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THE PÓLYA SUM PROCESS: A COX REPRESENTATION

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Abstract. In [1], Zessin constructed the so-called Pólya sum process via partial integration. Here we use the technique of integration by parts to the Pólya sum process to derive representations of the Pólya sum process as an infinitely divisible point process and a Cox process directed by an infinitely divisible random measure. This result is related to the question of the infinite divisibilty of a Cox process and the infinite divisibility of its directing measure. Finally we consider a scaling limit of the Pólya sum process and show that the limit satisfies an integration by parts formula, which we use to determine basic properties of this limit.

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1. Introduction

A huge class of point processes admit an integration by parts formula of their Campbell measure, that is for a point process P on a polish space X,

$$C_{\mathrm{P}}(h) = \iint h(x,\mu)\mu(dx)\,\mathrm{P}(d\mu) = \iint h(x,\mu+\delta_x)\,\eta(\mu,dx)\,\mathrm{P}(d\mu)$$

for nonnegative, measurable functions $h : X \times \mathcal{M}(X) \to \mathbb{R}$. In such a case, η is named conditional intensity or Papangelou kernel. They were introduced and studied systematically by Papangelou [2] in connection with point processes on spaces of lines and flats, Kallenberg [3], Nguyen and Zessin [4] in connection with Gibbs processes and Matthes, Warmuth and Mecke [5].

Recently, Zessin [1] gave a construction method for point processes by specifying a Papangelou kernel. As a fundamental example he introduced the Pólya sum process, which is the point process given by the Papangelou kernel

$$\eta(\mu,dx) = z\big(\rho+\mu\big)(dx), \tag{1.1}$$

where $z \in (0,1)$ and ρ is a fixed Radon measure on X. Furthermore he showed that in that way constructed point process has independent increments and that the number of points inside some bounded, measurable set follows a negative binomial distribution.

A point process with the last two properties is already mentioned in [6]. Here we want to demonstrate only by using the integration by parts technique that the Pólya sum process can be represented as a Cox process, where the underlying random intensity measure is a random measure with again independent increments, but with gamma distributed mass inside each bounded, measurable set. This particular directing random measure is given e.g. in Nehring and Zessin [7] as an example for a random measure which is given as the solution of a certain functional equation.

That point of view on the Pólya sum process is in spirit very close to the construction of the negative binomial process in Barndorff-Nielsen and Yeo [8] as a Cox process with some Gamma process as underlying random measure, but however, the latter Gamma processes properties differ to a large extend from those of the former directing measure.

In section 2 we briefly give the setting, review the characterisation of random measures by a functional equation as in [7]. The main theorem, the Cox representation of the Pólya sum process is stated and proven in section 3. In section 4 we demonstrate the more general principle behind that representation. Finally we turn in section 5 to a question of H. Zessin about the behaviour of the Pólya sum process as the parameter z tends to 1 and show that after a suitable scaling, the limit is an infinitely divisible random measure and show that the total mass in bounded regions is gamma distributed.

2. Preliminaries

Here and in the following sections let X be a polish space and denote by $\mathcal{B} = \mathcal{B}(X)$ its Borel sets as well as by $\mathfrak{B}_0 = \mathfrak{B}_0(X)$ the ring of bounded Borel sets of X. Furthermore let $\mathcal{M}(X)$ and $\mathcal{M}^{\cdot\cdot}(X)$ be the space of locally finite measures and locally finite point measures on X, respectively, each of which is known to be vaguely Polish with the σ-algebras generated by the mappings $\zeta_B(\mu) = \mu(B)$, $B \in \mathcal{B}_0$. We call a probability measure P on $\mathcal{M}(X)$ a random measure and if P is concentrated on $\mathcal{M}^{\cdot\cdot}(X)$ a point process. Finally let $F(X)$ be the set of bounded, non-negative and continuous functions on X and $F_b(X) \subset F(X)$ the subset of those functions in $F(X)$ with bounded support.

For a detailed construction of random measures solving the functional equation $(\Sigma_{L,\alpha})$ below see [7]. Let $\alpha \in \mathcal{M}(X)$ be a locally finite measure and denote by $\alpha(f) := \int f d\alpha$ for any $f \in F$. Let L be a measure on $\mathcal{M}(X) \setminus \{0\}$ satisfying

$$\int \left[1 - e^{-\nu(f)}\right] L(d\nu) < +\infty, \qquad f \in F. \tag{2.1}$$

We are interested in solution of the functional equation

$$C_{\mathrm{P}} = C_L \star \mathrm{P} + \alpha \otimes \mathrm{P}, \tag{$\Sigma_{L,\alpha}$}$$

where $\star$ is a kind of convolution,

$$(C_L \star \mathrm{P})(h) = \iint h(x, \mu + \nu) C_L(\underset{\cdot}{x}, d\nu) \mathrm{P}(d\mu) \quad \text{for} \quad h \in F(X \times \mathcal{M}(X)).$$

Theorem 2.1. *(Integration by parts [7]). Let $\alpha \in \mathcal{M}(X)$ and $L \in \mathcal{M}(\mathcal{M}(X) \setminus \{0\})$ satisfy condition (2.1). Then the functional equation $(\Sigma_{L,\alpha})$ has a unique solution which we denote by $\mathbf{D}_{L,\alpha}$. The random measure $\mathbf{D}_{L,\alpha}$ is infinitely divisible with canonical pair (α, L).*

Our main interest in the following section lies in measures L which are concentrated on measures of the form $r\delta_x$, $r > 0$, $x \in X$. L will even be given as the image of a product measure $\rho \otimes \tau$ on $X \times \mathbb{R}_+$ under the mapping $\chi : (x, r) \mapsto r\delta_x$. We only assume $\rho \in \mathcal{M}(X)$ and $\int r\tau(\underset{\cdot}{r}) < \infty$, in particular τ does not need to be a finite measure. This result is caused by the Pólya sum process itself, but the structure stays the same if we drop this restriction in section 4.

Given a kernel

$$\eta : \mathcal{M}(X) \times X \to \mathbb{R},$$

Zessin [1] gave sufficient conditions on η such that a point process P exists which satisfies the partial integration formula for its Campbell measure C_{P},

$$C_{\mathrm{P}}(h) = \iint h(x, \mu + \delta_x)\, \eta(\mu, dx) \mathrm{P}(d\mu), \qquad h \in F.$$

Such point processes are called Papangelou processes according to Zessin. One fundamental example he stated is the point process with conditional intensity (1.1).

Definition 2.1. *(Pólya Sum Process). Let $\rho \in \mathcal{M}(X)$ be a locally finite measure and $z \in (0, 1)$ a real number. Then the* Pólya sum process $\mathbf{S}_{z,\rho}$ *is the Papangelou process for the Papangelou kernel*

$$\eta : \mathcal{M}^{\cdot\cdot}(X) \times X \to \mathbb{R}, \qquad \eta(\mu, dx) := z\big(\rho + \mu\big)(dx).$$

Zessin showed by using partial integration that for $B \in \mathcal{B}_0(X)$, the number of points inside B obeys a negative binomial distribution,

$$\mathbf{S}_{z,\rho}(\zeta_B = k) = (1 - z)^{\rho(B)} \frac{z^k}{k!} \rho(B)^{[k]},$$

where $a^{[m]} = a(a + 1) \cdots (a + m - 1)$, and furthermore for each finite collection of disjoint, bounded, measurable sets $B_1, \dots, B_n$, $\zeta_{B_1}, \dots, \zeta_{B_n}$ is a family independent random variables.

3. The Cox representation

We begin with the main theorem of this section, which shows in which way the Lévy measure, the basic component of the random measure, transforms under the mapping to the corresponding Cox process.

Theorem 3.1. *Let $\rho \in \mathcal{M}(X)$, $z \in (0,1)$ and $L = \chi\gamma$ be the image of γ under the mapping $\chi : X \times \mathbb{R}_+ \to \mathcal{M}(X), (x,r) \mapsto r\delta_x$, where*

$$\gamma := \rho \otimes \tau, \qquad \tau(r) := \frac{1}{r} e^{-r\frac{1-z}{z}} 1_{(0,\infty)}(r)\, dr$$

and define the Cox process

$$\mathrm{P} := \int \mathbf{P}_k D_{L,0}(d\kappa).$$

Then $\mathrm{P} = \mathbf{S}_{z,\rho}$.

Note that since

$$\int r\tau(dr) = \int_0^\infty e^{-r\frac{1-z}{z}}\, dr = \frac{z}{1-z} < \infty,$$

L is of first order, hence $\mathbf{D}_{L,0}$ is well defined. The theorem follows directly from lemma 3.1 and proposition 3.1 below, in which we show that both processes satisfy the functional equation $(\Sigma_{L,0})$ for the same Lévy measure L. A second direct consequence of either theorem 3.1 or lemma 3.1 below is that the superposition of realizations of two independent Pólya sum processes is distributed according to a Pólya sum processes,

Corollary 3.1. *Let $z \in (0,1)$ and $\rho, \sigma \in \mathcal{M}(X)$. Then $\boldsymbol{S}_{z,\rho+\sigma} = \boldsymbol{S}_{z,\rho} * \boldsymbol{S}_{z,\sigma}$.*

A third consequence of lemma 3.1 is a characterization of the Palm distribution $\mathbf{S}^x_{z,\rho}$ of the Pólya sum processes. By integrating with respect to the Campbell measure functions h which depend on its first component only, one recovers the intensity measure $\nu_{\mathbf{S}_{z,\rho}}$ of the Pólya sum process $\mathbf{S}_{z,\rho}$ to be

$$\nu_{\mathbf{S}_{z,\rho}} = \frac{z}{1-z}\rho.$$

From equation (3.1) below we get immediately that $\mathbf{S}^x_{z,\rho}$ is $\mathbf{S}_{z,\rho}$ with a geometrically distributed number of points added at the site x.

Corollary 3.2. *Let $z \in (0,1)$ and $\rho \in \mathcal{M}(X)$. Then*

$$\boldsymbol{S}^x_{z,\rho} = \boldsymbol{S}_{z,\rho} * \sum_{j\geq 1} (1-z) z^{j-1} \Delta_x^{*j},$$

where Δ_x is the point process which realizes exactly one point at x.

The first step to prove theorem 3.1 is a successive integration by parts.

Lemma 3.1. *Let $\rho \in \mathcal{M}(X)$, $z \in (0,1)$ and $L' = \chi\gamma'$, where*

$$\gamma' := \rho \otimes \tau', \qquad \tau' := \sum_{j\geq 1} \frac{z^j}{j} \delta_j.$$

Then $\boldsymbol{S}_{z,\rho} = \boldsymbol{D}_{L',0}$.

Proof. Assume $0 \le h \le c 1_B \otimes 1$, then by applying the integration by parts formula repeatedly and bounded convergence,

$$C_{\mathbf{S}_{z,\rho}}(h) = \iint \sum_{j\ge 1} z^j h(x, \mu + j\delta_x)\rho(dx)\mathbf{S}_{z,\rho}(d\mu). \tag{3.1}$$

On the other hand, $\mathbf{D}_{L',0}$ is characterized by

$$\begin{aligned} C_{\mathbf{D}_{L',0}}(h) &= \iint h(x, \mu + \nu)C_{L'}(dx, d\nu)\mathbf{D}_{L',0}(d\mu) \\ &= \iint h\big(x, \mu + \chi(x, r)\big)\gamma'(dx, dr)\mathbf{D}_{L',0}(d\mu) \\ &= \iint \sum_{j\ge 1} z^j h(x, \mu + j\delta_x)\rho(dx)\mathbf{D}_{L',0}(d\mu). \end{aligned}$$

Therefore $\mathbf{S}_{z,\rho}$ and $\mathbf{D}_{L,0}$ agree. $\square$

Proposition 3.1. *Let* P *be the Cox process defined in theorem 3.1. Then* $\mathrm{P} = \boldsymbol{D}_{L',0}$*, where* $\boldsymbol{D}_{L',0}$ *is the point process defined in lemma 3.1.*

Proof. Let $h \in F$. We divide the proof into four steps:
I. At first we apply the partial integration to the inner Poisson process, which in fact is a partial integration formula for the Cox process, and observe that we get the Campbell measure $C_{\mathbf{D}_{L,0}}$ of $\mathbf{D}_{L,0}$,

$$\begin{aligned} C_{\mathrm{P}}(h) &= \iint h(x, \mu + \delta_x)\kappa(dx)\mathrm{P}(d\mu, d\kappa) \\ &= \iiint h(x, \mu + \delta_x)\kappa(dx)\mathbf{P}_\kappa(d\mu)\mathbf{D}_{L,0}(d\kappa). \end{aligned}$$

Let $g(x, \kappa) := \int h(x, \mu + \delta_x)\mathbf{P}_\kappa(d\mu)$ to get that on the rhs. we obtained $C_{\mathbf{D}_{L,0}}(g)$.
II. Applying partial integration to $C_{\mathbf{D}_{L,0}}$ yields

$$C_{\mathbf{D}_{L,0}}(g) = \iiint g(x, \kappa + \nu)\nu(dx)\chi\gamma(d\nu)\mathbf{D}_{L,0}(d\kappa) \tag{3.2}$$

$$= \iiint g(y, \kappa + r\delta_y)r\tau(dr)\rho(dy)\mathbf{D}_{L,0}(d\kappa). \tag{3.3}$$

Note that the factor r cancels the r^{-1} in the definition of the measure τ.
III. Consider the integrand g with its new arguments. We have to integrate w.r.t. a Poisson process with intensity measure $\kappa + r\delta_y$, but that is only a convolution of two Poisson processes with intensity measures κ and $r\delta_y$, respectively.

$$\begin{aligned} g(y, \kappa + r\delta_y) &= \int h(y, \mu + \delta_y)\mathbf{P}_{\kappa + r\delta_y}(d\mu) \\ &= \iint h(y, \mu + \nu + \delta_y)\mathbf{P}_{r\delta_y}(d\nu)\mathbf{P}_\kappa(d\mu). \end{aligned}$$

IV. We go back to equation (3.3) and observe that the Poisson process $\mathbf{P}_\kappa$ and $\mathbf{D}_{L,0}$ form the Cox process P. The remaining Poisson process $\mathbf{P}_{r\delta_y}$ is mixed with respect to τ and can be evaluated explicitly,

$$\iint h(y, \mu + \nu + \delta_y)\mathbf{P}_{r\delta_y}(d\nu)e^{-ar}1_{0\leq r<\infty}dr =$$
$$= \sum_{n\geq 0}\frac{1}{n!}h(y,\mu+(n+1)\delta_y)\int_0^\infty r^n e^{-r\frac{1-z}{z}}e^{-r}dr = \sum_{n\geq 1} z^n h(y,\mu+n\delta_y).$$

For the integrals we have $I_n := \int r^n e^{-r/z}dr = z^{n+1}n!$, which can easily be shown by observing that they satisfy the recursion

$$I_n = znI_{n-1}, \qquad I_0 = z. \qquad \square$$

4. General canonical pairs

In the previous section we focussed on Lévy measures which were concentrated on the set of measures $\{r\delta_x : r > 0, x \in X\}$, which caused the Cox process to have independent increments. However, we may drop this restriction on the Lévy measure and nevertheless obtain an infinitely divisible point process.

Theorem 4.1. *Let* $\mathrm{P}(d\mu) = \int \boldsymbol{P}_\kappa(d\mu)\boldsymbol{D}_{L,\alpha}(d\kappa)$ *for* $\alpha \in \mathcal{M}(X)$ *and a first-order measure* L *on* $\mathcal{M}(X)\setminus\{0\}$*. Then* P *satisfies the integration by parts formula*

$$C_{\mathrm{P}}(h) = \iint h(x,\mu+\nu)C_{\tilde{L}}(dx,d\nu)\mathrm{P}(d\mu).$$

In particular $\mathrm{P} = \boldsymbol{D}_{\tilde{L},0}$*, where* $\tilde{L}$ *is the image under the mapping*

$$\mathcal{M}(X)\times\mathcal{M}\big(\mathcal{M}(X)\setminus\{0\}\big) \to \mathcal{M}\big(\mathcal{M}(X)\setminus\{0\}\big), \qquad (\alpha, L) \mapsto \chi(\alpha\otimes\delta_1) + \int \boldsymbol{P}_\lambda L(d\lambda).$$

Proof. Assume $\alpha = 0$ for the moment. We adapt the proof of proposition 3.1 and outline the changes occurring in the different parts of that proof.
II. We do not make use of the particular form of L in equation (3.2), therefore

$$C_{\mathbf{D}_{L,0}}(g) = \iiint g(x,\kappa+\lambda)\lambda(dx)L(d\lambda)\mathbf{D}_{L,0}(d\kappa). \tag{4.1}$$

III. Consider now the integration of g wrt. λ, then by the partial integration of the Poisson process,

$$\begin{aligned}\int g(x,\kappa+\lambda)\lambda(dx) &= \iint h(x,\mu+\delta_x)\mathbf{P}_{\kappa+\lambda}(d\mu)\lambda(dx)\\ &= \iiint h(x,\mu+\nu+\delta_x)\lambda(dx)\mathbf{P}_\lambda(d\nu)\mathbf{P}_\kappa(d\mu)\\ &= \iiint h(x,\mu+\nu)\nu(dx)\mathbf{P}_\lambda(d\nu)\mathbf{P}_\kappa(d\mu).\end{aligned}$$

Putting all together we obtain

$$C_{\mathrm{P}}(h) = \iiint h(x, \mu + \nu)\nu(dx)\mathbf{P}_\lambda(d\nu)L(d\lambda)\mathrm{P}(d\mu).$$

V. Now we drop the restriction $\alpha = 0$. Because of the additive structure of the partial integration we only consider the additional summand to be introduced in equation (4.1),

$$\begin{aligned}\iint g(x,\kappa)\alpha(dx)\mathbf{D}_{L,\alpha} &= \iiint h(x, \mu + \delta_x)\alpha(dx)\mathbf{P}_\kappa(d\mu)\mathbf{D}_{L,\alpha}(d\kappa)\\ &= \iint h(x, \mu + \nu)\chi\big(\alpha \otimes \delta_1\big)(d\nu)\mathrm{P}(d\mu). \qquad \square\end{aligned}$$

Remark 4.1. *Consider again the measure L as the image of a measure γ on $X \times \mathbb{R}_+$ under the mapping $(x, r) \mapsto r\delta_x$, in which case* P *has independent increments. If $\gamma = \rho \otimes \tau$, we obtain $\tilde{L} = \bar{\alpha} + \chi\tilde{\gamma}$, where $\tilde{\gamma} = \rho \otimes \tilde{\tau}$ and*

$$\tilde{\tau} = \sum_{j \geq 1} \frac{I_j}{(j-1)!}\delta_j, \qquad I_j = \int_0^\infty r^j e^{-r}\tau(dr).$$

Theorem 4.1 states that if the directing measure of a Cox process is infinitely divisible, then this holds for the Cox process and furthermore its Lévy measure is known. This can be reversed directly: If we knew that the Cox process is infinetely divisible and its Lévy measure can be decomposed in a suitable way, then the directing measure is infinitely divisible itself.

Proposition 4.1. *Let* P *be an infinitely divisible Cox process with directing measure P and Lévy measure $\tilde{L}$. If there exist $\alpha \in \mathcal{M}(X)$ and a measure L on $\mathcal{M}(X) \setminus \{0\}$ such that $\tilde{L} = \bar{\alpha} + \int \boldsymbol{P}_\lambda L(d\lambda)$, then $P = \boldsymbol{D}_{L,\alpha}$.*

Proof. Reversing the calculations in the proof of theorem 4.1 with $\mathbf{D}_{L,\alpha}$ replaced by P, we get that P satisfies the integration by parts formula

$$C_P(h) = \iint g(x, \kappa + \lambda)C_L(dx, d\lambda)P(d\kappa) + \iint g(x, \kappa)\alpha(dx)P(d\kappa),$$

where $g(x, \rho) = \int h(x, \mu + \delta_x)\mathbf{P}_\rho(d\mu)$. If $A, B \in \mathcal{B}_0$, then at least for $h = 1_A \otimes e^{-\zeta_B}$ we have

$$g(x, \rho) = 1_A(x)e^{-1_B(x)}e^{-(1-e^{-1})\zeta_B(\rho)},$$

and the set of these functions generates $\mathcal{B} \otimes \sigma(\zeta_B : B \in \mathcal{B}_0)$. $\square$

5. Asymptotic of the Pólya sum process

We already identified the intensity measure $\nu_{\mathbf{S}_{z,\rho}}$ of the Pólya sum process $\mathbf{S}_{z,\rho}$ as

$$\nu_{\mathbf{S}_{z,\rho}} = \frac{z}{1-z}\rho.$$

Hence as $z \to 1$, the intensity measure of $\mathbf{S}_{z,\rho}$ explodes. But what happens if we weight each (including multiple) point configuration by $\frac{1-z}{z}$ and consider then the limit as $z \to 1$?

Definition 5.1. *Let $z \in (0,1)$ and $\rho \in \mathcal{M}(X)$. Then denote by $\boldsymbol{P}_z$ the image of $\boldsymbol{S}_{z,\rho}$ under the mapping*

$$\mathcal{M}(X) \to \mathcal{M}(X), \qquad \mu \mapsto \frac{1-z}{z}\mu.$$

Firstly we remark that $\mathbf{P}_z$ inherits the infinite divisibility from $\mathbf{S}_{z,\rho}$ and that its Lévy measure may be obtained directly from that of the Pólya sum process by a proper scaling.

Lemma 5.1. *$\boldsymbol{P}_z = \boldsymbol{D}_{K,0}$, where $K = \chi\gamma''$ and*

$$\gamma'' = \rho \otimes \tau_z, \qquad \tau_z = \sum_{j\geq 1} \frac{z^j}{j}\delta_{\frac{1-z}{z}j}. \tag{5.1}$$

Proof. Let $h \in F$, then by lemma 3.1,

$$\begin{aligned}
C_{\mathbf{P}_z}(h) &= \iint h\left(x, \frac{1-z}{z}\mu\right)\frac{1-z}{z}\mu(dx)\mathbf{S}_{z,\rho}(d\mu)\\
&= \iint \frac{1-z}{z}\sum_{j\geq 1} z^j h\left(x, \frac{1-z}{z}\mu + j\frac{1-z}{z}\delta_x\right)\rho(dx)\mathbf{S}_{z,\rho}(d\mu)\\
&= \iint h(x, \mu+\nu)\nu(dx)K(d\nu)\mathbf{P}_z(d\mu),
\end{aligned}$$

where $K = \chi\gamma''$ is given by (5.1). □

One can show that τ_z converges for a certain class of bounded and continuous function with a growth condition at the origin. We do not need these considerations if we use the Cox representation.

Theorem 5.1. *As $z \to 1$, $\boldsymbol{P}_z$ has a weak limit $\boldsymbol{P}$ which is the unique solution of the functional equation*

$$C_P = C_{\tilde{K}} \star P, \tag{$\Sigma_{\tilde{K},0}$}$$

where $\tilde{K} = \chi(\rho \otimes \tilde{\tau})$ and $\tilde{\tau}(dr) = \frac{1}{r}e^{-r}1_{(0,\infty)}dr$.

Since the unique solution of the functional equation $(\Sigma_{\tilde{K},0})$ is $\mathbf{D}_{K',0}$, we deduce that $P_z \to \mathbf{D}_{K',0}$ weakly as $z \to 1$. **P**roof. Let $h \in F$, then the Cox representation in theorem 3.1 and the argument in the previous proof show that

$$\begin{aligned}
C_{P_z}(h) &= \int h\left(x, \nu + \frac{1-z}{z}(\mu+\delta_x)\right) \mathbf{P}_\lambda(d\mu)\frac{1-z}{z}\lambda(dx)L(d\lambda)\, P_z(d\nu) \\
&= \int h\left(x, \nu + \frac{1-z}{z}(\mu+\delta_x)\right) P_{r\delta_x}(d\mu)e^{-\frac{1-z}{z}r}\frac{1-z}{z}dr\rho(dx)\, P_z(d\nu) \\
&= \int h\left(x, \nu + \frac{1-z}{z}(\mu+\delta_x)\right) \mathbf{P}_{\frac{z}{1-z}t\delta_x}(d\mu)e^{-t}\, dt\rho(dx)\, P_z(d\nu).
\end{aligned}$$

Thus it suffices to show that the inner integrals converge to the Campbell measure of a random measure $\tilde{K}$ [9]. Due to the substitution $t = \frac{1-z}{z}r$, the mixing measure of the inner Poisson process does not depend on z any more. Notice that, as $z \to 1$, $\mathbf{P}_{\frac{z}{1-z}t\delta_x}$ realizes a point at x with Poisson distributed weight with increasing intensity, but that weight is scaled in the same manner such that we have, since h is bounded,

$$\int h\left(x, \nu + \frac{1-z}{z}(\mu+\delta_x)\right) \mathbf{P}_{\frac{z}{1-z}t\delta_x}(d\mu) \to h(x, \nu + t\delta_x)$$

as $z \to 1$. Furthermore note that

$$\iint h(x, \nu + t\delta_x)e^{-t}dt\rho(dx) = \int h(x, \nu+\lambda)C_{\tilde{K}}(dx, d\lambda),$$

where $\tilde{K}$ is given in the theorem. Using [10, thm 1], one immediately sees that $C_{\tilde{K}}$ is indeed the Campbell measure of a random measure. Therefore Pr_z has a weak limit P as $z \to 1$ and P is a solution of the functional equation

$$C_P(h) = \iint h(x, \nu+\lambda)C_{\tilde{K}}(dx, d\lambda)P(d\nu). \qquad \square$$

Finally we address the distribution of the random variable ζ_B under P for $B \in \mathcal{B}_0$, which can be derived from the integration by parts formula $(\Sigma_{\tilde{K},0})$. Whenever $\rho(B) > 0$, $P(\zeta_B = 0) = 0$ since τ is an infinite measure and therefore $P(\zeta_B = 0) = \exp\left(-\tilde{K}(\zeta_B > 0)\right)$. Furthermore, since $\tilde{K}$ is concentrated on the set of measures of the form $r\delta_x$, any family $\zeta_{B_1}, \dots \zeta_{B_n}$ of pairwise disjoint Borel sets $B_1, \dots B_n$ is independent.

Next we apply $(\Sigma_{\tilde{K},0})$ to the function $h := 1_B \otimes 1_{\{t<\zeta_B \leq t+\varepsilon\}}$ for $t, \varepsilon > 0$ to get information about the distribution of ζ_B:

$$\begin{aligned}
P\big(\zeta_B \in (t, t+\varepsilon]\big) &= \iint \frac{1_B(x)}{\mu(B)} 1_{\{t<\mu(B)\leq t+\varepsilon\}}\mu(\text{x})P(d\mu) \\
&= \rho(B) \iint_{\mathbb{R}_+^2} \frac{1}{s+r} 1_{\{s+r \in (t,t+\varepsilon]\}} e^{-r}drP\zeta_B^{-1}(ds) \\
&= \rho(B) \int_0^{t+\varepsilon} \int_{t-s}^{t+\varepsilon-s} \frac{1}{s+r} 1_{\{s+r \in (t,t+\varepsilon]\}} e^{-r}drP\zeta_B^{-1}(ds) \\
&= \rho(B) \int_0^{t+\varepsilon} e^s \int_t^{t+\varepsilon} \frac{1}{u}e^{-u}\, duP\zeta_B^{-1}(ds).
\end{aligned}$$

The last line shows that ζ_B is a continuous random variable, therefore, if we denote by f_B the density of ζ_B, dividing by ε and $\varepsilon \to 0$ yields that f_B satisfies

$$f_B(t) = \int_0^t \rho(B)e^{s-t}\frac{1}{t}f_B(s)(ds),$$

which is a Volterra integral equation. By substituting $g(t) = e^{-t}f_B(t)$, we see that g satisfies

$$g(t) = \frac{\rho(B)}{t}\int_0^t g(s)ds,$$

whose solutions are multiples of $t^{\rho(B)-1}$. Using the fact that $P(\zeta_B = 0) = 0$, f_B is the density of the gamma distribution

$$f_B(t) = \frac{t^{\rho(B)-1}}{\Gamma\big(\rho(B)\big)}e^{-t}$$

is the desired solution.

Список литературы

[1] H. Zessin, Der Papangelou Prozess, Journal of Contemporary Mathematical Analysis, **44**, 36 – 44 (2009).

[2] F. Papangelou, The conditional intensity of general point processes and an application to line processes, Z. Wahrscheinlichkeitstheorie verw. Gebiete, **28**, 207 – 226 (1974).

[3] O. Kallenberg, On Conditional Intensities of Point Processes, Z. Wahrscheinlichkeitstheorie verw. Geb., **41**, 205 – 220 (1978).

[4] X. X. Nguyen and H. Zessin, Integral and differential characterisation of the Gibbs process, Math. Nachr., **88**, 105 – 115 (1979).

[5] K. Matthes, W. Warmuth and J. Mecke, Bemerkungen zu einer Arbeit von Nguyen Xuan Xanh and Hans Zessin, Math. Machr., **88**, 117 – 127 (1979).

[6] D. Daley and D. Vere-Jones, An Introduction to the Theory of Point Processes, 1st ed, Springer, (1988).

[7] B. Nehring and H. Zessin, A path integral representation of the moment measures of the general ideal Bose gas, Preprint (2010).

[8] O. Barndorff-Nielsen and G. Yeo, Negative Binomial Processes, J. Appl. Prob., **6**, 633 – 647 (1969).

[9] J. Kerstan, K. Matthes and J. Mecke, Infinitely Divisible Point Processes, John Wiley & Sons (1978).

[10] H. Wegmann, Characterization of Palm Distributions and Infinitely Divisible random measures, Z. Wahrscheinlichkeitstheorie verw. Gebiete **39**, 257 – 262 (1977).

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