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ON THE CONVERGENCE OF MULTIPLE WALSH-FOURIER SERIES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

U. GOGINAVA AND A. A. SAHAKIAN

Institute of Mathematics, Faculty of Exact and Natural Sciences, Tbilisi, Georgia
Yerevan State University, Armenia
E-mails: *zazagoginava@gmail.com* *sart@ysu.am*

Abstract. The convergence of multiple Walsh-Fourier series of functions of bounded generalized variation is investigated. The sufficient and necessary conditions on the sequence $\Lambda = \{\lambda_n\}$ are found for the convergence of multiple Walsh-Fourier series of functions of bounded partial Λ-variation.

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1. Classes of Functions of Bounded Generalized Variation

In 1881 Jordan [13] introduced the class of functions of bounded variation and applied it to the theory of Fourier series. Hereafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation ets., see [13, 22, 21, 14]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [12].

Let $I := [0, 1)$ and

$$J^k = \left(a^k, b^k\right) \subset I, \qquad k = 1, 2, \dots d.$$

Consider a measurable function $f(x)$ defined on R^d and 1-periodic with respect to each variable. For $d = 1$ we set

$$f\left(J^1\right) := f\left(b^1\right) - f(a^1).$$

If for a function of $d-1$ variables the expression $f\left(J^1 \times \cdots \times J^{d-1}\right)$ is already defined, then for a function of d variables the *mixed difference* is defined as follows:

$$f\left(J^1 \times \cdots \times J^d\right) := f\left(J^1 \times \cdots \times J^{d-1}, b^d\right) - f\left(J^1 \times \cdots \times J^{d-1}, a^d\right).$$

Let $E = \{J_k\}$ be a collection of nonoverlapping intervals from I ordered in arbitrary way and let Ω be the set of all such collections E.

For sequences of positive numbers $\Lambda^j = \{\lambda_n^j\}_{n=1}^{\infty}$, $j = 1, 2, \ldots, d$, the $\left(\Lambda^1, \ldots, \Lambda^d\right)$-*variation of f with respect to the index set*

$$D := \{1, 2, ..., d\}$$

is defined as follows:

$$V_{\Lambda^1,\ldots,\Lambda^d}^{D}(f) := \sup_{\{J_s^j\}_{s=1}^{k_j} \in \Omega} \sum_{i_1,\ldots,i_d} \frac{\left|f\left(J_{i_1}^1 \times \cdots \times J_{i_d}^d\right)\right|}{\lambda_{i_1} \cdots \lambda_{i_d}}.$$

For an index set $\alpha = \{j_1, ..., j_p\} \subset D$ and any $x = (x_1, ..., x_d) \in R^d$ we set $\widetilde{\alpha} := D \setminus \alpha$ and denote by x_α the vector of R^p consisting of components $x_j, j \in \alpha$, i.e.

$$x_\alpha = \left(x_{j_1}, ..., x_{j_p}\right) \in R^p.$$

By $V_{\Lambda^{j_1},\ldots,\Lambda^{j_p}}^{\alpha}(f, x_{\widetilde{\alpha}})$ and $f\left(J_{i_{j_1}}^1 \times \cdots \times J_{i_{j_p}}^p, x^{\widetilde{\alpha}}\right)$ we denote respectively the $\left(\Lambda^{j_1}, ..., \Lambda^{j_p}\right)$-variation and the mixed difference of f as a function of variables $x_{j_1}, ..., x_{j_p}$ over the p-dimensional cube I^p with fixed values $x_{\widetilde{\alpha}}$ of other variables. The $\left(\Lambda^{j_1}, ..., \Lambda^{j_p}\right)$ *−variation of f with respect to index set α* is defined as follows:

$$V_{\Lambda^{j_1},\ldots,\Lambda^{j_p}}^{\alpha}(f) = \sup_{x_{\widetilde{\alpha}} \in I^{d-p}} V_{\Lambda^{j_1},\ldots,\Lambda^{j_p}}^{\alpha}(f, x_{\widetilde{\alpha}}),$$

where $I^p := [0, 1)^p$.

Definition 1.1. *We say that the function f has Bounded total $\left(\Lambda^1, ..., \Lambda^d\right)$-variation on I^d and write*

$$f \in BV_{\Lambda^1,\ldots,\Lambda^d} := BV_{\Lambda^1,\ldots,\Lambda^d}(T^d),$$

if

$$V_{\Lambda^1,\ldots,\Lambda^d}(f) := \sum_{\alpha \subset D} V_{\Lambda^{j_1},\ldots,\Lambda^{j_p}}^{\alpha}(f) < \infty.$$

Definition 1.2. *We say that the function f is continuous in $\left(\Lambda^1, ..., \Lambda^d\right)$-variation on I^d and write*

$$f \in CV_{\Lambda^1,\ldots,\Lambda^d} := CV_{\Lambda^1,\ldots,\Lambda^d}(T^d),$$

if

$$\lim_{n\to\infty} V_{\Lambda^{j_1},\ldots,\Lambda^{j_{k-1}},\Lambda_n^{j_k},\Lambda^{j_{k+1}},\ldots,\Lambda^{j_p}}^{\alpha}(f) = 0, \qquad k = 1, 2, \ldots, p$$

for any $\alpha \subset D$, $\alpha := \{j_1, ..., j_p\}$, where $\Lambda_n^{j_k} := \left\{\lambda_s^{j_k}\right\}_{s=n}^{\infty}$.

Definition 1.3. *We say that the function f has Bounded Partial $\left(\Lambda^1, ..., \Lambda^d\right)$-variation and write*

$$f \in PBV_{\Lambda^1,\ldots,\Lambda^d} := PBV_{\Lambda^1,\ldots,\Lambda^d}(T^d),$$

if

$$PV_{\Lambda^1,\ldots,\Lambda^d}(f) := \sum_{i=1}^{d} V_{\Lambda^i}^{\{i\}}(f) < \infty.$$

In the case $\Lambda^1 = \cdots = \Lambda^d = \Lambda$ we denote

$$BV_\Lambda := BV_{\Lambda^1,\ldots,\Lambda^d}, \quad CV_\Lambda := CV_{\Lambda^1,\ldots,\Lambda^d},$$

and

$$PBV_\Lambda := PBV_{\Lambda^1,\ldots,\Lambda^d}.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \ldots$) the classes BV_Λ and PBV_Λ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n \to \infty$, and since the intervals in the collection $E = \{J_i\}$ are ordered arbitrarily, we suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$(1.1) \qquad 1 < \lambda_1 \le \lambda_2 \le \ldots, \qquad \lim_{n\to\infty} \lambda_n = \infty.$$

When $\lambda_n = n$ for all $n = 1, 2 \ldots$ we say *Harmonic Variation* instead of Λ-variation and write H instead of Λ (BV_H, PBV_H, CV_H, ets).

Remark 1.1. *The notion of Λ-variation was introduced by Waterman* [21] *in one dimensional case, by Sahakian* [19] *in two dimensional case and by Sablin* [18] *in the case of higher dimensions. The notion of bounded partial variation (class PBV) was introduced by Goginava in* [7]. *These classes of functions of generalized bounded variation play an important role in the theory Fourier series.*

Observe, that the number of variations in Definition 1.1 of total variation is $2^d - 1$, while the number of variations in Definition 1.3 of partial variation is only d.

The statements of the following theorem are known.

Theorem A. 1) (Dragoshanski [5]) *If $d = 2$, then $BV_H = CV_H$.*

2) (Bakhvalov [1]) *For any $d \ge 2$,*

$$CV_H = \bigcup_\Gamma BV_\Gamma,$$

where the union is taken over all sequences $\Gamma = \{\gamma_n\}_{n=1}^\infty$ with $\gamma_n = o(n)$ as $n \to \infty$.

The main result of this section is the following theorem.

Theorem 1.1. *Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ and $d \ge 2$. If*

$$(1.2) \qquad \frac{\lambda_n}{n} \downarrow 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} < \infty,$$

then there exists a sequence $\Gamma = \{\gamma_n\}_{n=1}^{\infty}$ *with*

$$\gamma_n = o(n) \quad as \quad n \to \infty, \tag{1.3}$$

such that $PBV_\Lambda \subset BV_\Gamma$.

Proof of Theorem 1.1. Choosing the sequence $\{A_n\}_{n=1}^{\infty}$ such that

$$A_n \uparrow \infty, \qquad \frac{\lambda_n A_n}{n} \downarrow 0, \qquad \sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n A_n^d}{n^2} < \infty, \tag{1.4}$$

we set

$$\gamma_n = \frac{n}{A_n}, \qquad n = 1, 2, \dots \tag{1.5}$$

We prove that there is a constant $C > 0$ such that

$$\sum_{i_1,\dots,i_p} \frac{\left| f\left(J_{i_1}^1 \times \dots \times J_{i_p}^p, x^{\widetilde{\alpha}}\right)\right|}{\gamma_{i_1}\cdots\gamma_{i_p}} < C \cdot PV_\Lambda(f), \tag{1.6}$$

for any $f \in PBV_\Lambda$, $\{J_{i_j}^j\}_{i_j=1}^{k_j} \in \Omega$, $j = 1, 2, \dots, d$, and $\alpha := \{i_1, ..., i_p\} \subset D$.

To prove (1.6) observe, that

$$\sum_{i_1,\dots,\, i_p} \frac{\left| f\left(J_{i_1}^1 \times \dots \times J_{i_p}^p, x^{\widetilde{\alpha}}\right)\right|}{\gamma_{i_1}\cdots\gamma_{i_p}} \tag{1.7}$$
$$= \sum_{\sigma} \sum_{i_{\sigma(1)}\le\dots\le i_{\sigma(p)}} \frac{\left| f\left(J_{i_1}^1 \times \dots \times J_{i_p}^p, x^{\widetilde{\alpha}}\right)\right|}{\gamma_{i_1}\cdots\gamma_{i_p}} < \infty,$$

where the sum is taken over all rearrangements $\sigma = \{\sigma(k)\}_{k=1}^{p}$ of the set $\{1, 2, \dots, p\}$.

Denoting $M = PV_\Lambda(f)$ and using (1.5), (1.4) and (1.2) we obtain:

$$\sum_{i_1\le i_2\le\dots\le i_p} \frac{\left| f\left(J_{i_1}^1 \times \dots \times J_{i_p}^p, x^{\widetilde{\alpha}}\right)\right|}{\gamma_{i_1}\cdots\gamma_{i_p}}$$
$$= \sum_{i_1\le i_2\le\dots\le i_{p-1}} \frac{A_{i_1}\cdots A_{i_{p-1}}}{i_1\cdots i_{p-1}} \sum_{i_p\ge i_{p-1}} \frac{\left| f\left(J_{i_1}^1 \times \dots \times J_{i_p}^p, x^{\widetilde{\alpha}}\right)\right|}{\lambda_{i_p}} \cdot \frac{\lambda_{i_p} A_{i_p}}{i_p}$$
$$\le M \sum_{i_1\le i_2\le\dots\le i_{p-1}} \frac{A_{i_{p-1}}^p \lambda_{i_{p-1}}}{i_{p-1}^2} \cdot \frac{1}{i_1\cdots i_{p-2}}$$
$$= M \sum_{i_{p-1}=1}^{\infty} \frac{A_{i_{p-1}}^p \lambda_{i_{p-1}}}{i_{p-1}^2} \sum_{i_{p-2}=1}^{i_{p-1}} \frac{1}{i_{p-2}} \sum_{i_{p-3}=1}^{i_{p-2}} \frac{1}{i_{p-3}} \cdots \sum_{i_1=1}^{i_2} \frac{1}{i_1}$$
$$\le M \sum_{i_{p-1}=1}^{\infty} \frac{A_{i_{p-1}}^p \lambda_{i_{p-1}}}{i_{p-1}^2} \left(\sum_{i=1}^{i_{p-1}} \frac{1}{i}\right)^{p-2} \le C \cdot M \sum_{n=1}^{\infty} \frac{A_n^p \lambda_n \log^{d-2} n}{n^2} < \infty.$$

Similarly we can prove that all other summands in the right hind side of (1.7) are finite. Theorem 1.1 is proved. □

2. Walsh functions

We denote the set of all non-negative integers by $\mathbf{N}$, the set of all integers by $\mathbf{Z}$ and the set of dyadic rational numbers in the unit interval $I := [0,1)$ by $\mathbf{Q}$. Each element of $\mathbf{Q}$ has the form $\frac{p}{2^n}$ for some $p, n \in \mathbf{N},\ 0 \le p < 2^n$.

By a dyadic interval in I we mean an interval $I_N^l := [l2^{-N}, (l+1)\,2^{-N})$ for some $l \in \mathbf{N},\ 0 \le l < 2^N$. Given $N \in \mathbf{N}$ and $x \in I$, we denote by $I_N(x)$ the dyadic interval of length 2^{-N} that contains x. We denote $I_N := [0, 2^{-N})$.

Let $r_0(x)$ be the function defined on the real line by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0,1/2) \\ -1, & \text{if } x \in [1/2,1) \end{cases}, \qquad r_0(x+1) = r_0(x), \qquad x \in \mathbf{R}.$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad x \in I, \quad n = 0, 1, \ldots .$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e. $w_0(x) \equiv 1$ and if $n = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s$ then

$$w_n(x) = r_{n_1}(x) \cdots r_{n_s}(x), \qquad x \in I.$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [20, 11]):

$$D_n(t) = w_n(t) \sum_{j=0}^{\infty} \delta_j w_{2^j}(t) D_{2^j}(t), \tag{2.1}$$

where $n = \sum_{j=0}^{\infty} \delta_j 2^j,\ \delta_j = 0$ or 1.

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 2^{-n}), \\ 0, & \text{if } x \in [2^{-n}, 1) \end{cases} \tag{2.2}$$

$$|D_n(x)| \le \min\left(n, \frac{1}{x}\right), \quad x \in (0,1), \tag{2.3}$$

$$|D_{m_A}(x)| \ge \frac{1}{4x}, \quad 2^{-2A-1} \le x < 1, \tag{2.4}$$

where

$$m_A := 2^{2A-2} + 2^{2A-4} + \cdots + 2^2 + 2^0. \tag{2.5}$$

Given $x \in I$, the expansion

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \tag{2.6}$$

where $x_k = 0$ or 1, is called the dyadic expansion of x. If $x \in I \backslash \mathbf{Q}$, then (2.6) is uniquely determined. For the dyadic expansion of $x \in \mathbf{Q}$ we choose the one with $\lim_{k \to \infty} x_k = 0$.

The dyadic sum of $x, y \in I$ in terms of the dyadic expansion of x and y is defined by

$$x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

We consider the multiple Walsh system

$$w_{n_1}(x_1) \times \cdots \times w_{n_d}(x_d), \qquad n_i \in \mathbf{N}, \quad i = 1, 2, ..., d$$

on the d-dimensional unit cube $I^d = [0,1) \times \cdots \times [0,1)$.

If $f \in L^1\left(I^d\right)$, then

$$\hat{f}\left(n_1, ..., n_d\right) = \int_{I^d} f\left(x_1, ..., x_d\right) w_{n_1}(x_1) \cdots w_{n_d}\left(x_d\right) dx_1 \cdots dx_d$$

is the $(n_1, ..., n_d)$-th Walsh-Fourier coefficient of f.

The rectangular partial sums of d-dimensional Fourier series with respect to the Walsh system are defined by

$$S_{m_1, \ldots, m_d} f(x_1, ..., x_d) = \sum_{n_1=0}^{m_1-1} \cdots \sum_{n_d=0}^{m_d-1} \hat{f}\left(n_1, ..., n_d\right) w_{n_1}(x_1) \cdots w_{n_d}\left(x_d\right).$$

Denoting

$$h_{\{i\}} := (0, ..., 0, h_i, 0, ..., 0) \in \mathbf{R}^d$$

and

$$\Theta\left(f, x, h_{\{i\}}\right) := f\left(x + h_{\{i\}}\right) - f\left(x\right), \qquad x \in \mathbf{R}^d,$$

the symbols $\Theta\left(f, x, h_{\{\alpha_1, \ldots, \alpha_p\}}\right)$ will stand for the expression which can be obtained by consecutive applying of Θ to the arguments with indices $\{\alpha_1, ..., \alpha_p\}$.

We denote by $C\left(I^d\right)$ the space of continuous, 1-periodic with respect to each variable functions defined on $\mathbf{R}^d$ with the norm

$$\|f\|_C := \sup_{x \in I^d} |f\left(x\right)|.$$

For $f \in C\left(I^d\right)$ the expressions

$$\omega_{\alpha_1, \ldots, \alpha_p}\left(\delta_{\alpha_1}, ..., \delta_{\alpha_1}; f\right)_C := \sup_{\left|h_{\alpha_i}\right| \leq \delta_{\alpha_i}, \ i=1, \ldots, p} \left\|\Theta\left(f, \cdot, h_{\{\alpha_1, \ldots, \alpha_p\}}\right)\right\|_C$$

are called moduli of continuity of function f.

3. Convergence of d-dimensional Walsh-Fourier series

In this paper we consider convergence of **only rectangular partial sums** (convergence in the sense of Pringsheim) of d-dimensional Walsh-Fourier series.

We say that $f(x_1,...,x_d)$ is continuous at $(x_1,...,x_d)$ if

$$\lim_{h_i\to 0+,\ i=1,\dots,d} f\left(x_1 \dotplus h_1,...,x_d \dotplus h_d\right) = f\left(x_1,...,x_d\right). \tag{3.1}$$

The well known Dirichlet-Jordan theorem (see [23]) states that the Fourier series of a function $f(x),\ x\in T$ of bounded variation converges at every point x to the value $\left[f(x+0)+f(x-0)\right]/2$.

Hardy [12] generalized the Dirichlet-Jordan theorem to the double Fourier series. He proved that if function $f(x,y)$ has bounded variation in the sense of Hardy ($f\in BV$), then $S[f]$ converges at any point (x,y) to the value $\frac{1}{4}\sum f(x\pm 0,y\pm 0)$.

Convergence of d-dimensional trigonometric Fourier series of functions of bounded Λ-variation was investigated in details by Sahakian [19], Dyachenko [2, 3, 4], Bakhvalov [1], Sablin [18], Goginava, Sahakian [10].

For the d-dimensional Walsh-Fourier series the convergence of partial sums of functions Harmonic bounded fluctuation and other bounded generalized variation were studied by Moricz [15, 16], Onnewer, Waterman [17], Goginava [8, 9].

For two-dimensional functions of bounded Harmonic variation Sargsyan [24] has proved the following

Theorem S. [Sargsyan [24]] *If $f\in BV_H(I^2)$, then the 2-dimensional Walsh-Fourier series of f converges to $f(x_1,x_2)$ at any point $(x_1,x_2)\in I^2$, where f is continuous.*

Now we formulate the main results of this paper.

Theorem 3.1. *Let $f\in CV_H(I^d)$, $d\geq 2$. Then the d-dimensional Walsh-Fourier series of f converges to $f(x)$ at any point $x\in I^d$, where the function f is continuous.*

The next theorem shows that Theorem S is not true for $d>2$.

Theorem 3.2. *Let $d>2$. Then there exists a continuous function $f\in BV_H(I^d)$ such that the d-dimensional Walsh-Fourier cubic partial sums of f diverge at some point.*

Note that similar results for the trigonometric system were proved by Bakhvalov [1].

In the next theorem we consider the behavior of the multidimensional Walsh-Fourier series of functions of bounded partial Λ-variation.

Theorem 3.3. *Let $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ and $d \geq 2$.*

a) If

$$\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} < \infty, \tag{3.2}$$

then the d-dimensional Walsh-Fourier series of a function $f \in PBV_{\Lambda}(I^d)$ converges to $f(x)$ at any point $x \in I^d$, where f is continuous.

b) If

$$\frac{\lambda_n}{n} = O\left(\frac{\lambda_{[n^{\delta}]}}{[n^{\delta}]}\right) \tag{3.3}$$

for some $\delta > 1$, and

$$\sum_{n=1}^{\infty} \frac{\lambda_n \log^{d-2} n}{n^2} = \infty, \tag{3.4}$$

then there exists a continuous function $f \in PBV_{\Lambda}(I^d)$ such that the d-dimensional cubic partial sums of its Walsh-Fourier series diverge at some point.

Theorem 3.3 implies

Corollary 3.1. *a) If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ with*

$$\lambda_n = \frac{n}{\log^{d-1+\varepsilon} n}, \quad n = 2, 3, \ldots, \qquad d \geq 2,$$

for some $\varepsilon > 0$, then the d-dimensional Walsh-Fourier series of a function $f \in PBV_{\Lambda}(I^d)$ converges to $f(x)$ at any point x, where f is continuous.

b) If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ with

$$\lambda_n = \frac{n}{\log^{d-1} n}, \quad n = 2, 3, \ldots, \qquad d \geq 2,$$

then there exists a continuous function $f \in PBV_{\Lambda}(I^d)$ such that the d-dimensional cubic partial sums of its Walsh-Fourier series diverge at some point.

4. Proofs of Main Results

Proof of Theorem 3.1. Let $n_i := 2^{N_i} + n_i',\ 0 \le n_i' < 2^{N_i},\ i = 1, 2, ..., d$. Since

$$D_{2^{N_i}+n_i'} = D_{2^{N_i}} + w_{2^{N_i}} D_{n_i'}$$

we can write

$$\begin{aligned}
(4.1) \qquad & S_{n_1,\ldots,n_d} f(x_1, ..., x_d) - f(x_1, ..., x_d) \\
&= \int_{I^d} \left[f\left(x_1 \dotplus s_1, ..., x_d \dotplus s_d\right) - f(x_1, ..., x_d)\right] \prod_{j=1}^{d} D_{n_j}(s_j)\, ds_1 \cdots ds_d \\
&= \sum_{\alpha \subset D} \int_{I^d} \left[f\left(x_1 \dotplus s_1, ..., x_d \dotplus s_d\right) - f(x_1, ..., x_d)\right] \\
&\quad \times \prod_{r \in D \backslash \alpha} D_{2^{N_r}}(s_r) \prod_{l \in \alpha} w_{2^{N_l}}(s_l) D_{n_l'}(s_l)\, ds_1 \cdots ds_d =: \sum_{\alpha \subset D} A_\alpha.
\end{aligned}$$

If $\alpha = \varnothing$, then from (2.2) we have

$$(4.2) \qquad A_\alpha = o(1), \quad \text{as} \quad \min\{n_1, \ldots, n_d\} \to \infty.$$

If $\alpha = D$, then we can write

$$\begin{aligned}
A_D &= \sum_{i_1=0}^{2^{N_1}-1} \cdots \sum_{i_d=0}^{2^{N_d}-1} \int_{I_{N_1}^{i_1} \times \cdots \times I_{N_d}^{i_d}} \left[f\left(x_1 \dotplus s_1, ..., x_d \dotplus s_d\right) - f(x_1, ..., x_d)\right] \\
&\quad \times \prod_{l=1}^{d} w_{2^{N_l}}(s_l) D_{n_l'}(s_l)\, ds_1 \cdots ds_d \\
&= \sum_{i_1=0}^{2^{N_1-1}} \cdots \sum_{i_d=0}^{2^{N_d-1}} \prod_{r=1}^{d} D_{n_l'}\left(\frac{i_r}{2^{N_r}}\right) \\
&\quad \times \int_{I_{N_1}^{i_1} \times \cdots \times I_{N_d}^{i_d}} \left[f\left(x_1 \dotplus s_1, ..., x_d \dotplus s_d\right) - f(x_1, ..., x_d)\right] \\
&\quad \times \prod_{l=1}^{d} w_{2^{N_l}}(s_l)\, ds_1 \cdots ds_d.
\end{aligned}$$

Since

$$\int_{I_{N_1}^{i_1}\times\cdots\times I_{N_d}^{i_d}} \left[f\left(x_1\dotplus s_1,...,x_d\dotplus s_d\right)-f\left(x_1,...,x_d\right)\right]$$
$$\times\prod_{l=1}^{d} w_{2^{N_l}}\left(s_l\right)ds_1\cdots ds_d.$$
$$=\int_{I_{N_1+1}^{2i_1}}\int_{I_{N_2}^{i_2}\times\cdots\times I_{N_d}^{i_d}} \Delta^{N_1+1}f\left(x_1\dotplus s_1,...,x_d\dotplus s_d\right)\prod_{l=2}^{d} w_{2^{N_l}}\left(s_l\right)ds_1\cdots ds_d$$
$$=\cdots$$
$$=\int_{I_{N_1+1}^{2i_1}\times\cdots\times I_{N_d+1}^{2i_d}} \Delta^{N_d+1}\left(\Delta^{N_{d-1}+1}\cdots\Delta^{N_1+1}\right.$$
$$\left.f\left(x_1\dotplus s_1,...,x_d\dotplus s_d\right)_1\cdots\right)_d ds_1\cdots ds_d,$$

where

$$\Delta^{N}f\left(x_1,...,x_d\right)_j \; : \; = f\left(x_1,...,x_d\right)$$
$$-f\left(x_1,...,x_{j-1},x_j\dotplus 2^{-N},x_{j+1},...,x_d\right).$$

From (2.3) we have

$$\left|A_D\right| \leq c\prod_{r=1}^{d}2^{N_r}\int_{I_{N_1+1}\times\cdots\times I_{N_d+1}}\sum_{i_1=0}^{2^{N_1}-1}\cdots\sum_{i_d=0}^{2^{N_d}-1}\prod_{r=1}^{d}\frac{1}{i_r+1}$$
$$\times \left|\Delta^{N_d+1}\left(\cdots\Delta^{N_1+1}\right.\right.$$
$$\left.\left.f\left(x_1\dotplus s_1\dotplus\frac{i_1}{2^{N_1}},...,x_d\dotplus s_d\dotplus\frac{i_d}{2^{N_d}}\right)_1\cdots\right)_d\right|ds_1\cdots ds_d$$

Set

$$\tau\left(N_1,...,N_d\right):=\left[\min\left\{N_1-2,...,N_d-2,\left(\theta\left(N_1,...,N_d\right)\right)^{-1}\right\}\right],$$

where

$$\theta\left(N_1,...,N_d\right)$$
$$=\sup_{0<s_i<N_i2^{-N_i},\; i=1,...,d}\left|f\left(x_1\dotplus s_1,...,x_d\dotplus s_d\right)-f(x_1,...,x_d)\right|.$$

Then we can write

$$
\begin{aligned}
|A_D| &\leq c\prod_{r=1}^{d} 2^{N_r} \int\limits_{I_{N_1+1}\times\cdots\times I_{N_d+1}} \sum_{i_1,\ldots,i_d=0}^{\tau(N_1,\ldots,N_d)} \prod_{r=1}^{d} \frac{1}{i_r+1} \\
&\times \left|\Delta^{N_d+1}\left(\cdots\Delta^{N_1+1}\right.\right. \\
&\qquad f\left(x_1 \dotplus s_1 \dotplus \frac{i_1}{2^{N_1}},\ldots,x_d \dotplus s_d \dotplus \frac{i_d}{2^{N_d}}\right)_1 \left.\left.\cdots\right)_d\right| ds_1\cdots ds_d \\
&+ c\prod_{r=1}^{d} 2^{N_r}\sum_{l=1}^{d} \int\limits_{I_{N_1+1}^{2i_1}\times\cdots\times I_{N_d+1}^{2i_d}} \sum_{i_1=0}^{2^{N_1}-1}\cdots\sum_{i_{l-1}=0}^{2^{N_{l-1}}-1} \\
&\qquad \sum_{i_l=\tau(N_1,\ldots,N_d)}^{2^{N_l}-1} \sum_{i_{l+1}=0}^{2^{N_{l+1}}-1}\cdots\sum_{i_d=0}^{2^{N_d}-1}\prod_{r=1}^{d}\frac{1}{i_r+1} \qquad (4.3)\\
&\times \left|\Delta^{N_d+1}\left(\cdots\Delta^{N_1+1}\right.\right. \\
&\qquad f\left(x_1 \dotplus s_1 \dotplus \frac{i_1}{2^{N_1}},\ldots,x_d \dotplus s_d \dotplus \frac{i_d}{2^{N_d}}\right)_1 \left.\left.\cdots\right)_d\right| ds_1\cdots ds_d \\
&\leq c\theta\left(N_1,\ldots,N_d\right)\log^d\left(\frac{1}{\theta\left(N_1,\ldots,N_d\right)}\right) \\
&+ c\sum_{l=1}^{d} V^D_{\{i_1\}\ldots\{i_{l-1}\}\{i_l+\tau(N_1,\ldots,N_d)\}\{i_{l+1}\}\ldots\{i_d\}}\left(f\right) = o\left(1\right),
\end{aligned}
$$

as $\min\left(n_1,\ldots,n_d\right)\to\infty$.

If $\alpha\subset D,\ \alpha\neq\varnothing,\ \alpha\neq D$, then we can prove similarly, that

$$A_\alpha = o\left(1\right) \text{ as } \min\left(n_1,\ldots,n_d\right)\to\infty. \tag{4.4}$$

Combining (4.1)-(4.4) we complete the proof of Theorem 3.1. □

Proof of Theorem 3.2. Let $\{A_k : k\geq 1\}$ be an increasing sequence of positive integers, satisfying

$$A_k > 2A_{k-1}, \tag{4.5}$$

$$\frac{A_k 2^{2dA_{k-1}}}{2^{A_k}} < \frac{1}{k^2}, \tag{4.6}$$

$$\frac{A_{k-1}^d}{A_k} < \frac{1}{k}. \tag{4.7}$$

Set

$$\varphi_k(x) := \begin{cases} 2\left(2^{2A_k}x - j\right), & \text{if } x \in \left[j2^{-2A_k}, (2j+1)\,2^{-2A_k-1}\right) \\ -2\left(2^{2A_k}x - j - 1\right), & \text{if } x \in \left[(2j+1)\,2^{-2A_k-1}, (j+1)\,2^{-2A_k}\right) \\ & \quad j = 1, 2, ..., 2^{2A_k - 2A_{k-1}} - 1 \\ 0, & \text{otherwise} \end{cases},$$

$$\psi_k(x) := \begin{cases} 2\left(2^{2A_k}x - 1\right), & \text{if } x \in \left[2^{-2A_k}, 3\cdot 2^{-2A_k-1}\right) \\ -2\left(2^{2A_k}x - 2\right), & \text{if } x \in \left[3\cdot 2^{-2A_k-1}, 2^{-2A_k+1}\right) \\ 0, & \text{otherwise} \end{cases}.$$

Let

$$g_k(x) := \varphi_k(x)\,\mathrm{sgn}\left(D_{m_{A_k}}(x)\right), \qquad g_k(x+l) = g_k(x), \quad l = 0, \pm1, \pm2, \ldots,$$

$$h_k(x) := \psi_k(x)\,\mathrm{sgn}\left(D_{m_{A_k}}(x)\right), \qquad h_k(x+l) = h_k(x), \quad l = 0, \pm1, \pm2, \ldots$$

Consider the function f defined by

$$f(x_1, ..., x_d) := \sum_{k=1}^{\infty} f_k(x_1, ..., x_d), \qquad f(0, ..., 0) = 0, \tag{4.8}$$

where

$$f_k(x_1, ..., x_d) := \frac{g_k(x_1)}{A_k} \prod_{j=2}^{d} h_k(x_j).$$

First, we prove that $f \in BV_H$. We consider several cases:

a) If $\alpha := \{\alpha_1, ..., \alpha_p\} \subset D \backslash \{1\}$ then by the construction of f we can write

$$V_H^{\alpha}(f) \le \frac{c}{A_k} \sum_{i_{\alpha_1}, ..., i_{\alpha_p}} \frac{\left|h_k\left(I_{i_{\alpha_1}}^{\alpha_1}, ..., I_{i_{\alpha_p}}^{\alpha_p}\right)\right|}{i_{\alpha_1} \cdots i_{\alpha_p}} \le c < \infty, \qquad k = 1, 2, ... \tag{4.9}$$

b) If $\alpha := \{1, \alpha_2, ..., \alpha_p\} \subset D$ and $p < d - 1$, then we have

$$V_H^{\alpha}(f) \le \frac{c}{A_k} \sum_{i_1, i_{\alpha_2}, ..., i_{\alpha_p}} \frac{\left|g_k\left(I_{i_1}^1\right)\right|}{i_1} \frac{\left|h_k\left(I_{i_{\alpha_2}}^{\alpha_2}, ..., I_{i_{\alpha_p}}^{\alpha_p}\right)\right|}{i_{\alpha_2} \cdots i_{\alpha_p}}, \qquad k = 1, 2, ... \tag{4.10}$$

On the other hand,

$$\sum_{i_1} \frac{\left|g_k\left(I_{i_1}^1\right)\right|}{i_1} \le cA_k \tag{4.11}$$

and

$$\sum_{i_{\alpha_2}, ..., i_{\alpha_p}} \frac{\left|h_k\left(I_{i_{\alpha_2}}^{\alpha_2}, ..., I_{i_{\alpha_p}}^{\alpha_p}\right)\right|}{i_{\alpha_2} \cdots i_{\alpha_p}} \le c < \infty. \tag{4.12}$$

From (4.10) – (4.12) we obtain

$$V_H^{\alpha}(f) \le c < \infty. \tag{4.13}$$

c) Let $\alpha = D$. Then by the construction of f we get

$$V_H^{\alpha}(f) \leq c\sum_{k=1}^{\infty}\frac{1}{A_k}\sum_{i_1,i_2,...,i_d}\frac{\left|g_k\left(I_{i_1}^1\right)\right|}{i_1}\frac{\left|h_k\left(I_{i_2}^2,...,I_{i_d}^d\right)\right|}{i_2\cdots i_d} \tag{4.14}$$

$$\leq c\sum_{k=1}^{\infty}\frac{1}{A_k}\frac{1}{k^{d-1}}\sum_{i_1=1}^{2^{2A_k-2A_{k-1}}}\frac{1}{i_1}$$

$$\leq c\sum_{k=1}^{\infty}\frac{1}{k^{d-1}} < \infty, \quad d > 2.$$

Combining (4.9), (4.13) and (4.14) we conclude that $f \in BV_H$.

Now, we prove that the d-dimensional cubic partial sums of Walsh-Fourier series of f diverge at the point $(0,...,0)$. By (4.8) we can write

$$S_{m_{A_k},...,m_{A_k}}f(0,...,0) = S_{m_{A_k},...,m_{A_k}}f_k(0,...,0) \tag{4.15}$$
$$+\sum_{i=1}^{k-1}S_{m_{A_k},...,m_{A_k}}f_i(0,...,0) + \sum_{i=k+1}^{\infty}S_{m_{A_k},...,m_{A_k}}f_i(0,...,0)$$
$$= J_1 + J_2 + J_3.$$

Since

$$\left|S_{m_{A_k},...,m_{A_k}}f_i(0,...,0)\right| \leq \|f_i\|_C\left(\log m_{A_k}\right)^d \leq \frac{cA_k^d}{A_i},$$

by (4.5) and (4.7) for J_3 we obtain

$$J_3 \leq cA_k^d\sum_{i=k+1}^{\infty}\frac{1}{A_i} \leq \frac{cA_k^d}{A_{k+1}} = o(1) \quad \text{as} \quad k \to \infty. \tag{4.16}$$

It is well-known [6] that

$$\left\|S_{m_{A_k},...,m_{A_k}}f_i - f_i\right\|_C$$
$$\leq c\sum_{\{\alpha_1,...,\alpha_p\}\subset D}\omega_{\alpha_1,...,\alpha_p}\left(\frac{1}{2^{2A_k}},...,\frac{1}{2^{2A_k}};f_i\right)_C A_k^p.$$

On the other hand,

$$\omega_{\alpha_1,...,\alpha_p}\left(\frac{1}{2^{2A_k}},...,\frac{1}{2^{2A_k}};f_i\right)_C \leq c\left(\frac{2^{2A_i}}{2^{2A_k}}\right)^p.$$

Consequently, taking into account (4.6) and the equality $f_i(0,...,0) = 0$, we obtain

$$J_2 \leq \sum_{i=1}^{k-1}\left|S_{m_{A_k},...,m_{A_k}}f_i(0,...,0)\right| \tag{4.17}$$
$$\leq \frac{cA_k}{2^{2A_k}}\sum_{i=1}^{k-1}2^{2dA_i} \leq \frac{cA_k2^{2dA_{k-1}}}{2^{2A_k}} = o(1), \quad \text{as} \quad k \to \infty.$$

Finally, by (2.4) we have

(4.18)
$$\begin{aligned}
&\left|S_{m_{A_k},\dots,m_{A_k}} f_k(0,\dots,0)\right| \\
&= \frac{1}{A_k}\left|\int_{I^d} f_k(x_1,\dots,x_d)\prod_{j=1}^{d} D_{m_{A_k}}(x_j)\,dx_1\cdots dx_d\right| \\
&= \frac{1}{A_k}\left|\int_{I} g_k(x_1)\,D_{m_{A_k}}(x_1)\,dx_1\right| \\
&\qquad\times\left|\int_{I^{d-1}}\prod_{j=2}^{d} h_k(x_j)\,D_{m_{A_k}}(x_j)\,dx_2\cdots dx_d\right| \\
&= \frac{1}{A_k}\int_{I}\varphi_k(x_1)\left|D_{m_{A_k}}(x_1)\right|dx_1 \\
&\qquad\times\prod_{j=2}^{d}\int_{I}\psi_k(x_j)\left|D_{m_{A_k}}(x_j)\right|dx_j \\
&= \frac{1}{A_k}\sum_{j=0}^{2^{2A_k-2A_{k-1}-1}}\int_{j\cdot 2^{-2A_k}}^{(j+1)2^{-2A_k}}\varphi_k(x_1)\left|D_{m_{A_k}}(x_1)\right|dx_1 \\
&\qquad\times\prod_{j=2}^{d}\int_{2^{-2A_k}}^{2^{-2A_k+1}}\psi_k(x_j)\left|D_{m_{A_k}}(x_j)\right|dx_j
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{16A_k}\sum_{j=0}^{2^{2A_k-2A_{k-1}-1}}\int_{j\cdot 2^{-2A_k}}^{(j+1)2^{-2A_k}}\frac{\varphi_k(x_1)}{x_1}dx_1\prod_{j=2}^{d}\int_{2^{-2A_k}}^{2^{-2A_k+1}}\frac{\psi_k(x_j)}{x_j}dx_j \\
&\geq \frac{1}{16A_k}\sum_{j=0}^{2^{2A_k-2A_{k-1}-1}}\frac{2^{2A_k}}{j+1}\int_{j\cdot 2^{-2A_k}}^{(j+1)2^{-2A_k}}\varphi_k(x_1)\,dx_1 \\
&\qquad\times\left(2^{2A_k-1}\right)^{d-1}\prod_{j=2}^{d}\int_{2^{-2A_k}}^{2^{-2A_k+1}}\psi_k(x_j)\,dx_j \\
&= \frac{1}{16A_k}\sum_{j=0}^{2^{2A_k-2A_{k-1}-1}}\frac{2^{2A_k}}{j+1}\frac{2^{(2A_k-1)(d-1)}}{2^{2A_k+1}}\left(\frac{1}{2^{2A_k+1}}\right)^{d-1}\geq c>0.
\end{aligned}$$

Combining (4.15)-(4.18) completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. Part a) immediately follows from Theorem 1.1, Theorem 3.1 and Theorem A.

To prove part b) we denote

$$A_{i_1,\ldots,i_d} := \left[\frac{i_1}{2^{2N}}, \frac{i_1+1}{2^{2N}}\right) \times \cdots \times \left[\frac{i_d}{2^{2N}}, \frac{i_d+1}{2^{2N}}\right),$$

$$W := \left\{(i_1,\ldots,i_d) : i_d < i_s < i_d + m_{i_d},\ 1 \le s < d,\ 1 \le i_d \le N_\delta\right\},$$

$$N_\delta = \left[4^{(N-1)/(\delta+1)}\right], \qquad t_j := \left(\sum_{i=1}^{s_j} \frac{1}{\lambda_i}\right)^{-1}, \qquad s_j := \left[j^{1+\delta}\right],$$

where $[x]$ is the integer part of x.

It is not hard to see, that for any sequence $\Lambda = \{\lambda_n\}$ satisfying (1.1) the class $C(I^d) \cap PBV_\Lambda(I^d)$ is a Banach space with the norm

$$\|f\|_{PBV_\Lambda} := \|f\|_C + PV_\Lambda(f).$$

For $N \in \mathbb{N}$ consider the following function

$$f_N(x_1,\ldots,x_d) := \sum_{(i_1,\ldots,i_d)\in W} t_{i_d} 1_{A_{i_1,\ldots,i_d}}(x_1,\ldots,x_d) \prod_{s=1}^{d} \xi_N(x_s)\,\mathrm{sgn}\left(D_{m_N}(x_s)\right),$$

where $1_A(x_1,\ldots,x_d)$ is the characteristic function of the set $A \subset T^d$, m_a is defined by (2.5) and

$$\xi_N(x) := \begin{cases} 1, \text{ if } x = (2j+1)\,2^{-(2N+1)},\ j = 1,2,\ldots,2^{2N}-1 \\ 0, \text{ if } x \in [0, 2^{-2N}),\ x = j\cdot 2^{-2N},\ j = 1,2,\ldots,2^{2N} \\ \text{linear and continuous on } [j2^{-(2N+1)}, (j+1)2^{-(2N+1)}],\ j = 2,3\ldots,2^{2N} \end{cases},$$

$$\xi_N(x+l) = \xi_N(x), \qquad l = \pm 1, \pm 2, \ldots.$$

First we show that the norms $\|f_N\|_{PBV_\Lambda}$ are uniformly bounded.

Let $(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_d)$ be fixed, where $k = 1,\ldots,d-1$. Then it is easy to show that

$$V_\Lambda^{\{k\}}(f_N) \le C\cdot t_{i_d}\left(\sum_{i_k=i_d+1}^{i_d+m_{i_d}} \frac{1}{\lambda_{i_k-i_d}}\right) \le C\cdot t_{i_d}\left(\sum_{i_k=1}^{m_{i_d}} \frac{1}{\lambda_{i_k}}\right) \le C < \infty.$$

If $(i_1,\ldots,i_{d-1})$ is fixed, the condition $(i_1,\ldots,i_d) \in W$ implies

$$\max\left\{i_d(i_s) : 1 \le s \le d-1\right\} < i_d < \min\left\{i_s : 1 \le s \le d-1\right\},$$

where

$$i_d(i_s) := \min\{i_d : i_d + m_{i_d} > i_s\}.$$

Consequently, by the definition of the function f_N we obtain that for any $s = 1, ..., d-1$

$$\begin{aligned} V_\Lambda^{\{d\}}(f_N) &\leq C \sum_{i_d=i_d(i_s)+1}^{i_s} \frac{t_{i_d}}{\lambda_{i_d-i_d(i_s)}} \\ &\leq C \cdot t_{i_d(i_s)} \sum_{i_d=i_d(i_s)+1}^{i_s} \frac{1}{\lambda_{i_d-i_d(i_s)}} \\ &= C \cdot t_{i_d(i_s)} \sum_{i_d=1}^{i_s-i_d(i_s)} \frac{1}{\lambda_{i_d}} \leq C \cdot t_{i_d(i_s)} \sum_{i_d=1}^{m_{i_d(i_s)}} \frac{1}{\lambda_{i_d}} = C < \infty. \end{aligned}$$

Hence $f_N \in PBV_\Lambda$ and

$$\|f_N\|_{PV_\Lambda} \leq C, \quad N = 1, 2, \ldots. \tag{4.19}$$

Observe, that by (3.3) we have

$$\frac{1}{t_j} = \sum_{i=1}^{m_j} \frac{1}{\lambda_i} = \sum_{i=1}^{m_j} \frac{1}{i} \cdot \frac{i}{\lambda_i} \leq C \frac{m_j}{\lambda_{m_j}} \log m_j \leq C \frac{j \log j}{\lambda_j}.$$

Hence

$$t_j \log j \geq c \frac{\lambda_j}{j}.$$

Consequently,

$$\begin{aligned} & S_{m_N,\cdots,m_N} q_N(0,\cdots,0) \\ &= \int_{I^d} q_N(x_1,\cdots,x_d) \prod_{s=1}^{d} D_{m_N}(x_s)\, dx_1 \cdots dx_d \\ &= \sum_{(i_1,\cdots,i_d)\in W} t_{i_d} \int_{A_{i_1,\cdots,i_d}} \prod_{s=1}^{d} \left|\xi N(x_s) D_{m_N}(x_s)\right| dx_1 \cdots dx_d \\ &\geq c \sum_{(i_1,\cdots,i_d)\in W} t_{i_d} \int_{A_{i_1,\cdots,i_d}} \frac{dx_1 \cdots dx_d}{x_1 \cdots x_d} \geq c \sum_{(i_1,\cdots,i_d)\in W} t_{i_d} \frac{1}{i_1 \cdots i_d} \\ &\geq c \sum_{i_d=1}^{N_\delta} \frac{t_{i_d}}{i_d} \sum_{i_1=i_d}^{i_d+m_{i_d}} \cdots \sum_{i_{d-1}=i_d}^{i_d+m_{i_d}} \frac{1}{i_1 \cdots i_{d-1}} \\ &\geq c \sum_{i_d=1}^{N_\delta} \frac{t_{i_d}}{i_d} \log^{d-1}\left(\frac{i_d + m_{i_d}}{i_d}\right) \geq c(\delta) \sum_{i_d=1}^{N_\delta} \frac{t_{i_d} \log i_d}{i_d} \log^{d-2} i_d \\ &\geq c(\delta) \sum_{n=1}^{N_\delta} \frac{\lambda_n \log^{d-2} n}{n^2} \to \infty, \end{aligned} \tag{4.20}$$

as $N \to \infty$, according to (3.4).

Applying the Banach-Steinhaus theorem, from (4.19) and (4.20) we obtain that there exists a continuous function $f \in PBV_\Lambda(I^d)$ such that

$$\sup_N |S_{N,\cdots,N} f(0, \cdots, 0)| = \infty.$$

□

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