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ON THE INSTABILITY OF THE RIEMANN HYPOTHESIS FOR VARIETIES OVER FINITE FIELDS

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Abstract. There exist perturbations of a rational function which remove zeroes and poles from a prescribed region as well as perturbations which add zeroes and poles to a prescribed region. We employ this to show the instability of the Riemann Hypothesis for zeta-functions of smooth projective varieties over finite fields.[1]

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Dedicated to Academician Norair Arakelian with profound admiration

1. Introduction

Various examples have been given of functions sharing many properties of the Riemann zeta-function and, in particular, satisfying a *similar* functional equation, but failing to satisfy the analogue of the Riemann hypothesis (see, for example [1, Remark 5, page 3]).

L. D. Pustil'nikov [7] innovated in two ways by showing the existence of such functions which satisfy the *same* functional equation as $\zeta(s)$ and moreover approximate $\zeta(s)$ arbitrarily well. The initiative of Pustil'nikov was refined by others and extended to other zeta functions. In [3], it is shown that zeta functions of curves over finite fields can be approximated by functions satisfying the same functional equation but failing to satisfy the analogue of the Riemann hypothesis. In the present paper, for zeta-functions of varieties over finite fields, we show that such approximations can even be obtained as *continuous perturbations* of the zeta-functions.

Deligne obtained the Fields Medal for proving the Riemann Hypothesis for zeta functions of varieties over finite fields. In the direction opposite to that of the previous paragraph, we show the existence of small perturbations of these zeta-functions which satisfy the same functional equation and continue to satisfy the analogue of the Riemann hypothesis.

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This paper contains no new results in number theory. Number theorists have worked hard to show that zeta-functions over finite fields are merely rational functions of a very explicit form. In the next section, we present this explicit form and thereafter study meromorphic functions having a similar form, with no subsequent reference to number theory. Our approach is rather from the viewpoint of complex approximation theory.

2. Analytic properties and symmetries

The following results can be found, for example, in [5]. For p prime, let $V = V(\mathbb{F}_p)$ be a smooth projective variety over the field $\mathbb{F}_p$ having p elements. The Zeta function $\zeta_V(s)$ associated to V was defined by Weil by the equations

$$\zeta_V(s) = Z_V(p^{-s}), \qquad Z_V(u) = \exp\left(\sum_{m\geq 1} N_m \frac{u^m}{m}\right), \tag{2.1}$$

where N_m is the number of points of $V(\mathbb{F}_{p^m})$.
Of course, this definition a priori only makes sense for u in the disc of convergence of the power series or, equivalently, for s in the half-plane of convergence of the corresponding Dirichlet series, and it is a nontrivial result that Z_V is a rational function and hence ζ_V extends meromorphically to all of $\mathbb{C}$. More precisely,

$$Z_V(u) = \frac{P_1(u)\cdots P_{2d-1}(u)}{P_0(u)\ldots P_{2d}(u)}, \tag{2.2}$$

where $d = \dim V$ and $P_j(u) \in \mathbb{Z}[u]$. Moreover, we have the functional equations

$$\overline{Z_V(\overline{u})} = Z_V(u), \qquad Z_V\left(\frac{1}{p^d u}\right) = \pm u^{\chi} p^{d\chi/2} Z_V(u), \tag{2.3}$$

where χ is the self-intersection number of the diagonal in $V \times V$. The Riemann hypothesis for all smooth projective varieties over finite fields was proven in full generality by Deligne, and it amounts to the equations

$$P_j(u) = \prod (1 - \alpha_{jk} u), \qquad |\alpha_{jk}| = p^{j/2}. \tag{2.4}$$

Hence the zeroes of the zeta-function $\zeta(V, s)$ lie on the lines

$$\Re(s) = \frac{1}{2}, \frac{3}{2}, \cdots, \frac{2d-1}{2}$$

and the poles on the lines

$$\Re(s) = 0, 1, 2, \cdots, d.$$

Let Z_j denote the divisor of the zeroes of the polynomial P_j. It follows from (2.4) that the support of Z_j is contained in the circle C_j of radius $p^{\frac{-j}{2}}$. We call C_d the central circle and for j odd, we call the circles C_j critical circles. The second equation in (2.3) implies that

$$Z_j = \frac{1}{p^d Z_{2d-j}}. \tag{2.5}$$

Therefore, the zeroes of P_j are the reflection of the zeroes of P_{2d-j} with respect to the central circle C_d. More precisely, from (2.4) we obtain the equations

$$\frac{1}{\alpha_{(2j-d)k}} = \frac{\alpha_{jk}}{p^d}. \tag{2.6}$$

It follows that the polynomials that make up the factors of the zeta functions satisfy the functional equation

$$P_{2d-j}(u) = (-1)^{N_j} A_j^{-1} p^{N_j d} u^{N_j} P_j(\frac{1}{p^d u}), \tag{2.7}$$

where $A_j = \prod_k \alpha_{jk}$, and N_j is the number of zeroes of P_j, counting multiplicity.

Remark 2.1. *If we compare the relations* (2.3) *and* (2.7) *we obtain the interesting formula:*

$$\chi = N_0 - N_1 + N_2 - \cdots + N_{2d}.$$

In fact the number χ has the interpretation as the Euler characteristic of a complex variety associated to V (see [4]*, Appendix C).*

For a positive number x and a positive integer j, the expression $x^{j/2}$ represents the positive determination of the square root of x^j. For a function $f : E \to \mathbb{C}$ defined on a set $E \subset \mathbb{C}$, we set

$$\|f\|_E = \sup_{z \in E} |f(z)|,$$

if f omits the value ∞ on E. Otherwise, we put $\|f\|_E = +\infty$. Moreover, we denote by $\mathcal{M}$ and $\mathcal{O}$ respectively the spaces of meromorphic and holomorphic functions on $\mathbb{C}$, and by $\mathcal{O}(E)$ the space of holomorphic functions defined on some open neighborhood of E. The following construction provides a metric d on $\mathcal{M}$ whose topology coincides with the topology of uniform convergence on compacta. Given an exhaustion of $\mathbb{C}$ by closed disks D_n, define

$$d(f, g) = \sum_{n=0}^{\infty} 2^{-n} \min(1, \|f - g\|_{D_n}).$$

We observe in particular that $\mathcal{O}$ is a locally convex space.

3. Approximation by Zeta functions

The first theorem does not rely on the functional equations of a zeta function Z_V, but on the fact [5, p.159] that $s = 0$ is a simple pole of ζ_V.

Theorem 3.1. *For a compact K, let $f \in \mathcal{O}(K)$. Then for any positive ϵ, there exist numbers a_k, b_k, λ_k, $k = 1, 2, \ldots n$, such that*

$$|f(s) - \sum_{k=1}^{n} \lambda_k \zeta_V(a_k s + b_k)| < \epsilon \quad \forall s \in K.$$

Proof. Let O be a bounded open set containing K such that f is holomorphic on O. After replacing f by $\chi_1 f$, where χ_1 is a smooth function supported in O such that $\chi_1(s) = 1$ on an open subset of O that contains K, we can assume that f extends smoothly on $\mathbb{C}$.
Let O'' be a bounded open set containing the closure of the set $O' = O \cup (O - O)$. Choose r_o so small that $\zeta_V(z)$ has no poles, other than 0, in the disc $D_o = (|z| < r_o)$ and choose t_o so small that the closure of O'' is contained in the disc $D_1 = (|s| < r_o/t_o)$. Fix t with $0 < t < t_o$. Then the poles of $\zeta_V(ts)$ other than zero lie outside the disc D_1 and hence outside the set O''. Since the pole at 0 of ζ_V is simple, setting $a_o = \pi \cdot res(\zeta_V, 0)$ and $a = a_o \pi/t$, all the poles of the meromorphic function

$$h(s) = \zeta_V(ts) - \frac{a}{\pi s}$$

lie outside the closure of O'.

Let χ_2 be a smooth function with support in O'' such that $\chi_2(s) = 1$ on a neighborhood of $\overline{O'}$: then $\widetilde{h} = \chi_2 h$ is a smooth function on $\mathbb{C}$ with compact support, and we can treat

$$\widetilde{\zeta}(s) = \frac{a}{\pi s} + \widetilde{h}(s)$$

as a distribution. Since $\psi(s) = \frac{1}{\pi s}$ is a fundamental solution for the $\overline{\partial}$-operator, and f is locally integrable and continuous, we have the following equalities:

$$f(s) = (f * \delta)(s) = (f * \overline{\partial}\psi)(s) = (\overline{\partial} f * \psi)(s) = \int\int (\overline{\partial} f)(z)\psi(s - z)dxdy =$$
$$a^{-1} \int\int (\overline{\partial} f)(z)\widetilde{\zeta}(s - z)dxdy - a^{-1} \int\int (\overline{\partial} f(z))\widetilde{h}(s - z)dxdy.$$

Since $f = 0$ off O, $s \in K \subset O$ and $\overline{\partial}_z \widetilde{h}(s - z) = 0$ on $O \times O$, integration by parts shows that

$$\int\int (\overline{\partial} f)(z)\widetilde{h}(s - z)dxdy = -\int\int f(z)\overline{\partial}_z \widetilde{h}(s - z)dxdy =$$
$$-\int\int_O f(z)\overline{\partial}_z \widetilde{h}(s - z)dxdy = 0. \tag{3.1}$$

Therefore if $s \in O$,

$$f(s) = a^{-1} \int\int (\overline{\partial} f)(z)\widetilde{\zeta}(s-z)dxdy = a^{-1} \int\int_{supp(\overline{\partial} f)} (\overline{\partial} f)(z)\widetilde{\zeta}(s-z)dxdy =$$

$$a^{-1} \int\int_{supp(\overline{\partial} f)} (\overline{\partial} f)(z)\zeta_V(t(s-z))dxdy, \tag{3.2}$$

where the last equality holds since $\zeta_V(ts) = \widetilde{\zeta}(s)$ for $s \in O'$. Since the integrand in (3.2) is smooth and uniformly continuous, we can approximate it uniformly by Riemann sums, and the result follows. □

4. Instability theorems

In this section we shall show two instability properties of the Riemann hypothesis. First, we prove that the functional equations (2.3) are not sufficient to characterize the zeta function of a variety. Indeed, we approximate the zeta function by functions which satisfy the same functional equation but fail to satisfy the analogue of the Riemann hypothesis, in that they have nontrivial zeroes off the critical axes. It is interesting to compare this with Hamburger's theorem which asserts more or less that the Riemann zeta function is characterized by its functional equation. Secondly, we shall construct functions, close (but not equal) to a given zeta function that satisfy the same functional equations (1) and have the same zeroes. Thus, among small perturbations of the zeta function satisfying the same functional equation, some do not and some do satisfy the analogue of the Riemann hypothesis. In this sense, the Riemann hypothesis is unstable.

Definition 4.1. *Let V be a smooth projective variety over $\mathbb{F}_p$ and $Z_V(u) = \zeta_V(s)$ the corresponding zeta function (where $u = p^{-s}$). Let $\mathcal{M}_V \subset \mathcal{M}$ be the subset of the meromorphic functions that can be written as*

$$f(s) = Z_f(u) = \frac{Q_1(u)Q_3(u)\dots Q_{2d-1}(u)}{Q_0(u)Q_2(u)\dots Q_{2d}(u)}, \tag{4.1}$$

where Q_j are holomorphic functions $\mathbb{C} \setminus \{0\}$ that satisfy the following properties:

1. *for j even, $Q_j = P_j$; for $i - j$ odd, Q_i and Q_j have no common zeroes;*
2. *if $\hat{u}$ is a pole of Z_f of order m, then $\hat{u}$ is a zero of order at least m for $Q_k - P_k$, for all k;*
3. *Q_j satisfy the same functional equations as the Zeta-function Z_V:*

$$Q_j(u) = \overline{Q_j(\overline{u})}, \qquad Q_{2d-j}(u) = (-1)^{N_j} A_j^{-1} p^{N_j d} u^{N_j} Q_j(\frac{1}{p^d u}), \tag{4.2}$$

where as in Section 1, A_j is the product of the inverses of the zeroes of P_j, and N_j denotes the number of zeroes of P_j.

We denote by $\mathcal{R}_V$ the class of rational functions in $\mathcal{M}_V$.

Remark 4.1. *With Definition 4.1 we have selected a class of functions that resemble the zeta-function ζ_V, since:*

(1) *if $f \in \mathcal{M}_V$, then Z_f satisfies the same functional equation (2.3) as the zeta-function Z_V;*
(2) *if $f \in \mathcal{M}_V$, then Z_f and Z_V have the same poles and moreover, at each pole they have the same principal part in the Laurent expansion; in particular, $Z_V - Z_f$ is holomorphic on $\mathbb{C} \setminus \{0\}$;*

The following theorem shows that every function in $\mathcal{M}_V$ has continuous perturbations which fail to satisfy the analog of the Riemann hypothesis.

Theorem 4.1. *The class $\mathcal{M}_V^-$ of functions in $\mathcal{M}_V$ which fail the "Riemann hypothesis"is an open dense subset of $\mathcal{M}_V$ ($\mathcal{M}_V$ endowed with the induced topology from $\mathcal{M}$). Moreover, for each $f \in \mathcal{M}_V$, there is a continuous curve $f_t \in \mathcal{M}_V^-, t \in (0,1]$, such that $f_t \to f$ in $\mathcal{M}_V$, as $t \to 0$. If $f \in \mathcal{R}_V$, we may suppose $f_t \in \mathcal{R}_V^-, t \in (0,1]$.*

In particular, we can approximate the zeta function ζ_V by continuous perturbations thereof which strongly resemble ζ_V but fail to satisfy the analogue of the Riemann hypothesis. The proof of Theorem 4.1 will be given after the introduction of the following technical lemma.

Lemma 4.1. *Let $f \in \mathcal{M}_V$. For j odd, $0 < j < 2d$, consider functions μ_j holomorphic on $\mathbb{C} \setminus \{0\}$ such that: (a) $\mu_j(\overline{u}) = \overline{\mu_j(u)}$, (b) $\mu_{2d-j}(u) = \mu_j(\frac{1}{p^d u})$, (c) if $\hat{u}$ is a pole of Z_V of order m, then $\mu_j - 1$ vanishes at $\hat{u}$ with order at least m. Then the functions $\widetilde{Q}_j(u) = \mu_j(u) Q_j(u)$ are holomorphic for all $u \neq 0$ and satisfy the functional equations (4.2). Hence the function $\widetilde{f}$, which is defined by*

$$\widetilde{f}(s) = \widetilde{Z}(u) = \frac{\widetilde{Q}_1(u)\widetilde{Q}_3(u)\ldots\widetilde{Q}_{2d-1}(u)}{Q_0(u)Q_2(u)\ldots Q_{2d}(u)}, \tag{4.3}$$

belongs to $\mathcal{M}_V$. If μ is rational and $f \in \mathcal{R}_V$, then $\widetilde{f} \in \mathcal{R}_V$.

Proof. It is simple to check that $\widetilde{Z}$ satisfies the functional equations (4.2). Then condition (c) guarantees that $\widetilde{Q}_k - P_k$ vanishes of order at least m at $\hat{u}$, if $\hat{u}$ is a pole of order m. □

We may now prove Theorem 4.1.

Proof. The fact that $\mathcal{M}_V^-$ is open follows immediately from Rouché's theorem . We give a proof of the Theorem for varieties of dimension $d \geq 2$, leaving to the reader to adjust the proof to the one-dimensional case. Given $f \in \mathcal{M}_V$, write as usual

$$f(s) = Z_f(u) = \frac{Q_1(u)Q_3(u)\ldots Q_{2d-1}(u)}{Q_0(u)Q_2(u)\ldots Q_{2d}(u)}.$$

Let P be the set of poles of $Z_f(u)$ and m_a be the order of a pole a of Z_f. From the functional equations (4.2) we deduce that, for all $a \in P$, $m_a = m_{1/a} = m_{\overline{a}}$. Consider the functions

$$\mu_{1,t}(u) = 1 + t \prod_{a \in P} (u-a)^{2m_a} \quad \text{and} \quad \mu_{2d-1,t}(u) = \mu_{1,t}(1/p^d u).$$

Since $P = \overline{P}$, the pair $\{\mu_{2d-1}, \mu_1\}$ satisfies condition (a) of Lemma 4.1, while (b) and (c) of the same lemma follow by definition. Hence, we set

$$Q_{1,t}(u) = \mu_{1,t}(u) Q_1(u) \quad \text{and} \quad Q_{2d-1,t}(u) = \mu_{2d-1,t}(u) Q_1(u)$$

and consider the family of function $f_t \in \mathcal{M}_V$

$$f_t(s) = Z_f(u) = \frac{Q_{1,t}(u) Q_3(u) \dots Q_{2d-3}(u) Q_{2d-1,t}(u)}{Q_0(u) Q_2(u) \dots Q_{2d}(u)}.$$

Now we show that there exists a positive ϵ such that if $t \in (0, \epsilon]$, then f_t has non-trivial zeroes outside the critical circles. Given $M > 0$, there exists $\epsilon > 0$ such that if $|t| < \epsilon$ and u is a zero of $\mu_{1,t}$, then $|u| > M$: in fact, since $\mu_{1,t}$ converges uniformly to 1 on compacta, the zero locus is pushed to infinity as t approaches zero. This shows in particular that for t sufficiently small the zeroes of $\mu_{1,t}$ do not belong to any of the critical circles. We are left to prove that $\mu_{1,t}$ has non-real zeroes for t sufficiently small. Denote by P^+ the set of poles of Z_f with positive imaginary part, by P' the set of real poles with absolute value greater than $p^{-d/2}$ and by P^d the set of real poles with absolute value equal to $p^{-d/2}$. Then,

$$\mu_{1,t}(u) = 1 + t \prod_{a \in P^+} (u-a)^{2m_a} (u-\overline{a})^{2m_a} \prod_{a \in P'} (u-a)^{2m_a} (u-1/p^d a)^{2m_a} \prod_{a \in P^d} (u-a)^{2m_a}.$$

From this expression for $\mu_{1,t}$ it is easy to see that if t is positive and real, $\mu_{1,t}$ can not have a real zero. Therefore, if $t \in (0, \epsilon]$, then $f_t \in \mathcal{M}_V^-$, and $f_t \in \mathcal{R}_V^-$ whenever $f_0 = f \in \mathcal{R}_V$. The fact that $f_t, t \in [0, \epsilon]$, depends continuously on t (that is, f_t is a continuous curve in $\mathcal{M}_V$) follows from the fact that $\mu_{1,t}$ and $\mu_{2d-1,t}$ converge to 1 uniformly on compacta and assume the value 1 at poles of Z_f. Of course, the curve $f_t, t \in (0, \epsilon]$ can be parameterized on the interval $(0, 1]$ rather than $(0, \epsilon]$. □ The next theorem is in the opposite direction of Theorem 4.1, namely we show that we can perturb elements in $\mathcal{M}_V$ while eliminating non-real zeroes off the critical circles.

Theorem 4.2. *Let $f \in \mathcal{M}_V$. Then, there exists an exhaustion of $\mathbb{C}$ by closed subsets $E_1 \subset E_2 \dots \subset E_n \subset \dots$ and a sequence of functions $f_n \in \mathcal{M}_V$ different from f satisfying the following properties:*

(1) f_n have the same zeroes as f on the critical axes (with the same multiplicity) and on the real axis and no other zeroes;

(2) $\lim ||f_n - f||_{E_n} = 0$; in particular the sequence f_n converges to f pointwise.

In particular, we can take as f the zeta function ζ_V itself. Each f_n resembles the zeta function ζ_V (because it is in $\mathcal{M}_V$) and f_n does satisfy the analogue of the

Riemann hypothesis, since it has no non-trivial zeroes off the critical axes. Let us start once again with the preparatory material.

Definition 4.2. *By a hole of a set $A \subset \mathbb{C}$, we mean any bounded component of $\mathbb{C} \setminus A$. A closed subset E without holes is said to be an Arakelian set if, for any closed disk D, the union of the holes of $D \cup E$ is a bounded set or, equivalently, if $\widehat{\mathbb{C}} \setminus E$ is connected and locally connected, where $\widehat{\mathbb{C}}$ denotes the Riemann sphere containing $\mathbb{C}$.*

Arakelian sets are extremely important in complex approximation. Given a function $f : E \to \mathbb{C}$ on a set $E \subset \mathbb{C}$, suppose we wish to approximate f uniformly by entire functions. Then, f must be continuous on E and holomorphic on the interior of E. Moreover, if a sequence f_n of entire functions converges uniformly to f on E, then this sequence is uniformly Cauchy on E and hence also on $\overline{E}$, so there is no loss of generality in assuming that E is closed. A famous theorem of N. U. Arakelian (see [8]) states that a necessary and sufficient condition on a closed set E, in order that each function continuous on E and holomorphic on the interior of E can be uniformly approximated by entire functions, is that $\widehat{\mathbb{C}} \setminus E$ be connected and locally connected. This theorem completely solves the problem of uniform approximation by entire functions.

For a divisor $D = \sum n_P(P)$ we define the conjugate divisor $\overline{D} = \sum n_{\overline{P}}(P)$. Also, we denote the support of D by $[D]$. Let $f \in \mathcal{M}_V$ be given. For all k odd, let W_k^+ (resp. W_k^-) denote the divisor of the zeroes of Q_k off the critical circles, outside (resp. inside) the central circle C_d and above the real axis, and let

$$W^+ = \sum_k W_k^+ \quad , \quad W^- = \sum_k W_k^-$$

$$W_k = W_k^+ + \overline{W_k^+} + W_k^- + \overline{W_k^-}.$$

The divisor W of non-trivial zeroes of Z_f off the critical circles is given by

$$W = \sum_k W_k = W^+ + \overline{W^+} + W^- + \overline{W^-}$$

and the set of non-trivial zeroes of Z_f off the critical circles is $[W]$.

Remark 4.2. *There is no relation between W_k^+ and W_k^- (unless $k = d$), but the functional equations (4.2) imply that*

$$\frac{1}{p^d W_{2d-k}^+} = \overline{W_k^-} \quad \textit{and} \quad \frac{1}{p^d W_{2d-k}^-} = \overline{W_k^+}. \tag{4.4}$$

Lemma 4.2. *For each $u \in [W^+]$ and each $n = 1, 2, \ldots$, there exists an unbounded domain (open connected set) $U_{n,u}$ whose boundary consists of two disjoint arcs, each of which goes monotonically from 0 to ∞, having the following properties:*

(1) *for each $u \in [W^+]$ and each n, we have $u \in U_{n,u}$;*

(2) *for each fixed* n, *the sets* $\mathrm{cl}(U_{n,u})$, *with* $u \in [W^+]$, *are disjoint;*
(3) *for each* $u \in [W^+]$ *and each* n, *we have* $U_{n,u} = 1/p^d\overline{U}_{n,u}$;
(4) *for each* $u \in [W^+]$, *the sets* $U_{n,u}$ *are decreasing and*

$$\bigcap_{n=1}^{\infty} U_{n,u} = \{u, 1/p^d\overline{u}\};$$

(5) *for each* $u \in [W^+]$, *the sets* $U_{n,u}$ *are in the upper half-plane and uniformly bounded away from the polar set of* Z_f.
(6) *for each* n,

$$meas \bigcup_{u\in[W^+]} U_{n,u} < 1/n.$$

For each odd value of k, we consider the following sets:

$$U_{n,k} = \bigcup_{u\in[W_k^+]} U_{n,u} \tag{4.5}$$

$$A_{n,k} = \mathbb{C} \setminus \left(U_{n,k} \cup \overline{U}_{n,k}\right). \tag{4.6}$$

Remark 4.3. (1) *If* $u \in W_k$ *for some* k, *then* u *is not a pole of* Z_f.
(2) *It follows from condition (4) of Lemma 4.2 that*

$$\bigcap_{n=1}^{\infty} \bigcup_{u\in[W_k^+]} U_{u,n} = \bigcup_{u\in[W_k^+]} \bigcap_{n=1}^{\infty} U_{u,n} = [W_k^+] \cup [W_{2d-k}^-].$$

The proof of Theorem 4.2 relies on an approximation-interpolation lemma, which is similar to Theorem 40 in [2], but the statement we provide here is stronger.

Lemma 4.3. *Let* X *be an Arakelian set,* $\epsilon > 0, m \in \mathbb{Z}^+$ *and the following data be given:*

(1) *a possibly finite sequence* Λ *in* $\mathbb{C} \setminus X$ *without limit points in* $\mathbb{C}$ *and for each* $\lambda \in \Lambda$ *an integer* $\nu(\lambda) > 0$ *and a non-zero complex number* β_λ;
(2) *a finite sequence* $\{b_1, b_2, \cdots, b_k \in X^o\}$.

Then there exists an entire function H *such that* $||1 - H||_X < \epsilon$, H *has zeroes only at the* λ'*s with order* $\nu(\lambda)$, $H^{(\nu(\lambda)}(\lambda) = \beta_\lambda$, *and* $H - 1$ *has a zero of order at least* m *at* $b_j, j = 1, \cdots, k$.

Proof. Let F be an entire function whose zeroes are precisely the points of $\lambda \in \Lambda$, with order $\nu(\lambda)$ and with $F^{(\nu(\lambda))}(\lambda) = \beta_\lambda$. Then, on X we may write $F = e^{-f}$, with $f \in \mathcal{O}(X)$. Set $E = X \cup \Lambda$ and put $f = 0$ on Λ. Then, E is again an Arakelian set and $f \in A(E)$. It follows from [6] that there is an entire function g such that $|f - g| < \min\{1, \epsilon/e\}$ on E, and $g(\lambda) = 0$, for each $\lambda \in \Lambda$. Moreover, we may stipulate that $g^{(\nu)}(b_j) = f^{(\nu)}(b_j), \nu = 0, 1, \cdots, m$, for each $j = 1, 2, \cdots, k$.

Set $G = e^g$ and $H = GF$. Then, on X we have

$$|H-1| = |e^{g-f}-1| \leq |g-f|\sum_{n=1}^{\infty}\frac{|g-f|^{n-1}}{n!} \leq |g-f|\sum_{n=0}^{\infty}\frac{1}{n!} \leq \epsilon.$$

The only zeros of H are those of F, that is, the points $\lambda \in \Lambda$. Near such a λ, we have

$$H(z) = G(z)F(z) = \left(1+\sum_{j=1}^{\infty} a_j(z-\lambda)^j\right)\left(\frac{\beta_\lambda}{\nu(\lambda)!}(z-\lambda)^{\nu(\lambda)}+\cdots\right).$$

Hence, these zeros are still of order $\nu(\lambda)$ and $H^{(\nu(\lambda))}(\lambda) = \beta_\lambda$.

At each $b_j, j = 1, 2, \cdots, k$, the function $g - f$ has a zero of order at least m and, since the exponential function is a local homeomorphism, at each such b_j, the function $H = e^{g-f}$ assumes the value 1 with multiplicity at least m. □

Proof of Theorem 4.2. Let

$$A_n = \bigcap_k A_{n,k} = \mathbb{C} \setminus \bigcup_k (U_{n,k} \cup \overline{U}_{n,k}),$$

where k runs over odd values. Choose an unbounded increasing sequence $\{r_n\}$ of positive numbers, with $r_1 > p^{-d}$ such that all poles in $\mathbb{C} \setminus \{0\}$ are in the annulus $\{1/(r_1 p^d) < |u| < r_1\}$. Then, there are no poles on the boundaries of the compact subsets

$$K_n = \{1/(p^d r_n) \leq |u| \leq r_n\} \cap A_n.$$

We define the finite set

$$\mathcal{W}_n = \{1/(p^d r_n) \leq |u| \leq r_n\} \cap [W]$$

and set

$$Y_{n,k} = A_n \cup (\mathcal{W}_n \setminus [W_k^+]).$$

Let $\delta_n > 0$; for each odd k we can apply Lemma 4.3, with $X = Y_{n,k}$ to construct a non-constant entire function $h_{n,k}$ such that

a) the zero divisor of $h_{n,k}$ equals W_k^+ and

$$\frac{Q_k(u)}{h_{n,k}(u)} = \delta_n, \quad \forall u \in [W_k^+]; \tag{4.7}$$

$$h_{n,k}(u) = 1, \quad \forall u \in \mathcal{W}_n \setminus [W_k^+]; \tag{4.8}$$

b) $||h_{n,k} - 1||_{Y_{n,k}} < \delta_n$;

c) if u is a pole of multiplicity m, then $h_{n,k}(u) - 1$ vanishes to order at least m at u.

For each k let

$$F_{n,k}(u) = h_{n,k}(u)\overline{h_{n,k}(\overline{u})}h_{n,2d-k}(1/p^d u)\overline{h_{n,2d-k}(1/p^d\overline{u})}. \tag{4.9}$$

Then the zero divisor of $F_{n,k}$ clearly coincides with W_k; moreover, since the polar divisor of Z_f is symmetric with respect to the real axis and the central circle C_d, the functions $F_{n,k}$ satisfy condition c) above. Hence, if we set

$$\widetilde{Q}_{n,k} = \frac{Q_k}{F_{n,k}} \tag{4.10}$$

and

$$Z_{f_n} = \frac{\widetilde{Q}_{n,1}\widetilde{Q}_{n,3}\ldots\widetilde{Q}_{n,2d-1}}{Q_0Q_2\ldots Q_{2d}}, \tag{4.11}$$

then the function f_n defined by $f_n(s) = Z_{f_n}(u)$ belongs to the class $\mathcal{M}_V$ and satisfies condition (1) of Theorem 4.2.

The validity of condition (2) of Theorem 4.2 follows from the following fact: for any $\epsilon_n > 0$, then there exists $\delta_n > 0$ such that if $h_{n,k}$ are defined by a) b) and c) as above, then

$$||Z_{f_n} - Z_f||_{K_n \cup \mathcal{W}_n} < \epsilon_n. \tag{4.12}$$

Indeed, consider a sequence $\epsilon_n \to 0$ for $n \to \infty$, and the collection of closed sets

$$E_n = \{s : p^{-s} \in K_n\} \cup \{s : p^{-s} \in \mathcal{W}_n\}. \tag{4.13}$$

which clearly satisfies $\bigcup_n E_n = \mathbb{C}$. Then $f_n \neq f$, since $h_{n,k} \neq 1$, and

$$||f - f_n||_{E_n} = ||Z_f - Z_{f_n}||_{K_n \cup \mathcal{W}_n} < \epsilon_n$$

which implies that $\lim_{n\to\infty} ||f_n - f||_{E_n} = 0$, which is condition (2) of Theorem 4.2.

We prove (4.12) first on K_n, then on $\mathcal{W}_n$. Observe that K_n is symmetric with respect to the central circle C_d and the real axis; moreover, $K_n \subset Y_{n,k}$ for all k. Given a collection $h_{n,k}$ of functions satisfying a), b) and c), we therefore have

$$||h_{n,k} - 1||_{K_n} \leq ||h_{n,k} - 1||_{Y_{n,k}} < \delta_n, \quad \forall k. \tag{4.14}$$

Condition c) imposed on $h_{n,k}$ implies that Z_f and Z_{f_n} have the same poles and the same principal part at each pole. Therefore $Z_f - Z_{f_n}$ is holomorphic for all $u \neq 0$, and the maximum principle implies that

$$||Z_f - Z_{f_n}||_{K_n} \leq ||Z_f - Z_{f_n}||_{\partial K_n}.$$

Since Z_f has no poles on ∂K_n, it is bounded on ∂K_n. Since K_n is a compact set, and each $F_{n,k}$ is a finite product of $h_{n,k}$ and its conjugates, then given $\epsilon_n > 0$ there exists $\delta_n > 0$ such that

$$||1 - \Pi_k F_{n,k}^{-1}||_{\partial K_n} \leq ||1 - \Pi_k F_{n,k}^{-1}||_{K_n} \leq \frac{\epsilon_n}{||Z_f||_{\partial K_n}}.$$

Therefore we can estimate

$$||Z_f - Z_{f_n}||_{K_n} \leq ||Z_f - Z_{f_n}||_{\partial K_n} = ||Z_f||_{\partial K_n}||1 - \Pi_k F_{n,k}^{-1}||_{K_n} < \epsilon_n. \tag{4.15}$$

Next, we want to show that (4.12) holds on $\mathcal{W}_n$; since $Z_f = 0$ on $\mathcal{W}_n$, we must show that

$$||Z_{f_n}||_{\mathcal{W}_n} < \epsilon_n. \tag{4.16}$$

We first prove the estimate (4.16) for $u \in \mathcal{W}_n \cap [W^+]$. The proof for the other three cases, namely $u \in \mathcal{W}_n \cap [W^-]$, $u \in \mathcal{W}_n \cap [\overline{W^-}]$, $u \in \mathcal{W}_n \cap [\overline{W^+}]$ will follow from the functional equations (4.2).

From equation (4.7), we have, with $v = 1/p^d u$ and abbreviating $h_{n,j}$ to h_j :

$$Z_{f_n}(u) = \frac{\delta_n}{h_1(\overline{u})h_{2d-1}(v)\overline{h_{2d-1}(\overline{v})}} \times \cdots \times \frac{\delta_n}{h_{2d-1}(\overline{u})h_1(v)\overline{h_1(\overline{v})}} \times \frac{1}{Q_0(u)\dots Q_{2d}(u)}.$$

If $u \in \mathcal{W}_n \cap [W^+]$, then, by (4.8), all the factors in the denominator, other than $Q_0(u), \cdots, Q_{2d}(u)$, are equal to 1 and so, for $u \in \mathcal{W}_n \cap [W^+]$,

$$Z_{f_n}(u) = \frac{\delta_n \times \cdots \times \delta_n}{Q_0(u)\dots Q_{2d}(u)}.$$

Since $Q_0(u), \cdots, Q_{2d}(u)$ are different from zero for u in the finite set $\mathcal{W}_n$, it follows that, if we choose δ_n sufficiently close to zero, we have $|Z_{f_n}(u)| < \epsilon_n$.

Suppose now that $u \in \mathcal{W}_n \cap [\overline{W^-}]$. Then, $v \in \mathcal{W}_n \cap [W^+]$ so $|Z_{f_n}(v)| < \epsilon_n$, by the previous case. Moreover, by the functional equation, $Z_{f_n}(u) = Cu^N Z_{f_n}(v)$, where N is plus or minus some N_j. Since u is restricted to the finite set $\mathcal{W}_n$, we may assume that δ_n is sufficiently small that $|Z_{f_n}(u)| < \epsilon_n$.

The remaining cases $u \in \mathcal{W}_n \cap [\overline{W^+}]$ and $u \in \mathcal{W}_n \cap [W^-]$ follow from the previous two and the functional equation $Z_{f_n}(\overline{u}) = \overline{Z_{f_n}(u)}$.

This concludes the proof of the estimate (4.16) and of Theorem 4.2. □

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