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# ON A SUBCLASS OF THE CLASS OF GENERALIZED DOUGLAS-WEYL METRICS

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Abstract. We study the class of Finsler metrics whose Douglas curvature is constant along any Finslerian geodesics. This class of Finsler metrics is a subclass of the class of generalized Douglas-Weyl metrics and contains the class of Douglas metrics as a special case. We find a condition under which this class of Finsler metrics reduces to the class of Landsberg metrics. Then we show this class of Landsberg metrics contains the class of $R$-quadratic metrics.

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**Keywords:** Douglas space; Landsberg metric; R-quadratic metric.

## 1. Introduction

In Finsler geometry, every Finsler metric $F$ on a manifold $M$ induces a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ which determines the geodesics, where $G^i = G^i(x,y)$ are called the spray coefficients of $\mathbf{G}$. A Finsler metric $F$ is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y \in T_xM$ for any $x \in M$ (see [17], [18], [11]). Let

$$D^i{}_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right).$$

It is easy to verify that $\mathcal{D} := D^i{}_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle $TM_0$. We call $\mathcal{D}$ the Douglas tensor. The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant, namely, if two Finsler metrics $F$ and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x,y)$ is positively y-homogeneous of degree one, then the Douglas tensor of $F$ is the same as that of $\bar{F}$ (see [9]). Finsler metrics with vanishing Douglas tensor are called Douglas metrics. The notion of Douglas curvature was proposed by B*ácsó* and Matsumoto as a generalization of Berwald curvature [3].

On the other hand, there is another projective invariant in Finsler geometry, namely ${D^i}_{jkl|m}y^m = T_{jkl}y^i$ that is hold for some tensor $T_{jkl}$, where ${D^i}_{jkl|m}$ denotes the horizontal covariant derivatives of ${D^i}_{jkl}$ with respect to the Berwald connection of Finsler metric $F$. This equation implies that the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic [7]. It is known that this class is closed under projective change and all metrics with vanishing Douglas curvature or vanishing Weyl curvature belong to it. Thus Finsler metrics in this class are called generalized Douglas-Weyl metrics [4].

In this paper, we study the class of Finsler metrics whose Douglas curvature satisfies

$$
{D^i}_{jkl|s}y^s = 0. \tag{1.1}
$$

The geometric meaning of (1.1) is that on this new class of Finsler spaces, the Douglas tensor is constant along any geodesics. It is easy to see that, this class of Finsler metrics is a subclass of the class of generalized Douglas-Weyl metrics. Here, we show that this condition is not projectively invariant. To prove this let two Finsler metrics $F$ and $\bar{F}$ are projectively equivalent, i.e. $G^i = \bar{G}^i + Py^i$, where $P = P(x,y)$ is positively $y$-homogeneous of degree one. Then we have

$$
G^i_j = \bar{G}^i_j + P_j y^i + P\delta^i_j, \tag{1.2}
$$

$$
G^i_{jk} = \bar{G}^i_{jk} + P_{jk}y^i + P_j\delta^i_k + P_k\delta^i_j. \tag{1.3}
$$

Let ${D^i}_{jkl|s}y^s = 0$. Then, we have

(1.4)
$$
\left[\frac{\partial {D^i}_{jkl}}{\partial x^s} - \frac{\partial {D^i}_{jkl}}{\partial y^m} G^m_s + G^i_{sm}{D^m}_{jkl} - G^m_{sj}{D^i}_{mkl} - G^m_{sk}{D^i}_{jml} - G^m_{sl}{D^i}_{jkm}\right] y^s = 0.
$$

Putting (1.2) and (1.3) in (1.4) imply that

$$
\begin{aligned}
\Big[\frac{\partial {D^i}_{jkl}}{\partial x^s} &- \frac{\partial {D^i}_{jkl}}{\partial y^m}(\bar{G}^m_s + P_m y^s + P\delta^m_s) + (\bar{G}^i_{sm} + P_{sm}y^i + P_s\delta^i_m + P_m\delta^i_s){D^m}_{jkl} \\
&- (\bar{G}^m_{sj} + P_{sj}y^m + P_s\delta^m_j + P_j\delta^m_s){D^i}_{mkl} \\
&- (\bar{G}^m_{sk} + P_{sk}y^m + P_s\delta^m_k + P_k\delta^m_s){D^i}_{jml} \\
&- (\bar{G}^m_{sl} + P_{sl}y^m + P_s\delta^m_l + P_l\delta^m_s){D^i}_{jkm}\Big] y^s = 0.
\end{aligned} \tag{1.5}
$$

Since the Douglas tensor is invariant under any projective relation, i.e., $D^i{}_{jkl} = \bar{D}^i{}_{jkl}$, then (1.5) reduces to the following equality:

$$\bar{D}^i{}_{jkl|s}y^s + P_m\bar{D}^m_{jkl}y^i = 0. \tag{1.6}$$

Thus by (1.6), we conclude that the class of Finsler metrics satisfies (1.1) is not closed under projective relations.

Other than Douglas curvature, there are several important non-Riemannian quantities: the Cartan torsion **C**, the Berwald curvature **B**, the mean Berwald curvature **E** and the Landsberg curvature **L**, etc (see [12], [14] – [16], [19]). The study shows that the above mentioned non-Riemannian quantities are closely related to the Douglas metrics, namely Bácsó-Matsumoto proved that every Douglas metric with vanishing Landsberg curvature is a Berwald metric [2]. Is there any other interesting non-Riemannian quantity with such property?

In [12], Shen find a new non-Riemannian quantity for Finsler metrics that is closely related to the **E**-curvature and call it $\bar{\mathbf{E}}$-curvature. Recall that $\bar{\mathbf{E}}$ is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. In this paper, we prove that every complete Finsler space satisfies (1.1) with bounded mean Cartan tensor and vanishing $\bar{\mathbf{E}}$-curvature is a Landsberg metric. More precisely, we prove the following statement.

**Theorem 1.1.** *Let $(M, F)$ be a complete Finsler space satisfying (1.1) with bounded Cartan tensor. If $\bar{\mathbf{E}}$-curvature of $F$ is vanishing, then $F$ is a Landsberg metric. In particular, every compact Finsler space satisfying (1.1) with $\bar{\mathbf{E}} = \mathbf{0}$ is a Landsberg space.*

The converse of Theorem 1.1 is not true. See the following example.

**Example 1.** *Consider the following Finsler metric on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$,*

$$F(y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)}}{1 - |x|^2} + \frac{<x, y>}{1 - |x|^2}, \quad y \in T_x\mathbb{B}^n = \mathbb{R}^n$$

*where $|.|$ and $<,>$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. $F$ is called the Funk metric which is a positively complete Finsler metric on $\mathbb{B}^n$ with bounded Cartan tensor* [12]. *The Funk metric is a Douglas metric with vanishing $\bar{\mathbf{E}}$-curvature while is not Landsbergian.*

For every weakly Berwald metric, the $\bar{\mathbf{E}}$-curvature is vanishing. Then by Theorem 1.1, we have the following result.

**Corollary 1.1.** *Let $(M, F)$ be a compact Finsler space satisfying (1.1). Suppose that $F$ is a weakly Berwald metric. Then $F$ is a Landsberg metric.*

For a Randers metric $F = \alpha + \beta$, the Cartan tensor is bounded $\|\mathbf{C}\| \leq \frac{3}{\sqrt{2}}$ (see [12]). In [6], Matsumoto showed that $F = \alpha + \beta$ is a Landsberg metric if and only if $\beta$ is parallel. In [5], M. Hashiguchi and I. Ichijyō showed that for a Randers metric $F = \alpha + \beta$, if $\beta$ is parallel, then $F$ is a Berwald metric. Then by Theorem 1.1, we obtain the following corollary.

**Corollary 1.2.** *Let $(M, F)$ be a Finsler space satisfying (1.1). Suppose that $F$ is a complete Randers metric on $M$. Then $F$ is a Berwald metric if and only if it is a weakly Berwald metric.*

It is known that on a Douglas manifold $(M, F)$, the Finsler metric $F$ is a Landsberg metric if and only if it is a Berwald metric (see [1], [2]). Hence, by Theorem 1.1, we get the following assertion.

**Corollary 1.3.** *Every compact Douglas metric with vanishing $\bar{\mathbf{E}}$-curvature is a Berwald metric.*

For a vector $y \in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) := R^i{}_k(y)u^k \frac{\partial}{\partial x^i}$, where

$$R^i{}_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $R := \{R_y\}_{y \in TM_0}$ is called the Riemann curvature [10], [12]. A Finsler metric $F$ is said to be *R-quadratic* if $R_y$ is quadratic in $y \in T_xM$ at each point $x \in M$. In this paper, we prove the following theorem.

**Theorem 1.2.** *Every R-quadratic Finsler metric satisfies (1.1).*

There are many connections in Finsler geometry [13]. In this paper, we set the Berwald connection on Finsler manifolds. The $h$- and $v$- covariant derivatives of a Finsler tensor field are denoted by " $|$ "and ", "respectively.

## 2. Preliminaries

Let $M$ be a n-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of $M$, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on $M$. A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ which has the following properties: (i) $F$ is $C^\infty$ on $TM_0$; (ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on $T_xM$ is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of $F_x$, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if $F$ is Riemannian.

For $y \in T_xM_0$, define $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k,$$

where $L_{ijk} := C_{ijk|s}y^s$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called Landsberg curvature. $F$ is called Landsberg metric if $\mathbf{L} = 0$.

Given a Finsler manifold $(M, F)$, a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, where $G^i(y)$ are local functions on $TM$ given by

$$G^i := \frac{1}{4}g^{il}\Big\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l}y^k - \frac{\partial[F^2]}{\partial x^l}\Big\}, \quad y \in T_xM.$$

$\mathbf{G}$ is called associated spray to $(M, F)$. The projection of an integral curve of $\mathbf{G}$ is called geodesic in $M$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$ (see [14]).

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{B}_y(u, v, w) := B^i_{j\ kl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x, \quad \mathbf{E}_y(u, v) := E_{jk}(y)u^j v^k$$

where

$$B^i{}_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y), \quad E_{jk}(y) := \frac{1}{2}B^m{}_{jkm}(y),$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. **B** and **E** are called the Berwald curvature and mean Berwald curvature respectively. A Finsler metric is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$, respectively [12].

The quantity $\mathbf{H}_y = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of **E** along geodesics [8]. More precisely

$$H_{ij} := E_{ij|m} y^m$$

The Riemann curvature $\mathbf{R}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

For a flag $P = \mathrm{span}\{y, u\} \subset T_xM$ with flagpole $y$, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{R}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2},$$

where $\mathbf{g}_y = g_{ij}(x, y) dx^i \otimes dx^j$. We say that a Finsler metric $F$ is of scalar curvature if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent bundle $TM_0$. If $\mathbf{K} = constant$, then $F$ is said to be of constant flag curvature. A Finsler metric $F$ is said to be *R-quadratic* if $R_y$ is quadratic in $y \in T_xM$ at each point $x \in M$. Let

$$R^i{}_{jkl}(x, y) := \frac{1}{3} \frac{\partial}{\partial y^j} \left\{ \frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k} \right\},$$

where $R^i{}_{jkl}$ is the Riemann curvature of Berwald connection. Then we have $R^i_k = R^i_{j\ kl}(x, y) y^j y^l$. Therefore $R^i_k$ is quadratic in $y \in T_xM$ if and only if $R^i_{j\ kl}$ are functions of position alone. Indeed a Finsler metric is R-quadratic if and only if the h-curvature of Berwald connection depends on position only in the sense of B*á*cs*ó*-Matsumoto [3]. By means of **E**-curvature, we can define $\bar{\mathbf{E}}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\bar{\mathbf{E}}_y(u, v, w) := \bar{E}_{jkl}(y) u^i v^j w^k,$$

where $\bar{E}_{ijk} := E_{ij|k}$. We call it $\bar{\mathbf{E}}$-curvature. From a Bianchi identity, we have

$$B^i{}_{jml|k} - B^i{}_{jkm|l} = R^i{}_{jkl.m}$$

where $R^i_{jkl}$ is the Riemannian curvature of Berwald connection [12]. This implies that $\bar{E}_{jlk} - \bar{E}_{jkl} = 2R^m_{jkl,m}$. Then $\bar{E}_{ijk}$ is not totally symmetric in all three of its indices. It is easy to see that, on R-quadratic Finsler metrics, $\bar{E}_{ijk} = \bar{E}_{ikj}$ holds. By definition, if $\bar{\mathbf{E}} = 0$, then $\mathbf{E}$-curvature is covariantly constant along all horizontal directions on $TM_0$.

## 3. Proof of Theorem 1.1

To prove Theorem 1.1, we need to prove that every complete Finsler metric with $L_{ijk|m}y^m = 0$ and bounded Cartan torsion must be a Landsberg metric. First, we remark the following.

**Remark 1.** *Let $(M, F)$ be a Finsler space and $c : [a, b] \to M$ be a geodesic. For a parallel vector field $V(t)$ along c,*

$$g_{\dot{c}}(V(t), V(t)) = constant. \tag{3.1}$$

Now, we consider the Finsler metrics with Landsberg curvatures satisfying $L_{ijk|m}y^m = 0$.

**Lemma 3.1.** *Let $(M, F)$ be a complete Finsler space with bounded Cartan tensor. Suppose that the Landsberg curvature of $F$ satisfies*

$$L_{ijk|s}y^s = 0. \tag{3.2}$$

*Then $F$ is a Landsberg space.*

**Proof.** Take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Let $c(t)$ be the geodesic with $\dot{c}(0) = y$ and $V(t)$ be the parallel vector field along $c$ with $V(0) = v$. Define $\mathbf{C}(t)$ and $\mathbf{L}(t)$ as following

$$\mathbf{C}(t) = \mathbf{C}_{\dot{c}}(V(t), V(t), V(t)), \quad \mathbf{L}(t) = \mathbf{L}_{\dot{c}}(V(t), V(t), V(t)).$$

By definition of $\mathbf{L}_y$, we get:

$$\mathbf{L}(t) = \mathbf{C}^{'}(t).$$

It follows from (3.2) that:

$$\mathbf{L}^{'}(t) = 0. \tag{3.3}$$

The equation (3.3) implies:

$$\mathbf{L}(t) = \mathbf{L}(0).$$

Then we have

$$\mathbf{C}(t) = t\,\mathbf{L}(0) + \mathbf{C}(0).$$

Suppose that $\mathbf{C}_y$ is bounded, i.e., there is a constant $Q < \infty$ such that

$$||\mathbf{C}||_x := \sup_{y \in T_xM_0} \sup_{v \in T_xM} \frac{\mathbf{C}_y(v,v,v)}{[g_y(v,v)]^{\frac{3}{2}}} \le Q.$$

By (3.1), $T := g_{\dot{c}}(V(t), V(t)) = constant$ is positive constant. Thus

$$|\mathbf{C}(t)| \le QT^{\frac{3}{2}} < \infty,$$

and $\mathbf{C}(t)$ is a bounded function on $[0, \infty)$. This implies

$$\mathbf{L}_y(v,v,v) = \mathbf{L}(0) = 0.$$

Therefore $\mathbf{L} = 0$ and $F$ is a Landsberg metric. □

**Lemma 3.2.** *Let $(M, F)$ be a Finsler space satisfies (1.1) with $\bar{\mathbf{E}} = 0$. Then the Landsberg curvature of $F$ satisfies (3.2).*

**Proof.**

$$D^i{}_{jkl} = B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\}.$$

Then

$$D^i{}_{jkl|m}y^m = B^i{}_{jkl|m}y^m - \frac{2}{n+1}\{H_{jk}\delta^i{}_l + H_{kl}\delta^i{}_j + H_{lj}\delta^i{}_k + E_{jk,l|m}y^m y^i\}. \tag{3.4}$$

On the other hand, the following Ricci identity for $E_{ij}$ holds:

$$E_{jk,l|k} - E_{ij|k,l} = E_{pj}B^p{}_{ikl} + E_{ip}B^p{}_{jkl}. \tag{3.5}$$

It follows from (3.5) that:

$$E_{jk,l|m}y^m = E_{jk|m,l}y^m = [E_{jk|m}y^m]_{,l} - E_{jk|l}.$$

This yield that:

$$E_{jk,l|m}y^m = H_{jk,l} - \bar{E}_{jkl}. \tag{3.6}$$

By (3.4) and (3.6), we get:

$$B^i{}_{jkl|m}y^m = \frac{2}{n+1}\{H_{jk}\delta^i{}_l + H_{kl}\delta^i{}_j + H_{lj}\delta^i{}_k + H_{jk,l}y^i - \bar{E}_{jkl}y^i\}. \tag{3.7}$$

From the assumption, we have:

$$B^i{}_{jkl|m}y^m = 0. \tag{3.8}$$

(3.8) with $y_i$ implies that $F$ satisfies (3.2). □

**Proof of Theorem 1.1**: By the Lemmas 3.1 and 3.2, we get the proof. □

**Corollary 3.1.** *Let $(M, F)$ be a compact Finsler space satisfying (1.1). Then* $\bar{\mathbf{E}} = 0$ *if and only if* $\mathbf{L} = 0$ *and* $\mathbf{H} = 0$.

**Proof.** By definition, if $\bar{\mathbf{E}} = 0$ then $\mathbf{H} = 0$ and by Theorem 1.1, every compact Finsler metric satisfying (1.1) with $\bar{\mathbf{E}} = 0$ is a Landsberg metric. Conversely, let $F$ be a Finsler metric satisfying (1.1) with $\mathbf{H} = 0$ and $\mathbf{L} = 0$. By (3.7), we have

$$B^i{}_{jkl|m}y^m = \frac{2}{n+1}\{H_{jk}\delta^i{}_l + H_{kl}\delta^i{}_j + H_{lj}\delta^i{}_k + H_{jk,l}y^i - \bar{E}_{jkl}y^i\},$$

which implies

$$B^i{}_{jkl|m}y^m = \frac{-2}{n+1}\bar{E}_{jkl}y^i. \tag{3.9}$$

Contacting (3.9) with $y_i$ and using $y_{i|m} = 0$ and $y_iB^i{}_{jkl} = -2L_{jkl}$ yields

$$L_{jkl|m}y^m = \frac{F^2}{n+1}\bar{E}_{jkl}.$$

Since $\mathbf{L} = 0$ then $\bar{\mathbf{E}} = 0$. □

## 4. Proof of Theorem 1.2

In this section, we prove that every R-quadratic metric is a Finsler metric satisfies (1.1). To prove this, we need the following.

**Lemma 4.1.** *Let $(M, F)$ be a Finsler manifold and $F$ is R-quadratic. Then the Berwald curvature of $F$ is constant along any geodesics.*

**Proof.** The curvature form of Berwald connection is:

$$\Omega^i{}_j = d\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k = \frac{1}{2}R^i{}_{jkl}\omega^k \wedge \omega^l - B^i{}_{jkl}\omega^k \wedge \omega^{n+l}. \tag{4.1}$$

For the Berwald connection, we have the following structure equation:

$$dg_{ij} - g_{jk}\Omega^k{}_i - g_{ik}\Omega^k{}_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}. \tag{4.2}$$

Differentiating (4.2) yields the following Ricci identity:

$$\begin{aligned} g_{pj}\Omega^p{}_i - g_{pi}\Omega^p{}_j = & -\ 2L_{ijk|l}\omega^k \wedge \omega^l - 2L_{ijk,l}\omega^k \wedge \omega^{n+l} - 2C_{ijl|k}\omega^k \wedge \omega^{n+l} \\ & -\ 2C_{ijl,k}\omega^{n+k} \wedge \omega^{n+l} - 2C_{ijp}\Omega^p{}_l y^l. \end{aligned} \tag{4.3}$$

Differentiating of (4.1) yields:

$$d\Omega_i{}^j - \omega_i{}^k \wedge \Omega_k{}^j + \omega_k{}^j \wedge \Omega_i{}^k = 0. \tag{4.4}$$

Define $B^i_{j\ kl|m}$ and $B^i_{j\ kl,m}$ by:

(4.5) $dB^i_{jkl} - B^i_{mkl}\omega^m_i - B^i_{jml}\omega^m_k - B^i_{jkm}\omega^m_l + B^i_{jkl}\omega^i_m = B^i_{jkl|m}\omega^m + B^i_{jkl,m}\omega^{n+m}.$

Similarly, we define $R^i{}_{jkl|m}$ and $R^i{}_{jkl,m}$ by:

(4.6) $dR^i_{jkl} - R^i_{mkl}\omega^m_i - B^i_{jml}\omega^m_k - R^i_{jkm}\omega^m_l + R^i_{jkl}\omega^i_m = R^i_{jkl|m}\omega^m + R^i_{jkl,m}\omega^{n+m}.$

From (4.3) – (4.6) we obtain

$$\begin{aligned} &R^i{}_{jkl|m} + R^i{}_{jlm|k} + R^i{}_{jmk|l} = B^i{}_{jku}R^u{}_{lm} + B^i{}_{jlu}R^u{}_{km} + B^i{}_{klu}R^u{}_{jm},\\ &B^i{}_{jml|k} - B^i{}_{jkm|l} = R^i{}_{jkl,m},\\ &B^i{}_{jkl,m} = B^i{}_{jkm,l}. \end{aligned} \tag{4.7}$$

By assumption and (4.7) we have:

$$B^i{}_{jkl|m} = B^i{}_{jmk|l},$$

which contacting with $y^m$, we conclude that:

$$B^i{}_{jkl|m}y^m = 0.$$

This means that the Berwald curvature of $F$ is constant along any geodesics. □

By Lemma 4.1, we have the following result.

**Corollary 4.1.** *Let $(M, F)$ be a Finsler manifold. If $F$ is R-quadratic then $\mathbf{H} = \mathbf{0}$.*

**Proof of Theorem 1.2**:

$$D^i{}_{jkl} = B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\}.$$

Then

$$D^i{}_{jkl|m}y^m = B^i{}_{jkl|m}y^m - \frac{2}{n+1}\{E_{jk|m}y^m\delta^i{}_l + E_{kl|m}y^m\delta^i{}_j + E_{lj|m}y^m\delta^i{}_k + E_{jk,l|m}y^my^i\}.$$

It follows from (4.7) that

$$B^i{}_{jkl|m}y^m = R^i{}_{jml,k}y^m.$$

Then we have

$$E_{jk|m}y^m = R^p{}_{jmp,k}y^m.$$

Therefore, we get

$$D^\alpha{}_{jkl|m}y^m = R^\alpha{}_{jml,k}y^m - \frac{2}{n+1}\{R^p{}_{jmp,k}y^m\delta^\alpha_l + R^p{}_{lmp,j}y^m\delta^\alpha_k + R^p{}_{kmp,l}y^m\delta^\alpha{}_j\}.$$

$F$ is R-quadratic, then we have:

$$D^{\alpha}{}_{jkl|m}y^m = 0.$$

This implies that $F$ satisfies (1.1). □

Hence, on R-quadratic metrics, for any linearly parallel vector fields $U = U(t)$, $V = V(t)$ and $W = W(t)$ along a geodesic $c(t)$, we have

$$\frac{d}{dt}[D_{\dot{c}}(U, V, W)] = 0.$$

The geometric meaning of the above identity is that on R-quadratic metrics the Douglas curvature along a geodesic is constant.

**Corollary 4.2.** *Let $(M, F)$ be a R-quadratic manifold. Then $\bar{\mathbf{E}} = 0$.*

**Proof.**

$$D^i{}_{jkl} = B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\}.$$

Then

$$D^i{}_{jkl|m}y^m = B^i{}_{jkl|m}y^m - \frac{2}{n+1}\{H_{jk}\delta^i_l + H_{kl}\delta^i_j + H_{lj}\delta^i_k + E_{jk,l|m}y^m y^i\}.$$

It follows from Lemma 4.1 and Theorem 1.2 that

$$E_{jk,l|m}y^m y^i = 0,$$

and contracting with $y_i$ yields $E_{jk,l|m}y^m = 0$. By considering (3.6), we conclude that $\bar{E}_{ijk} = 0$. This completes the proof. □

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