Известия НАН Армении. Математика, том 47, н. 5, 2012, стр. 65-76.

GABOR FRAMES ON A HALF-LINE

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Abstract. The objective of this paper is to construct Gabor frame on a positive half-line. A necessary condition and two sufficient conditions for Gabor frame on a positive half-line are given in the time domain.

MSC2000 numbers: 42C15; 42C40; 42A38; 41A17.
Keywords: Frame; Gabor frame; Riesz basis; Walsh function.

1. Introduction

Frames were first introduced by Duffin and Schaeffer [7] in the context of non-harmonic Fourier series. Outside signal processing, frames did not seem to generate much interest until the seminal work by Daubechies, Grossmann, and Meyer [6]. They showed that Duffin and Schaeffer's definition is an abstraction of a concept given by Gabor [11] in 1946 for doing signal analysis. The frames introduced by Gabor now are called *Gabor frames* and have been widely used in communication theory, quantum mechanics and many other fields. For more about Gabor frames and their applications to signal and image processing, we refer to the monographs [9, 10,13].

Gabor systems $\{M_{mb}T_{na}g(x)\}_{m,n\in\mathbb{Z}}$ are generated by modulations and translations of a single function $g(x) \in L^2(\mathbb{R})$ and hence, can be viewed as the set of time-frequency shifts of $g(x)$ along the lattice $a\mathbb{Z}\times b\mathbb{Z}$ in $\mathbb{R}^2$. Gabor systems that form frames for $L^2(\mathbb{R})$ have a wide variety of applications. An important problem in practice is therefore to determine conditions for Gabor systems to be frames.

Many results in this area, including necessary conditions and sufficient conditions, have been established during the last two decades. For example, in 1990, Daubechies [5] proved the first result on the necessary and sufficient conditions for the Gabor system $\{M_{mb}T_{na}g(x)\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$, Chui and Shi improved the result of Daubechies in [4], Casazza and Christenson [1, 2] established a stronger version of Daubechies sufficient condition for wavelet frames. Recently, Shi and Chen

[17] have established a set of necessary conditions for Gabor frames and showed that these conditions are also sufficient for tight frames. Li *et al.* [14] gave two sufficient conditions for Gabor frames in $L^2(\mathbb{R})$ in terms of Fourier transform and showed the conditions are better than those of Daubechies [5].

Although there are many results for Gabor frame on the real-line $\mathbb{R}$, the counterparts on positive half-line $\mathbb{R}^+$ are not yet reported. So this paper is concerned with Gabor frame on positive half-line $\mathbb{R}^+$. Concerning the construction of wavelets on a half-line, Farkov [8] has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$ and proved necessary and sufficient conditions for scaling filters with p^n many terms $(p, n \geq 2)$ to generate a p-MRA analysis in $L^2(\mathbb{R}^+)$.

Recently, Shah and Debnath [16], have constructed dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ using the Walsh-Fourier transform and have established a necessary condition and a sufficient condition for the system

$$\left\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x \ominus k) : j \in \mathbb{Z}, k \in \mathbb{Z}^+\right\}$$

to be a frame for $L^2(\mathbb{R}^+)$. The objective of this paper is to prove the existence of the Gabor system $\{M_{mb}T_{na}g(x)\}_{m,n\in\mathbb{Z}}$ that forms a frame for $L^2(\mathbb{R}^+)$. We also establish a necessary condition and two sufficient conditions for the system $\{M_{mb}T_{na}g(x)\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R}^+)$.

The paper is structured as follows. In Section 1, we introduce some notations and preliminaries related to the operations on positive half-line $\mathbb{R}^+$, and some lemmas to be used throughout the paper. In Section 2, we prove the existence of the system $\{M_{mb}T_{na}g(x)\}_{m,n\in\mathbb{Z}}$ that forms a frame for $L^2(\mathbb{R}^+)$. In Section 3 we establish one necessary condition and two sufficient conditions for Gabor frame on the positive half-line $\mathbb{R}^+$.

As usual, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{0, 1, 2, ...\}$ and $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Denote by $[x]$ the integer part of x. Let a be a fixed natural number greater than 1. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [a^j x](\mathrm{mod}\, a), \quad x_{-j} = [a^{1-j} x](\mathrm{mod}\, a). \tag{1.1}$$

We consider on $\mathbb{R}^+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \zeta_j a^{-j-1} + \sum_{j>0} \zeta_j a^{-j}$$

with $\zeta_j = x_j + y_j (\text{mod } a)$ $(j \in \mathbb{Z} \setminus \{0\})$, where $\zeta_j \in \{0, 1, 2, ..., a-1\}$ and x_j, y_j are calculated by (1.1). Note that $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo a in $\mathbb{R}^+$.

For $x \in [0, 1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & \text{if } \ x \in [0, 1/a) \\ \varepsilon_a^\ell, & \text{if } x \in [\ell a^{-1}, (\ell+1)a^{-1}), \quad \ell = 1, 2, ..., a-1, \end{cases}$$

where $\varepsilon_a = \exp(2\pi i/a)$. The extension of the function r_0 to $\mathbb{R}^+$ is given by the equality $r_0(x+1) = r_0(x),\ x \in \mathbb{R}^+$. Then, the generalized Walsh functions $\{w_m(x) : m \in \mathbb{Z}^+\}$ are defined by

$$w_0(x) \equiv 1, \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(a^j x)\right)^{\mu_j},$$

where $m = \sum_{j=0}^{k} \mu_j a^j$, $\mu_j \in \{0, 1, 2, ..., a-1\}$, $\mu_k \neq 0$.
For $x, y \in \mathbb{R}^+$, let

$$\chi(x, y) = \exp\left(\frac{2\pi i}{a} \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)\right), \tag{1.2}$$

where x_j, y_j are given by (1.1). Note that $\chi(x, m/a^{n-1}) = \chi(x/a^{n-1}, m) = w_m(x/a^{n-1})$ for all $x \in [0, a^{n-1}),\ m \in \mathbb{Z}^+$. If $x, y, \xi \in \mathbb{R}^+$ and $x \oplus y$ is a-adic irrational, then

$$\chi(x \oplus y, \xi) = \chi(x, \xi)\, \chi(y, \xi). \tag{1.3}$$

It was shown by Golubov *et al.* [12] that both the systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(., \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$.

By a-adic interval $I \subset \mathbb{R}^+$ of range n, we mean interval of the form

$$I = I_n^k = \left[ka^{-n}, (k+1)a^{-n}\right), \quad k \in \mathbb{Z}^+.$$

The a-adic topology is generated by the collection of a-adic intervals and each a-adic interval is both open and closed under the a-adic topology (see [15]). Therefore, for each $0 \leq j, k < a^n$, the Walsh function $w_j(x)$ is piecewise constant and hence continuous. Thus $w_j(x) = 1$ for $x \in I_n^0$.

Let $\mathcal{E}_n(\mathbb{R}^+)$ be the space of a-adic entire functions of order n, that is, the set of all functions which are constant on all a-adic intervals of range n. Thus, for every $f \in \mathcal{E}_n(\mathbb{R}^+)$, we have

$$f(x) = \sum_{k \in \mathbb{Z}^+} f(a^{-n}k)\chi_{I_n^k}(x),\ x \in \mathbb{R}^+. \tag{1.4}$$

Clearly each Walsh function of order a^{n-1} belong to $\mathcal{E}_n(\mathbb{R}^+)$. The set $\mathcal{E}(\mathbb{R}^+)$ of a-adic entire functions on $\mathbb{R}^+$ is the union of all the spaces $\mathcal{E}_n(\mathbb{R}^+)$. It is clear that $\mathcal{E}(\mathbb{R}^+)$ is dense in $L^p(\mathbb{R}^+), 1 \leq p < \infty$ and each function in $\mathcal{E}(\mathbb{R}^+)$ is of compact support. Thus, we will consider the following set of functions:

$$\mathcal{E}^0(\mathbb{R}^+) = \left\{ f \in \mathcal{E}(\mathbb{R}^+) : \operatorname{supp} f \subset \mathbb{R}^+ \setminus \{0\} \right\}. \tag{1.5}$$

Definition 1.1. A function f defined on $\mathbb{R}^+$ is said to be with period a if $f(x \oplus ka) = f(x)$ for all $x \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$.

Definition 1.2. Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathbb{H}$ is called a *frame* for $\mathbb{H}$ if there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} \left|\langle f, f_k \rangle\right|^2 \leq B\|f\|^2 \text{ for all } f \in \mathbb{H}. \tag{1.6}$$

The largest constant A and the smallest constant B satisying (1.6) are called the *upper* and the *lower frame bound* respectively. The sequence $\{f_k\}_{k=1}^{\infty}$ is called a *tight frame* for $\mathbb{H}$ if the upper frame bound A and the lower frame bound B coincide.

The sequence $\{f_k\}_{k=1}^{\infty}$ is called a *Bessel sequence* in $\mathbb{H}$ if only the right-hand side inequality in (1.6) holds and is called *Riesz basis* for $\mathbb{H}$ if there exists a linear, bounded bijective operator $T : \mathbb{H} \to \mathbb{H}$ such that $\{f_k\}_{k=1}^{\infty} = \{Te_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathbb{H}$. For the fundamentals of frame theory and its applications we refer to [3, 13].

Let $\{f_k\}_{k=1}^{\infty}$ be a frame in $\mathbb{H}$. The operator $S : \mathbb{H} \to \mathbb{H}$ is called a *frame operator* associated with the frame $\{f_k\}_{k=1}^{\infty}$ if $Sf = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k$. It is well known (see [3]) that S is linearly bounded, invertible, self-adjoint and positive, and the system $\left\{S^{-1} f_k\right\}_{k=1}^{\infty}$ is also a frame in $\mathbb{H}$ with bounds B^{-1}, A^{-1}, which is called the *canonical dual* of the frame $\{f_k\}_{k=1}^{\infty}$. Moreover, this provides the reconstruction formula

$$f = SS^{-1} = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k \rangle f_k = \sum_{k=1}^{\infty} \langle f, f_k \rangle S^{-1} f_k, \tag{1.7}$$

where both series converge unconditionally for all $f \in \mathbb{H}$. Thus, a frame $\{f_k\}_{k=1}^{\infty}$ allows every $f \in \mathbb{H}$ to be written as a series expansion of the frame elements, which is similar to the property of basis; the main difference is that the frame coefficients $\langle f, S^{-1} f_k \rangle$ in (1.7) can, generally, be replaced by other coefficients. Further, it should be noted that $\left\{S^{-1/2} f_k\right\}_{k=1}^{\infty}$ is a tight frame with bound 1 and hence

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1/2} f_k \rangle S^{-1/2} f_k. \tag{1.8}$$

In order to prove theorems to be presented in next section, we need the following lemmas whose proofs can be found in [3].

Lemma 1.3. Let $\{f_k\}_{k=1}^{\infty}$ be a frame for a Hilbert space $\mathbb{H}$. Then $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis for $\mathbb{H}$ if and only if $\{f_k\}_{k=1}^{\infty}$ is complete in $\mathbb{H}$, and there exist constants $A, B > 0$ such that for every finite scalar sequence $\{c_k\}$,

$$A\sum_k |c_k|^2 \leq \left\| \sum_k c_k f_k \right\|^2 \leq B\sum_k |c_k|^2.$$

Lemma 1.4. Let $\{f_k\}_{k=1}^{\infty}$ be a Bessel sequence with bound B for a Hilbert space $\mathbb{H}$. Then

(i) $\|f_k\|^2 \leq B,\ k = 1, 2, ...$

(ii) $\langle f_j, f_k\rangle = 0,\ j \neq k$, whenever $\|f_k\|^2 = B$, for some k.

Lemma 1.5. Suppose that $\mathbb{H}$ is a Hilbert space. Let $T : \mathbb{H} \to \mathbb{H}$ be a bounded operator, and assume that $\langle Tx, x\rangle = 0$, for all $x \in \mathbb{H}$. If $\mathbb{H}$ is a complex Hilbert space, then $T = 0$; if $\mathbb{H}$ is a real Hilbert space and T is self-adjoint, then $T = 0$.

2. EXISTENCE OF A GABOR FRAME IN $L^2(\mathbb{R}^+)$

Let p and q be two given positive real numbers. For any fixed function $g \in L^2(\mathbb{R}^+)$, the family of functions of the form

$$\left\{M_{mq}T_{np}g(x) = w_{mq}(x)g(x \ominus np), \quad m, n \in \mathbb{Z}^+, x \in \mathbb{R}^+\right\} \tag{2.1}$$

is called a Gabor frame for $L^2(\mathbb{R}^+)$ if there exist constants A and B, $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+} |\langle f, M_{mq}T_{np}g\rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in L^2(\mathbb{R}^+), \tag{2.2}$$

where $M_{mq}f(x) = w_{mq}f(x)$ and $T_{np}f(x) = f(x \ominus np)$ are the modulation and translation operators defined on $L^2(\mathbb{R}^+)$, respectively.

We first show that the operator S associated with the Gabor frame $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is commutative, i.e.,

$$SM_{mq}T_{np} = M_{mq}T_{np}S. \tag{2.3}$$

Indeed, since $M_qT_pf(x) = w_q(x)f(x \ominus p)$ and $T_pM_qf(x) = \overline{w_q(x)}M_qT_pf(x)$, we have

$$\begin{aligned} SM_{mq}T_{np}f &= \sum_{k\in\mathbb{Z}^+}\sum_{\ell\in\mathbb{Z}^+} \langle M_{mq}T_{np}f, M_{kq}T_{\ell p}g\rangle M_{kq}T_{\ell p}g \\ &= \sum_{k\in\mathbb{Z}^+}\sum_{\ell\in\mathbb{Z}^+} \langle f, w_{(k-m)pqn}M_{(k-m)kq}T_{(\ell-n)p}g\rangle M_{kq}T_{\ell p}g \end{aligned}$$

$$= \sum_{k\in\mathbb{Z}^+}\sum_{\ell\in\mathbb{Z}^+} \overline{w_{kpqn}}\langle f, M_{kq}T_{\ell p}g\rangle M_{(k+m)q}T_{(\ell+n)p}g$$

$$= \sum_{k\in\mathbb{Z}^+}\sum_{\ell\in\mathbb{Z}^+} \langle f, M_{kq}T_{\ell p}g\rangle M_{mq}T_{np}M_{kq}T_{\ell p}g = M_{mq}T_{np}Sf.$$

This commutative property of S implies that S^{-1} commutes with the operator $M_{mq}T_{np}$. Consequently, $S^{-1/2}$ also commutes with $M_{mq}T_{np}$ and as a consequence, we have the following theorem.

Theorem 2.1. Let $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ be a Gabor frame for $L^2(\mathbb{R}^+)$. Then, Gabor structure of the canonical dual is given by $\left\{M_{mq}T_{np}S^{-1}g\right\}_{m,n\in\mathbb{Z}^+}$, where $g \in L^2(\mathbb{R}^+)$ and $p, q \in \mathbb{R}^+\setminus\{0\}$. Moreover, the canonical tight frame associated with $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is $\left\{M_{mq}T_{np}S^{-1/2}g\right\}_{m,n\in\mathbb{Z}^+}$.

First we prove two lemmas, which will be used in the proofs of the main results.

Lemma 2.2. Let $f, g \in L^2(\mathbb{R}^+), p, q \in \mathbb{R}^+\setminus\{0\}$ and $k \in \mathbb{Z}^+$ be given. Then the series

$$\sum_{n\in\mathbb{Z}^+} f(x\ominus np)\overline{g(x\ominus np\ominus q^{-1}k)}, \qquad x\in\mathbb{R}^+ \tag{2.4}$$

converges absolutely for almost all $x \in \mathbb{R}^+$. Furthermore, for any $m \in \mathbb{Z}^+$, we have

$$\langle f, M_{mq}T_{np}g\rangle = \int_0^p G_n(x)\overline{w_{mq}(x)}dx, \tag{2.5}$$

where

$$G_n(x) = \sum_{k\in\mathbb{Z}^+} f(x\ominus q^{-1}k)\overline{g(x\ominus np\ominus q^{-1}k)}. \tag{2.6}$$

Proof. Since $f, T_{q^{-1}k}g(x) \in L^2(\mathbb{R}^+)$, we have $f.\,\overline{T_{q^{-1}k}g(x)} \in L^1(\mathbb{R}^+)$. Thus

$$\int_0^p \sum_{n\in\mathbb{Z}^+} |f(x\ominus np)\overline{g(x\ominus np\ominus q^{-1}k)}|dx = \int_{\mathbb{R}^+} |f(x)\overline{g(x\ominus q^{-1}k)}|dx < \infty,$$

and hence

$$\sum_{n\in\mathbb{Z}^+} |f(x\ominus np)\overline{g(x\ominus np\ominus q^{-1}k)}| < \infty, \quad \text{for almost all } x\in[0,p).$$

Since the series in (2.4) converges a.e. on $[0,p)$, therefore, it converges absolutely a.e. on $x \in \mathbb{R}^+$ and by the Definition 1.1, it defines a function with period p. Further, we have

$$\langle f, M_{mq}T_{np}g\rangle = \int_{\mathbb{R}^+} f(x)\overline{g(x\ominus np)}.\,\overline{w_{mq}(x)}dx$$

$$= \sum_{k\in\mathbb{Z}^+}\int_0^{q^{-1}} f(x\ominus q^{-1}k)\overline{g(x\ominus np\ominus q^{-1}k)}.\,\overline{w_{mq}(x)}dx = \int_0^{q^{-1}} G_n(x)\overline{w_{mq}(x)}dx.$$

This completes the proof.

To prove the existence of the system (2.1) that forms a Gabor frame in $L^2(\mathbb{R}^+)$, we set

$$D(x) = \sum_{n\in\mathbb{Z}^+} |g(x \ominus np)|^2, \quad x \in \mathbb{R}^+, \tag{2.7}$$

$$E_k(x) = \sum_{n\in\mathbb{Z}^+} g(x \ominus np)\overline{g(x \ominus np \ominus q^{-1}k)}, \quad x \in \mathbb{R}^+, k \in \mathbb{Z}^+. \tag{2.8}$$

Note that $D(x)$ and $E_k(x)$ are bounded functions with period p and $D(x) = E_0(x)$.

Lemma 2.3. Let f be in $\mathcal{E}^0(\mathbb{R}^+)$ and g be in $L^2(\mathbb{R}^+)$ with $p, q \in \mathbb{R}^+ \setminus \{0\}$. Then

$$\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+} |\langle f, M_{mq}T_{np}g\rangle|^2 = \frac{1}{|q|}\int_{\mathbb{R}^+} |f(x)|^2 D(x)dx + R_g(f), \tag{2.9}$$

where

$$R_g(f) = \frac{1}{|q|}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^+} \overline{f(x)}f(x \ominus q^{-1}k)E_k(x)dx. \tag{2.10}$$

Proof. Since $f \in \mathcal{E}^0(\mathbb{R}^+)$, the function $f(x \ominus q^{-1}k)$ can be non-zero only for finitely many values of k. The number of values of k, for which $f(x \ominus q^{-1}k) \neq 0$ is uniformly bounded. Consequently, each $G_n(x)$ defined by (2.6) is bounded, and hence $G_n \in L^1[0, q^{-1}) \cap L^2[0, q^{-1})$. By Lemma 1.2, for all $m, n \in \mathbb{Z}^+$, we have

$$\langle f, M_{mq}T_{np}g\rangle = \int_0^{q^{-1}} G_n(x)\overline{w_{mq}(x)}dx. \tag{2.11}$$

Applying Parseval's theorem and using the fact that $\left\{q^{1/2}w_{mq}(x)\right\}_{m\in\mathbb{Z}^+}$ forms an orthonormal basis for $L^2[0, q^{-1})$, we obtain

$$\sum_{m\in\mathbb{Z}^+}\left|\int_0^{q^{-1}} G_n(x)\overline{w_{mq}(x)}dx\right|^2 = \frac{1}{|q|}\int_0^{q^{-1}} |G_n(x)|^2 dx. \tag{2.12}$$

By virtue of (2.11) and (2.12), we get

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+} |\langle f, M_{mq}T_{np}g\rangle|^2 &= \sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}\left|\int_0^{q^{-1}} G_n(x)\overline{w_{mq}(x)}dx\right|^2 \\
&= \frac{1}{|q|}\sum_{n\in\mathbb{Z}^+}\int_0^{q^{-1}} |G_n(x)|^2 dx = \frac{1}{|q|}\sum_{n\in\mathbb{Z}^+}\int_0^{q^{-1}} G_n(x)\overline{G_n(x)}dx \\
&= \frac{1}{|q|}\sum_{n\in\mathbb{Z}^+}\int_0^{q^{-1}} \sum_{k\in\mathbb{Z}^+} \overline{f(x \ominus q^{-1}k)}g(x \ominus np \ominus q^{-1}k)G_n(x)dx \\
&= \frac{1}{|q|}\sum_{n\in\mathbb{Z}^+}\int_{\mathbb{R}^+} \overline{f(x)}g(x \ominus np)G_n(x)dx
\end{aligned}$$

$$= \frac{1}{|q|} \sum_{n\in\mathbb{Z}^+} \int_{\mathbb{R}^+} \overline{f(x)} g(x \ominus np) \sum_{k\in\mathbb{Z}^+} f(x \ominus q^{-1}k) \overline{g(x \ominus np \ominus q^{-1}k)} dx$$

$$= \frac{1}{|q|} \int_{\mathbb{R}^+} |f(x)|^2 \sum_{n\in\mathbb{Z}^+} |g(x \ominus np)|^2 dx$$

$$+ \frac{1}{|q|} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^+} \overline{f(x)} f(x \ominus q^{-1}k) \sum_{n\in\mathbb{Z}^+} g(x \ominus np) \overline{g(x \ominus np \ominus q^{-1}k)}$$

$$= \frac{1}{|q|} \int_{\mathbb{R}^+} |f(x)|^2 D(x) dx + \frac{1}{|q|} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^+} \overline{f(x)} f(x \ominus q^{-1}k) E_k(x) dx.$$

This completes the proof.

Theorem 2.4 (*Existence*). Let $g \in L^2(\mathbb{R}^+)$ and $p, q \in \mathbb{R}^+ \setminus \{0\}$ be given. Then the following holds:

(i). If $|pq| > 1$, then $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is not a frame for $L^2(\mathbb{R}^+)$.

(ii). If $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is a frame, then $|pq| = 1$ if and only if $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is a Riesz basis.

Proof. Suppose that $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is a Gabor frame for $L^2(\mathbb{R}^+)$ and let $\left\{M_{mq}T_{np}S^{-1/2}g\right\}_{m,n\in\mathbb{Z}^+}$ be the canonical tight frame associated with $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$. Then, by Lemma 2.3, we have

$$\int_{\mathbb{R}^+} |f(x)|^2 dx = \sum_{m\in\mathbb{Z}^+} \sum_{n\in\mathbb{Z}^+} |\langle f, M_{mq}T_{np}S^{-1/2}g\rangle|^2$$

$$= \frac{1}{|q|} \int_{\mathbb{R}^+} |f(x)|^2 \sum_{n\in\mathbb{Z}^+} |S^{-1/2}g(x \ominus np)|^2 dx$$

Thus, this gives that $\sum_{n\in\mathbb{Z}^+} |S^{-1/2}g(x \ominus np)|^2 = |q|$, a.e. on $x \in \mathbb{R}^+$ and consequently, we have

$$\|S^{-1/2}g\|^2 = \int_{\mathbb{R}^+} |S^{-1/2}g|^2 dx = \int_0^p \sum_{n\in\mathbb{Z}^+} |S^{-1/2}g(x \ominus np)|^2 dx = |pq|.$$

Now, to prove (i), it is sufficient to prove that $|pq| \le 1$ for given frame $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$. Since $\left\{M_{mq}T_{np}S^{-1/2}g\right\}_{m,n\in\mathbb{Z}^+}$ is a tight frame with bound 1, therefore, it follows by Lemma 1.4 that $\|S^{-1/2}g\|^2 = |pq| \le 1$, and the result follows.

To prove (ii), we first assume that $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is a Riesz basis. Then by Lemma 2.3, $\left\{M_{mq}T_{np}S^{-1/2}g\right\}_{m,n\in\mathbb{Z}^+}$ is also a Riesz basis with bounds $A = B = 1$. This implies that $\|S^{-1/2}g\|^2 = |pq| = 1$.

Now, assume that $|pq| = 1$, which implies that $\|S^{-1/2}g\|^2 = \|M_{mq}T_{np}S^{-1/2}g\|^2 = 1$, for all $m, n \in \mathbb{Z}^+$. Using Lemma 1.4(ii), we conclude that $\left\{M_{mq}T_{np}S^{-1/2}g\right\}_{m,n\in\mathbb{Z}^+}$ is an orthonormal basis for $L^2(\mathbb{R}^+)$ and hence, by Definition 1.2, it follows that $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is a Riesz basis. This completes the proof.

3. NECESSARY CONDITION AND SUFFICIENT CONDITIONS

In this section, we first establish a necessary condition for the system (2.1), i.e. $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ to be a frame in $L^2(\mathbb{R}^+)$.

Theorem 3.1. Suppose that $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is a Gabor frame for $L^2(\mathbb{R}^+)$ with bounds A and B, then

$$|q|A \le D(x) \le |q|B, \quad \text{a.e. on } x\in\mathbb{R}^+, \tag{3.1}$$

where $D(x)$ is defined by (2.7).

Proof. We use the contradiction method. Assume that the second inequality in (3.1) is not true. Then, there exists a measurable set $\Omega\subseteq\mathbb{R}^+$ with $\text{meas}(\Omega)>0$ such that $D(x)>|q|B$ on Ω. Suppose that Ω is contained in $[0,q^{-1})$ with diameter of $|q|^{-1}$ and let $\Omega_0=\{x\in\Omega: D(x)\ge 1+|q|B\}$. Further, for each $k\in\mathbb{N}$, we define the sets Ω_k as

$$\Omega_k=\left\{x\in\Omega:\frac{1}{1+k}+|q|B\le D(x)<\frac{1}{k}+|q|B\right\}.$$

Clearly, $\{\Omega_k\}_{k\in\mathbb{N}}$ forms a sequence of mutually disjoint measurable sets such that $\Omega=\bigcup_{k\in\mathbb{N}}\Omega_k$. Therefore, there exist atleast one set say Ω_ℓ with property that $\text{meas}(\Omega_\ell)>0$, $\ell\in\mathbb{N}$. Let $f=\chi_{\Omega_\ell}$ be the characteristic function on Ω_ℓ. Then, clearly each $f.T_{np}g$ has a compact support in Ω_ℓ. Since Ω_ℓ is contained in an interval of length $|q|^{-1}$ and $\left\{q^{1/2}w_{mq}(x)\right\}_{m\in\mathbb{Z}^+}$ constitutes an orthonormal basis for $L^2[0,q^{-1})$, we have

$$\sum_{m\in\mathbb{Z}^+}|\langle f,M_{mq}T_{np}g\rangle|^2=\sum_{m\in\mathbb{Z}^+}|\langle f\overline{T_{np}g},M_{mq}\rangle|^2=\frac{1}{|q|}\int_{\mathbb{R}^+}|f(x)|^2|g(x\ominus np)|^2dx.$$

Thus

$$\begin{aligned}\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}|\langle f,M_{mq}T_{np}g\rangle|^2&=\frac{1}{|q|}\sum_{n\in\mathbb{Z}^+}\int_{\mathbb{R}^+}|f(x)|^2|g(x\ominus np)|^2dx\\&=\frac{1}{|q|}\int_{\Omega_\ell}|f(x)|^2D(x)dx\ge\left(B+\frac{1}{|q|(\ell+1)}\right)\|f\|^2.\end{aligned}$$

Consequently, B can not be an upper frame bound for $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$. A similar arguments can be used to show that if the first inequality in (4.1) is violated, then A cannot be a lower frame bound for $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$. The proof is completed.

Now, we are going to derive two sufficient conditions of a Gabor frame for $L^2(\mathbb{R}^+)$. The first sufficient condition is given in the next theorem.

Theorem 3.2. Let f be in $\mathcal{E}^0(\mathbb{R}^+)$ and g be in $L^2(\mathbb{R}^+)$ with $p,q\in\mathbb{R}^+\setminus\{0\}$. Suppose that there are constants $A,B>0$ such that

$$A\le D(x)\le B,\quad \text{a.e. on } x\in\mathbb{R}^+ \tag{3.2}$$

and

$$\sum_{k\in\mathbb{N}} \|E_k\|_\infty < A. \tag{3.3}$$

Then $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is a Gabor frame for $L^2(\mathbb{R}^+)$.

Proof. By Lemma 2.3, we have

$$|R_g(f)| = \left|\frac{1}{|q|}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^+}\overline{f(x)}f(x\ominus q^{-1}k)E_k(x)dx\right| \le \frac{1}{|q|}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^+}|f(x)|.|f(x\ominus q^{-1}k)|\,|E_k(x)|\,dx$$

$$\le \frac{1}{|q|}\sum_{k\in\mathbb{N}}\left\{\int_{\mathbb{R}^+}|f(x)|^2\,|E_k(x)|\,dx\right\}^{1/2}\left\{\int_{\mathbb{R}^+}|f(x\ominus q^{-1}k)|^2\,|E_k(x)|\,dx\right\}^{1/2}$$

$$\le \frac{1}{|q|}\left\{\int_{\mathbb{R}^+}|f(x)|^2\left[\sum_{k\in\mathbb{N}}|E_k(x)|\right]dx\right\}^{1/2}\left\{\int_{\mathbb{R}^+}|f(x\ominus q^{-1}k)|^2\left[\sum_{k\in\mathbb{N}}|E_k(x)|\right]dx\right\}^{1/2}$$

$$\le \frac{1}{|q|}\left\{\int_{\mathbb{R}^+}|f(x)|^2\left[\sum_{k\in\mathbb{N}}|E_k(x)|\right]dx\right\}^{1/2}\left\{\int_{\mathbb{R}^+}|f(x)|^2\left[\sum_{k\in\mathbb{N}}|E_k(x\oplus q^{-1}k)|\right]dx\right\}^{1/2}$$

$$\le \frac{1}{|q|}\left\{\int_{\mathbb{R}^+}|f(x)|^2\left[\sum_{k\in\mathbb{N}}|E_k(x)|\right]dx\right\}^{1/2}\left\{\int_{\mathbb{R}^+}|f(x)|^2\left[\sum_{k\in\mathbb{N}}|E_k(x)|\right]dx\right\}^{1/2}$$

$$\le \frac{1}{|q|}\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^+}|f(x)|^2\,|E_k(x)|\,dx \le \frac{1}{|q|}\sum_{k\in\mathbb{N}}\|E_k\|_\infty\,\|f\|^2. \tag{3.4}$$

It follows from (2.9) and (3.4) that

$$\sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}|\langle f, M_{mq}T_{np}g\rangle|^2 \le \frac{1}{|q|}\|f\|^2\left\{\|D\|_\infty + \sum_{k\in\mathbb{N}}\|E_k\|_\infty\right\}$$

and

$$\frac{1}{|q|}\|f\|^2\left\{\|D\|_\infty - \sum_{k\in\mathbb{N}}\|E_k\|_\infty\right\} \le \sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}|\langle f, M_{mq}T_{np}g\rangle|^2.$$

Taking into account (3.2) and (3.3), we get

$$\frac{1}{|q|}\|f\|^2\left\{A - \sum_{k\in\mathbb{N}}\|E_k\|_\infty\right\} \le \sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+}|\langle f, M_{mq}T_{np}g\rangle|^2$$

$$\le \frac{1}{|q|}\|f\|^2\left\{B + \sum_{k\in\mathbb{N}}\|E_k\|_\infty\right\}.$$

Hence $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is a Gabor frame for $L^2(\mathbb{R}^+)$.

Now we establish the second sufficient condition for Gabor frame for $L^2(\mathbb{R}^+)$.

Theorem 3.3. Let $g \in L^2(\mathbb{R}^+)$ be such that

$$A_g = \frac{1}{|q|} \inf_{x\in[0,p)} \left\{ D(x) - \sum_{k\in\mathbb{N}} |E_k(x)| \right\} > 0, \tag{3.5}$$

and

$$B_g = \frac{1}{|q|} \sup_{x\in[0,p)} \sum_{k\in\mathbb{Z}^+} |E_k(x)| < +\infty. \tag{3.6}$$

Then $\{M_{mq}T_{np}g\}_{m,n\in\mathbb{Z}^+}$ is a Gabor frame for $L^2(\mathbb{R}^+)$ with bounds A_g and B_g.

Proof. Using (2.8), we have

$$\sum_{k\in\mathbb{N}} \left|T_{-q^{-1}k}E_k(x)\right| = \sum_{k\in\mathbb{N}} |T_{-q^{-1}k} \sum_{n\in\mathbb{Z}^+} g(x\ominus np)\overline{g(x\ominus np\ominus q^{-1}k)}|$$

$$= \sum_{k\in\mathbb{N}} |T_{-q^{-1}k} \sum_{n\in\mathbb{Z}^+} T_{np}g(x)\,\overline{T_{np+q^{-1}k}g(x)}| = \sum_{k\in\mathbb{N}} |\sum_{n\in\mathbb{Z}^+} T_{np-q^{-1}k}g(x)\,\overline{T_{np}g(x)}|$$

$$= \sum_{k\in\mathbb{N}} |\sum_{n\in\mathbb{Z}^+} T_{np+q^{-1}k}g(x)\,\overline{T_{np}g(x)}| = \sum_{k\in\mathbb{N}} |\sum_{n\in\mathbb{Z}^+} \overline{T_{np+q^{-1}k}g(x)}\,T_{np}g(x)| = \sum_{k\in\mathbb{N}} |E_k(x)|\,.$$

We now estimate the term $R_g(f)$ given by (2.10), using Cauchy-Schwartz's inequality

$$\left|\sum_{k\in\mathbb{N}} \int_{\mathbb{R}^+} \overline{f(x)}f(x\ominus q^{-1}k)E_k(x)dx\right| = \left|\sum_{k\in\mathbb{N}} \int_{\mathbb{R}^+} \overline{f(x)}T_{q^{-1}k}f(x)E_k(x)dx\right|$$

$$\le \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^+} |f(x)|\,|T_{q^{-1}k}f(x)|\,|E_k(x)|dx$$

$$\le \sum_{k\in\mathbb{N}} \left\{\int_{\mathbb{R}^+} |f(x)|^2|E_k(x)|dx\right\}^{1/2} \left\{\int_{\mathbb{R}^+} |T_{q^{-1}k}f(x)|^2|E_k(x)|dx\right\}^{1/2}$$

$$\le \left\{\sum_{k\in\mathbb{N}} \int_{\mathbb{R}^+} |f(x)|^2|E_k(x)|dx\right\}^{1/2} \left\{\sum_{k\in\mathbb{N}} \int_{\mathbb{R}^+} |T_{q^{-1}k}f(x)|^2|E_k(x)|dx\right\}^{1/2}$$

$$= \left\{\int_{\mathbb{R}^+} |f(x)|^2 \sum_{k\in\mathbb{N}} |E_k(x)|dx\right\}^{1/2} \left\{\int_{\mathbb{R}^+} |T_{q^{-1}k}f(x)|^2 \sum_{k\in\mathbb{N}} |E_k(x)|dx\right\}^{1/2}$$

$$= \int_{\mathbb{R}^+} |f(x)|^2dx \sum_{k\in\mathbb{N}} |E_k(x)|. \tag{3.7}$$

Combining (3.7) and (2.9), we get

$$\frac{1}{|q|} \int_{\mathbb{R}^+} |f(x)|^2 \left\{D(x) - \sum_{k\in\mathbb{N}} |E_k(x)|\right\} dx \le \sum_{m\in\mathbb{Z}^+} \sum_{n\in\mathbb{Z}^+} |\langle f, M_{mq}T_{np}g\rangle|^2 \tag{3.8}$$

and

$$\sum_{m\in\mathbb{Z}^+} \sum_{n\in\mathbb{Z}^+} |\langle f, M_{mq}T_{np}g\rangle|^2 \le \frac{1}{|q|} \int_{\mathbb{R}^+} |f(x)|^2 \left\{D(x) + \sum_{k\in\mathbb{N}} |E_k(x)|\right\} dx. \tag{3.9}$$

Taking infimum in (3.8) and supremum in (3.9), respectively, we obtain

$$A_g\|f\|_2^2 \le \sum_{m\in\mathbb{Z}^+}\sum_{n\in\mathbb{Z}^+} |\langle f, M_{mq}T_{np}g\rangle|^2 \le B_g\|f\|_2^2,$$

which holds for all $f \in \mathcal{E}^0(\mathbb{R}^+)$, where A_g and B_g are given by (3.5) and (3.6), respectively. This completes the proof of the theorem.

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Поступила 24 ноября 2011