\}$ be as above and let $U \subset \Omega$ be an open set with $F \subset \widetilde{F} \subset U$. It follows that there exists an open set $V \subset \Omega$, such that $F \subset V \subset U$ and every component of $(\Omega \cup \{\alpha\}) \smallsetminus V$ intersects A. Further, we have $\widetilde{F} \subset V \cup (\bigcup_{n=1}^{\infty} B_n) \subset U$. Observe that $\widetilde{V} = V \cup (\bigcup_{n=1}^{\infty} B_n)$ is open and $(\Omega \cup \{\alpha\}) \smallsetminus \widetilde{V}$ is the component of $(\Omega \cup \{\alpha\}) \smallsetminus V$ which intersects A at ζ. Thus, $(\Omega \cup \{\alpha\}) \smallsetminus \widetilde{V}$ is connected and by Theorem 2.2 $\widetilde{F}$ is an Arakelian set in Ω.

Suppose that there is a compact set $K \subset \Omega$ such that the union of the holes of $F \cup K$ in Ω, which do not intersect A, is not contained in any compact subset of Ω. Since $\widetilde{F}$ is an Arakelian set in Ω, Proposition 2.1 implies that there are infinitely many of these holes, which accumulate on α and are contained in corresponding holes of F in Ω. Let $C_n \subset B_{k_n}$, $n \in \mathbb{N}$, be a sequence of such holes and $x_n \in C_n$ with $x_n \to \alpha$, as $n \to +\infty$. We consider the open set $U = \Omega \smallsetminus \{x_1, x_2, \ldots\}$, which obviously contains F. Based on our assumption, there exists an open set $V \subset \Omega$ such that $F \subset V \subset U$, each component of $(\Omega \cup \{\alpha\}) \smallsetminus V$ intersects A and the collection of all components of $\Omega \smallsetminus V$ is locally finite in Ω. Note that each component of $(\Omega \cup \{\alpha\}) \smallsetminus V$ is contained in some component of $(\Omega \cup \{\alpha\}) \smallsetminus F$.

Since $x_n \in \Omega \smallsetminus U \subset \Omega \smallsetminus V$, $n \in \mathbb{N}$, the previous fact yields that the component of $(\Omega \cup \{\alpha\}) \smallsetminus V$ which contains x_n, is contained in B_{k_n} and consequently intersects A at b_{k_n}. However, the holes C_n (of $F \cup K$), $n \in \mathbb{N}$, do not intersect A. Therefore, either $b_{k_n} \in \partial C_n$ or the latter component intersects the interior and exterior of C_n. Whatever the case, there exist $y_n \in (\Omega \smallsetminus V) \cap \partial C_n \cap \partial K$ for all $n \in \mathbb{N}$. Also, it is clear that the terms of the sequence $(y_n)_{n \in \mathbb{N}}$ are contained in distinct components of $(\Omega \cup \{\alpha\}) \smallsetminus V$. Thus, K intersects infinitely many components of $\Omega \smallsetminus V$ and we obtain a contradiction. The proof is complete. $\square$

Remark 3.2. *It is sufficient to require condition* 2. *only for an exhausting sequence* $\{K_m\}_{m \in \mathbb{N}}$ *of compact subsets of* Ω*, as in Remark 2.1.*

With our previous result, we managed to give a precise connection of Lemma 3.1 with complex approximation. However, we notice that there is an essential difference between Theorem 3.1 and Theorems 2.1, 2.2. That is because in Theorem 3.1 we considered a smaller class of approximable functions, namely, $H(F)$ instead of $A(F)$. Now we would like to characterize the closed sets F (in $\Omega \subset \mathbb{C}$) for which every element in $A(F)$ admits the corresponding approximation.

Observe that in this case both topological conditions 2. and 3. of Theorem 3.1 fail to imply the stronger approximation we seek. The reason behind this issue is that the analogue of Mergelyan's theorem [17] for rational approximation does not hold in general. Indeed, there is an interesting "Swiss cheese"example, where for certain compact sets $K \subset \Omega$ we have $R(K) \neq A(K)$ (see A. Roth's construction [9], p. 110). Thus, we need to impose an additional assumption, which will ensure that F has no such problematic compact subsets.

Theorem 3.2. *Let Ω, F, $\{B_n\}_{n=1}^{\infty}$ and A be as in Theorem 3.1. Then the following are equivalent:*

(1) *Every function $f \in A(F)$ can be uniformly approximated on F by meromorphic functions $g \in M(\Omega)$, all of whose poles lie in $A \cap \Omega$;*

(2) (a) $A(F \cap \overline{D}) = R(F \cap \overline{D})$, *for each disk D such that $\overline{D} \subset \Omega$ and*

(b) *for every compact set $K \subset \Omega$ the union of the holes of $F \cup K$ in Ω, which do not intersect A, is contained in a compact subset of Ω;*

(3) (a) $A(F \cap \overline{D}) = R(F \cap \overline{D})$, *for each disk D such that $\overline{D} \subset \Omega$,*

(b) *for every open set $U \subset \Omega$, which contains F, there exists an open set $V \subset \Omega$ such that $F \subset V \subset U$, every component of $(\Omega \cup \{\alpha\}) \setminus V$ intersects A and the collection of all components of $\Omega \setminus V$ is locally finite in Ω.*

Proof. The proof of (1) $\Leftrightarrow$ (2) follows from Theorem 1 in [24], if we take into consideration Theorem 2 from [9] (p. 136). As for (2) $\Leftrightarrow$ (3), by Theorem 3.1, we have (2) (b) $\Leftrightarrow$ (3) (b). $\square$

Remark 3.3. *If F has no holes in Ω, that is, $(\Omega \cup \{\alpha\}) \setminus F$ is connected, then one can easily see that condition* 2. (a) *is satisfied thanks to Mergelyan's theorem (see* [9], *p. 135). In this case, for $A = \{\alpha\}$, we notice that Theorem 3.2 is in fact a combination of Theorem 2.1, Proposition 2.1, Theorem 2.2 and Remark 2.2.*

Acknowledgements: I would like to thank V. Nestoridis for his valuable suggestions and his interest in this work. Also, I would like to thank P. M. Gauthier for useful comments on preliminary versions of the present article and for bringing to my attention references I was not aware of.

Список литературы

[1] N. U. Arakelian, "Uniform approximation on closed sets by entire functions" [in Russian], Izv. Akad. Nauk SSSR, **28**, 1187 – 1206 (1964).

[2] N. U. Arakelian, "Uniform and tangential approximations by analytic functions" [in Russian], Izv. Akad. Nauk SSR, **3**, 273 – 285 (1968); Translation in American Mathematical Society Translations (2), **122**, 85 – 97 (1984).

[3] D. H. Armitage, P. M. Gauthier, "Recent developments in harmonic approximation with application", Results in Math., **29**, 1 – 15 (1996).

[4] G. Costakis, "Some remarks on universal functions and Taylor series", Math. Proc. Cambr. Philos. Soc., **128**, 157 – 175 (2000).

[5] A. A. Danielyan and L. A. Rubel, "Uniform approximation by entire functions that are all bounded on a given set", Constr. Approx., **14**, 469 – 473 (1998).

[6] E. Diamantopoulos, Ch. Mouratides and N. Tsirivas, "Universal Taylor series on unbounded open sets", Analysis (Munich), **26**, 323 – 336 (2006).

[7] G. Fournodavlos, "On a characterization of Arakelian sets", arXiv:1107.0393 (2011).

[8] G. Fournodavlos and V. Nestoridis, "Generic approximation of functions by their Padé approximants", arXiv: 1106.0169 (2011).

[9] D. Gaier, Lectures on Complex Approximation, Birkhäuser (1987).

[10] P. M. Gauthier, "Meromorphic uniform approximation on closed subsets of open Riemann surfaces", Approx. Theory and Funct. Anal. (Proc. Internat. Sympos. Approximation Theory, Univ. Estadual de Campinas, Campinas, 1977), North-Holland Pub. Co., 139 – 158 (1979).

[11] P. M. Gauthier, "Subharmonic extensions and approximations", Can. Math. Bull., **37**, 46 – 53 (1994).

[12] P. M. Gauthier, M. Goldstein and W. H. OW, "Uniform approximation on closed sets by harmonic functions with Newtonian singularities", J. London Math. Soc. (2), **28**, 71 – 82 (1983).

[13] P. M. Gauthier and W. Hengartner, "Uniform approximation on closed sets by functions analytic on a Riemann surface", Approx. Theory (Proc. Conf. Inst. Math., Adam Mickiewicz Univ., Poznan, 1972), Reidel, Dordrecht, 63 – 69 (1975).

[14] P. M. Gauthier and M. R. Pouryayevali, "Approximation by Meromorphic Func- tions with Mittag-Leffler Type Constraints", Can. Math. Bull., **44**, 420 – 428 (2001).

[15] P. M. Got'e, "Remarks on a theorem of Keldysh and Lavrent'ev" [in Russian], Translated from the English by A. Nersesyan. Uspekhi Mat. Nauk, **40**, no. 4(244), 157 – 158 (1985).

[16] K.-G. Grosse-Erdmann, "Holomorphe Monster und universelle Funktionen", Mitt. Math. Sem. Giessen, **176**, 1 – 84 (1987).

[17] S. N. Mergelyan, "Uniform approximations to functions of a complex variable" [in Russian], Uspehi Mat. Nauk (N. S.) **7**, no. 2 (48), 31 – 122 (1952). Translation in Amer. Math. Soc. Transl. Ser. 1, **3**, 294 – 391 (1962).

[18] R. Narasimhan, Analysis on Real and Complex Manifolds, 2nd ed., North-Holland Pub. Co. (1973).

[19] J.-P. Rosay and W. Rudin, "Arakelian's approximation theorem", Amer. Math. Monthly, **96**, 432 – 434 (1989).

[20] A. Roth, "Uniform and tangential approximation by meromorphic functions on closed sets", Canad. J. Math., **28**, 104 – 111 (1976).

[21] W. Rudin, Real and Complex Analysis, 3rd ed., McGraw-Hill (1986).

[22] C. Runge, "Zur Theorie der eindeutigen analytischen Functionen", Acta Math., **6**, 229 – 244 (1885).

[23] S. Scheinberg, "Uniform approximation by functions analytic on a Riemann surface", Annals of Math., **108**, 257 – 298 (1978).
[24] S. Scheinberg, "Uniform approximation by meromorphic functions having prescribed poles", Math. Ann., **243**, 83 – 93 (1979).
[25] N. Tsirivas, Master's Thesis, Universal Taylor series, University of Athens (Greek) (2004).

Поступила 3 сентября 2011