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THE PAPANGELOU PROCESS. A CONCEPT FOR GIBBS, FERMI AND BOSE PROCESSES

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Abstract. This note is a revised and enlarged version of the german article [16] in a slightly different framework. We here correct a serious mistake in the first version and generalize the class of Polya sum processes considered there. (A corrected version of the same results can be found already in the thesis of Mathias Rafler [12].) Moreover, the class of Polya difference processes is constructed here for the first time. In analogy to classical statistical mechanics we propose a theory of interacting Bosons and Fermions. We consider Papangelou processes. These are point processes specified by some kernel which represents the conditional intensity of the process. The main result is a general construction of a large class of such processes which contains Cox, Gibbs processes of classical statistical mechanics, but also interacting Bose and Fermi processes.

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1. Introduction and Generalities

The analysis of the ideal quantum mechanical gases of Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac in [1] shows that the corresponding point processes are qualitatively different; the associated random fields have different distributions. Nevertheless they are ideal gases in the sense that they are of first order and have independent increments. Thus it is natural for the construction of the corresponding interacting particle systems to take as a starting point the corresponding ideal, i.e. non-interacting processes. For the Gibbsian theory of classical statistical mechanics this is the Poisson process. We propose here to construct interacting Bosons by means of the ideal Bose process and interacting Fermions by means of the ideal Fermi process. This is not done in the spirit of the DLR-approach but in the spirit of the equivalent theory of integration by parts formulas (cf. [7]) which represent an abstract version of the classical approach by Kirkwood-Salzburg equations. And this means that the starting point of the whole theory is the appropriate Boltzmann kernel determined by

the underlying interaction potential together with the appropriate ideal gas; and these kernels differ for the Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac processes. In the language of modern point process theory the Boltzmann kernels represent conditional intensities of the corresponding point processes, which are called Polya sum in case of the ideal Bose process resp. Polya difference processes for the ideal Fermi process and which have to be modified in case of interacions by an appropriate Boltzmann factor.

Thus the main task is first to identify these conditional intensities for the three ideal gas processes. And this has been done in [1]. We take them here as a starting point to build up the first steps of a unifying general theory which can explain the appearance of all three quantum mechanical gases. Conceptually this theory exists since a long time and is the theory of Papangelou processes (cf. [3, 6, 7, 9, 13, 15]). The historical point of departure of this theory has been described in [10, 16].

The main result is a construction theorem which was missing until now. As examples we present the ideal Bose process, i.e. the Polya sum process, and the ideal Fermi process, i.e. the Polya difference process, and then indicate how one can add an interaction between the particles. All this is done in analogy to the Gibbsian theory which can be considered as a theory for quantum particles obeying Maxwell-Boltzmann statistics.

Before starting our approach we indicate shortly how one can use the construction theory of the DLR-approach for the construction of Papangelou processes. This can be done by combining the important work of Rauchenschwandtner [13] with the fundamental work of Preston [11]: Given a kernel π one can define a socalled specification $\mathcal{V}_\pi$ such that the associated collection $\mathcal{G}(\mathcal{V}_\pi)$ of all abstract Gibbs states specified by $\mathcal{V}_\pi$ coincides with the collection of all Papangelou processes with kernel π. This result can be found in [13]. Therefore, if we start with a kernel π such that the assumptions of Preston's general existence theorem $\mathcal{G}(\mathcal{V}_\pi) \neq \emptyset$ are satisfied then we are done. Our approach here is more direct but also weaker in the sense that we construct for a given π a Papangelou process for some locally modified kernel $\tilde{\pi}$ which in general does not coinside with π.

We shall work in the following *general setting*. X denotes a Polish state space, $\mathcal{B}(X)$ resp. $\mathcal{B}_0(X)$ its Borel resp. bounded Borel sets. $\mathcal{M}(X)$ is the vaguely Polish space of locally finite measures on X (i.e. of Radon measures on X). $\mathcal{M}^{\cdot\cdot}(X)$ denotes the

subspace of all Radon point measures on X, and $\mathcal{M}_{\ddot{B}}(X), B \in \mathcal{B}_0(X)$, the collection of all point measures with support in B. $\mathcal{M}_{\dot{f}}(X)$ is the space of all finite point measures. We need also the space $\mathcal{M}_{\cdot}(X)$ of all *simple* Radon point measures on X, i.e. of all locally finite subsets of X. All these spaces are given the Borel σ-fields generated by the vague topology in $\mathcal{M}(X)$, and are denoted by $\mathcal{F}^{\cdot\cdot}, \mathcal{F}_{\dot{f}}$ and $\mathcal{F}_{\ddot{B}}$. For some underlying measurable space S we denote by $F_+(S)$ the collection of all non-negative, measurable real functions defined on S. We consider random measures in X, i.e. random elements ξ in or, their laws P on $\mathcal{M}(X)$, for which we write $P \in \mathcal{PM}(X)$. If such a P is concentrated on the measurable subset $\mathcal{M}^{\cdot\cdot}(X)$ then P is called a point process in X. The *Campbell measure* of a point process P is defined by $\mathbb{C}_P(h) = \int\int h(x,\mu)\mu(dx)P(d\mu), h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$, whereas the *reduced Campbell measure* of P is given by $\mathbb{C}'_P(h) = \int\int h(x,\mu-\delta_x)\mu(dx)P(d\mu), h \in F_+$. (We shall use freely these and related concepts of the theory of random measures and point processes and refer to the standard monographies [3] and [6] for details.)
The point of departure is a kernel $\pi(\varrho,\eta;dx)$ from $(\mathcal{M}\times\mathcal{M}^{\cdot\cdot}, \mathcal{F}\otimes\mathcal{F}^{\cdot\cdot})$ to the set of all Radon measures on X. (Here and in the sequel we shall skip X if possible.) We are interested in point processes P in X for which the kernel π is a conditional intensity. This means that P is a solution of the following equation

$$\mathbb{C}_P(h) = \int\int h(x,\mu+\delta_x)\pi(\mu,dx)P(d\mu), h \in F_+(X\times\mathcal{M}^{\cdot\cdot}(X)).$$

We then call P a *Papangelou process for* π. For such a Papangelou process the kernel π is a.s. uniquely determined; moreover, P is a Papangelou process for some kernel π iff the measure $\mathbb{C}'_P(B\times(.))$ is dominated by P for any $B \in \mathcal{B}_0(X)$. (All this can be found in [5, 7, 9].) In the scholion at the end we develop the notion of a Papangelou process in more detail in a discrete setting to relieve the understanding of the following abstract developements.
Given a kernel π we define for every $m \in \mathbb{N}, \varrho \in \mathcal{M}$ and $\eta \in \mathcal{M}^{\cdot\cdot}$ the following kernels on X^m:

$$\pi^{(m)}(\varrho,\eta;dx_1\dots dx_m) = \pi(\varrho,\eta;dx_1)\pi(\varrho,\eta+\delta_{x_1};dx_2)\dots\pi(\varrho,\eta+\delta_{x_1}+\dots+\delta_{x_{m-1}};dx_m)$$

For $m=0$ we denote by $\pi^{(0)}(\varrho,\eta;.)$ the kernel on $X^0=\{\emptyset\}$ which gives mass 1 to $\{\emptyset\}$.

2. The finite Papangelou process

Finite Papangelou processes can be constructed if the kernel π satisfies the following *integrability condition*: For any given (ϱ, η) the following series is convergent.

$$\Xi(\varrho, \eta) = \sum_{m \geq 0} \frac{1}{m!} \pi^{(m)}(\varrho, \eta; X^m). \tag{2.1}$$

Under this condition the *finite Papangelou process with kernel* π is well defined for a given $\varphi \in F_+(\mathcal{M}_f^\cdot)$ by

$$P_\pi^{(\varrho,\eta)}(\varphi) = \frac{1}{\Xi(\varrho, \eta)} \sum_{m \geq 0} \frac{1}{m!} \int_{X^m} \varphi(\delta_{x_1} + \cdots + \delta_{x_m}) \pi^{(m)}(\varrho, \eta; dx_1 \ldots dx_m). \tag{2.2}$$

We add another condition on the kernel π which plays a fundamental role in the whole theory. We require that π satisfies the following *cocycle condition*: For all ϱ, η, x, y

$$\pi(\varrho, \eta; dx)\pi(\varrho, \eta + \delta_x; dy) = \pi(\varrho, \eta; dy)\pi(\varrho, \eta + \delta_y; dx)$$

This condition implies the symmetry of the kernels $\pi^{(m)}(\varrho, \eta; .)$. The next result plays the role of a main lemma. A proof can be found in [16].

Lemma 2.1. *If π is an integrable kernel satisfying the cocycle condition then every $P_\pi^{(\varrho,\eta)}$ is a solution $P \in \mathcal{P}\mathcal{M}_f^\cdot$ of the following integration by parts formula.*

$$\mathbb{C}_P(h) = \int_{\mathcal{M}_f^\cdot} \int_X h(x, \mu + \delta_x) \pi(\varrho, \eta + \mu; dx) P(d\mu), h \in F_+(X \times \mathcal{M}_f^\cdot).$$

Here $\mathbb{C}_P$ denotes the *Campbell measure of* P.

3. The general Papangelou process

Here we give a construction of a large class of infinitely extended Papangelou processes, thereby correcting a mistake in [16]. The following comment is in order here: In theorem 1 of [16] the statement is that ξ is a Papangelou process for π. This is false. The correct statement can be found in theorem 3 of the present paper: ξ is a Papagelou process for $\tilde{\pi} = p_\varrho \pi$. Thus the factor p_ϱ was missing in theorem 1 of [16]. We start with a kernel $\pi(\varrho, \eta; .)$ from $\mathcal{M} \times \mathcal{M}^{\cdot\cdot}$ to the set $\mathcal{M}$ of Radon measures on X. We now require that there exists a *locally finite partition* $\Delta = (X_n)_{n \geq 0}$ of X. This means that every bounded Borel set has a non-empty intersection with only finitely many elements of the partition. In addition every X_n has to be a bounded Polish subspace of X. Recall that every Polish subspace of X is a $\mathcal{G}_\delta$−set and thereby Borelian.

We need the following notations. η_{X^n} is the restriction of η to $X^n = X_0 \cup \cdots \cup X_n$; and η_k its restriction to X_k. We then say that π is *locally integrable* (with respect to any Δ) if for all ϱ, η, n the following series converge:

$$\Xi_n(\varrho, \eta) = \sum_{m \geq 0} \frac{1}{m!} \pi^{(m)}(\varrho, \eta_{X^{n-1}}; X_n^m).$$

By our considerations above the following finite process in X_n is then well defined for all ϱ, η. If $\varphi \in F_+(\mathcal{M}_{\ddot{f}}(X_n))$

$$\Pi^{\varrho}_{X_n}(\eta; \varphi) = \frac{1}{\Xi_n(\varrho, \eta)} \sum_{m \geq 0} \frac{1}{m!} \int_{X_n^m} \varphi(\delta_{x_1} + \cdots + \delta_{x_m}) \pi^{(m)}(\varrho, \eta_{X^{n-1}}; dx_1 \ldots dx_m).$$

Note that the kernel $\Pi^{\varrho}_{X_n}$ depends only on the σ-field $\mathcal{F}_{\ddot{X}^{n-1}}$ of events happening in X^{n-1}. This will be important in the sequel.

The aim now is to construct by means of these kernels and suitable initial and boundary conditions an infinitely extended Papangelou process which has a modification of π as its kernel. For this purpose we use the theorem of Ionescu Tulcea which enables us to construct processes by means of an initial condition and conditional distributions (see [2] e.g.).

Given $\varrho \in \mathcal{M}(X)$ and $(\eta_0, \ldots, \eta_{m-1}) \in \mathcal{M}_{\ddot{X}_0} \times \cdots \times \mathcal{M}_{\ddot{X}_{m-1}}$, consider the Markovian kernels

$$Q^{\varrho}_m(\eta_0, \ldots, \eta_{m-1}; d\eta_m) = \Pi^{\varrho}_{X_m}(\eta_0 + \cdots + \eta_{m-1}; d\eta_m)$$

from $\mathcal{M}_{\ddot{X}_0} \times \cdots \times \mathcal{M}_{\ddot{X}_{m-1}}$ to $\mathcal{M}^{\cdot\cdot}(X_m)$. Given also a random measure $P_0 \in \mathcal{PM}$ then by the theorem of Ionescu Tulcea there exist random elements ζ in $\mathcal{M}(X)$ and random elements ξ_n in $\mathcal{M}^{\cdot\cdot}(X_n), n \geq 0$, with the property that the corresponding finite-dimensional distributions are given by

$$\mathcal{L}(\zeta, \xi_0, \ldots, \xi_n) = P_0(d\varrho) \Pi^{\varrho}_{X_0}(0; d\eta_0) Q^{\varrho}_1(\eta_0; d\eta_1) \ldots Q^{\varrho}_n(\eta_0, \ldots, \eta_{n-1}; d\eta_n).$$

We are now in the position to construct the following random element in $\mathcal{M}^{\cdot\cdot}(X)$.

$$\xi = \sum_{n \geq 0} \xi_n.$$

Note that ξ is locally finite because the underlying partition has this property. The distribution P_π of ξ is the point process in X we are interested in. (We do not indicate the initial condition P_0.) We shall show now that P_π is a Papangelou process specified by a modification of the kernel π. For later use we remark that P_π has the following

disintegration with respect to ζ.

$$P_\pi = \int_{\mathcal{M}} P_\pi^\varrho \; P_0(d\varrho).$$

We observe also that this implies for the Campbell measure of P_π the disintegration

$$\mathbb{C}_{P_\pi} = \int_{\mathcal{M}} \mathbb{C}_{P_\pi^\varrho} \; P_0(d\varrho).$$

We now formulate the conditions for the existence of infinitely extended Papangelou processes.

We assume that π is a kernel from $\mathcal{M}(X) \times \mathcal{M}^{\cdot\cdot}(X)$ to $\mathcal{M}(X)$ which is *locally integrable* with respect to any partition. Moreover, we assume that π is *dominated with a symmetric density*, i.e. on the complement of any $\{y\}$ each $\pi(\varrho, \eta + \delta_y; dx)$ has a *symmetric* density f_π with respect to $\pi(\varrho, \eta; dx)$ whith does not depend on ϱ, η. Thus

$$1_{\{y\}^c}(x)\pi(\varrho, \eta + \delta_y; dx) = 1_{\{y\}^c}(x) f_\pi(x, y)\pi(\varrho, \eta; dx),$$

with f_π being symmetric. We remark that this condition implies the cocycle condition for π and thus is a bit stronger.

Finally we need the following *finite-range property*. There exists a positive constant $R \geq 0$ such that

$$1_B(x)\pi(\varrho, \eta; dx) = 1_B(x)\pi(\varrho, \eta_{\partial_R B} + \eta_{C^c}; dx) \text{ for any } B, C \in \mathcal{B}_0, B \subset C.$$

Here $\partial_R B = \{x \in X | d(x, B) \leq R\}$, where d denotes some fixed metric compatible with the Polish topology in X. Note that for configurations η with compact support one has for any $B \in \mathcal{B}_0$

$$1_B(x)\pi(\varrho, \eta; dx) = 1_B(x)\pi(\varrho, \eta_{\partial_R B}; dx) \text{ for any } B, C \in \mathcal{B}_0.$$

Theorem 3.1. *If π is a kernel from $\mathcal{M}(X) \times \mathcal{M}^{\cdot\cdot}(X)$ to $\mathcal{M}(X)$ which is locally integrable, dominated and of finite range, then any P_π^ϱ, $\varrho \in \mathcal{M}$, is a Papangelou process with kernel*

$$\tilde{\pi}(\varrho, \eta; dx) = p_\varrho(x, \eta) \cdot \pi(\varrho, \eta; dx), \tag{3.1}$$

where

$$p_\varrho(x, \eta) = \prod_{j=0}^{\infty} \frac{\Xi_j(\varrho, \eta)}{\Xi_j(\varrho, \eta + \delta_x)} \tag{3.2}$$

The kernel $\tilde{\pi}$ is a random Radon measure.

Proof. We consider the point process ξ constructed by means of $P_0 = \delta_\varrho$ and start to compute its Campbell measure $\mathbb{C}_\xi$. For a given $h \in F_+$

$$\mathbb{C}_\xi(h) = \sum_{n\geq 0}^{\infty} \int h(x,\xi)\ \xi_n(dx)dP.$$

(Here P denotes the underlying probability which should not be confused with the law of ξ.) We follow the arguments of Mecke [8] and assume first that h has the form $h = g \otimes \varphi$, where $g \in F_+(X)$ is identically 0 outside some $X_k, 0 \leq k \leq n$, and $\varphi \in F_+(\mathcal{M}^{\cdot})$ is a random variable which is measurable with respect to $\mathcal{F}_{\ddot{X}^n}$. In this case

$$\mathbb{C}_\xi(h) = \int \xi_k(g)\ \varphi(\xi_{X^n})dP$$

and thus

$$\mathbb{C}_\xi(h) = \int \Pi^\varrho_{X_0}(0, d\eta_0)\dots\Pi^\varrho_{X_k}(\eta_{X^{k-1}}, d\eta_k)\ \eta_k(g)\cdot\psi^\varrho(\eta_{X^{k-1}}+\eta_k),$$

where

$$\psi^\varrho(\eta_{X^k})=\int \Pi^\varrho_{X_{k+1}}(\eta_{X^k}, d\eta_{k+1})\dots\Pi^\varrho(\eta_{X^k}+\eta_{k+1}+\dots+\eta_{n-1}, d\eta_n)\,\varphi(\eta_{X^k}+\eta_{k+1}+\dots+\eta_n).$$

Applying here the main lemma 2.1 to the inner integral of $\mathbb{C}_\xi(h)$ and using the cocycle property one obtains

$$\int \Pi^\varrho_{X_k}(\eta_{X^{k-1}}, d\eta_k)\ \eta_k(g)\cdot\psi^\varrho(\eta_{X^{k-1}}+\eta_k) =$$
$$\int \Pi^\varrho_{X^k}(\eta_{X^{k-1}}, d\eta_k)\int_{X^k}\pi(\varrho,\eta_{X^k}, dx)\ g(x)\cdot\psi(\varrho,\eta_{X^{k-1}}+\eta_k+\delta_x).$$

On the other hand, using again that the kernel π is dominated with a symmetric density f_π, we obtain for $k \leq l \leq n-1, x \in X_k$,

$$\Pi^\varrho_{X_{l+1}}(\eta_{X^l}+\delta_x; d\eta_{l+1}) = \frac{\Xi_{l+1}(\varrho,\eta)}{\Xi_{l+1}(\varrho,\eta+\delta_x)} f_\pi(x,\eta_{l+1})\Pi^\varrho_{X_{l+1}}(\eta_{X^l}, d\eta_{l+1}).$$

Here $f_\pi(x,\eta_{l+1}) = \prod_{y\in supp\,\eta_{l+1}} f_\pi(x,y)^{\eta_{l+1}(y)}$.
Thus we have

$$\psi(\varrho,\eta_{X^k}+\delta_x) = \int \Pi^\varrho_{X_{k+1}}(\eta_{X^k}, d\eta_{k+1})\dots\Pi^\varrho_{X_n}(\eta_{X^k}+\eta_{k+1}+\dots+\eta_{n-1}, d\eta_n)$$
$$\prod_{l=k}^{n-1}\frac{\Xi_{l+1}(\varrho,\eta)}{\Xi_{l+1}(\varrho,\eta+\delta_x)}\prod_{l=k}^{n-1} f_\pi(x,\eta_{l+1})\varphi(\eta_{X^k}+\eta_{k+1}+\dots+\eta_n+\delta_x),$$

and consequently, using again that the kernel is dominated with a symmetric density,

$$\mathbb{C}_\xi(h) = \int\int_{X_k} h(x,\xi+\delta_x)\prod_{l=k}^{n-1}\frac{\Xi_{l+1}(\varrho,\xi)}{\Xi_{l+1}(\varrho,\xi+\delta_x)}\ \pi(\varrho,\xi_{X^n}; dx)dP.$$

Changing slightly the parameter l and observing that Ξ_l depends only on that part of ξ which lies in the 'past' X^{l-1}, whereas δ_x lies in the 'future' X_k, one has

$$\mathbb{C}_\xi(h) = \int \int_{X_k} h(x, \xi + \delta_x) \prod_{l=0}^{n} \frac{\Xi_l(\varrho, \xi)}{\Xi_l(\varrho, \xi + \delta_x)} \; \pi(\zeta, \xi_{X^n}; dx) dP,$$

where in this equation ξ_{X^n} can be replaced by ξ because of the special choice of φ and the finite range property of π.

The finite range property is needed again and will enable us to replace the finite by the infinite product. Consider the terms of the infinite product

$$\prod_{l=0}^{\infty} \frac{\Xi_l(\varrho, \xi)}{\Xi_l(\varrho, \xi + \delta_x)}.$$

Here the assumption of *finite range* implies that only finitely many terms of the product can be different from 1. This shows the first part of the theorem for the special class of variables h. Standard arguments then complete the proof of this part. That $\tilde{\pi}$ is a random Radon measure is obvious. □

Some consequences of the theorem. As a first consequence we obtain for the distribution of ξ with a general initial condition P_0 that $P_\pi = \int_{\mathcal{M}} P_\pi^\varrho \; P_0(d\varrho)$ is a solution of the equation

$$\mathbb{C}_{P_\pi}(h) = \int_{\mathcal{M}} \int_{\mathcal{M}^{\cdot\cdot}} \int_X h(x, \eta + \delta_x) \; \tilde{\pi}(\varrho, \eta; dx) P_\pi^\varrho(d\eta) P_0(d\varrho),$$

where $h \in F_+(X \times \mathcal{M}^{\cdot\cdot}(X))$.

Furthermore, the above theorem gives a construction of a very large class of Papangelou processes. We first discuss the special case where the infinite product appearing in $\tilde{\pi}$ is identically 1: If the normalizing constants $\Xi_l(\varrho, \xi)$ do not change if ξ is locally modified (i.e. if they are tail-measurable) then the infinite product is 1. A sufficient condition for this to hold is for instance the tail-measurability of π.

Consider the following additional condition on the kernel π: For a given ϱ the kernel π *has zero range in the strict sense* if

$$Res_{B^m} \pi^{(m)}(\varrho, (.); dy), B \in \mathcal{B}_0, m \geq 1, \text{ are measurable with respect to } \mathcal{F}_{\ddot{B}}.$$

In this case π is dominated with density $f \equiv 1$, and has range $R = 0$. Moreover, the normalizing constants Ξ_n depend only on ϱ but not on η. As a consequence, if π satisfies this condition and is locally integrable then ξ is a Papangelou process for the kernel π. Moreover, by construction, ξ has independent increments, and the

distribution of the field variables $\xi_B, B \in \mathcal{B}_0$, is given by

$$P_\pi^\varrho(\xi_B = k) = \frac{1}{\Xi_B(\varrho)} \frac{1}{k!} \pi^{(k)}(\varrho, B^k),$$

where we skipped η because there is no dependence on it. This raises the question for which class of kernels π with strictly positive range the above construction leads to Papangelou processes with kernel π, so that $p_\varrho \equiv 1$.
Consider now a kernel π which satisfies the assumptions of the theorem and does not depend on ϱ for simplicity. We are interested in the kernel

$$\gamma(\eta, dx) = V(x, \eta) \cdot \pi(\eta, dx), \eta \in \mathcal{M}^{\cdot\cdot}(X).$$

We shall make several assumptions on V which will assure the existence of Papangelou processes $P_{\tilde{\gamma}}$ for the modified kernel $\tilde{\gamma}$. This will be our model for interacting Bosons. $V(x, \eta)$ is to be understood as a *Boltzmann factor* $\exp(-E(x, \eta))$, where $E(x, \eta)$ denotes the energy of a particle in x, given the configuration η. (Usually E is defined by means of some potential.) The first assumption on V is that $\gamma(\eta, .)$ is always a Radon measure. Moreover we need a symmetry condition:

$$V(x, \eta) \cdot V(y, \eta + \delta_x) = V(y, \eta) \cdot V(x, \eta + \delta_y) \text{ for any x,y, } \eta. \tag{3.3}$$

The next is a finite-range property:
Denoting $B_r(x)$ the ball centered in x with radius r, this means

$$V(x, \eta) = V(x, \eta_{B_r(x)}), x \in X, \text{ for some } r \geq 0. \tag{3.4}$$

The following condition will guaranty that the Boltzmann kernel γ is dominated by some symmetric density: There exists a symmetric function $U \in F_+(X \times X)$ such that

$$V(x, \eta + \delta_y) = V(x, \eta) \cdot U(x, y), \text{ for all } x, y, \eta. \tag{3.5}$$

In applications U is given by the exponential of the negative of a pairpotential.
Finally we are looking for a sufficient condition on V which implies local integrability of γ. We assume that there exists for any configuration η some constant $0 < C(\eta)$ such that for any $m \geq 1$ and any $x_1, \ldots, x_m \in X$

$$V(x_1, \eta)V(x_2, \eta + \delta_{x_1}) \ldots V(x_m, \eta + \delta_{x_1} + \cdots + \delta_{x_{m-1}}) \leq \frac{C(\eta)^m}{m!}. \tag{3.6}$$

In this case

$$\gamma^{(m)}(\eta_{X^{n-1}}, X_n^m) \leq C(\eta)^m \frac{1}{m!} \pi^{(m)}(\eta_{X^{n-1}}, X_n^m),$$

and thus local integrability of π implies the one of γ. We remark that the stability condition (3.6) is a version of *superstability.*

Given such a pair (π, V) there exists a Papangelou process for the modified kernel $\tilde{\gamma}$.

We now comment the case of simple Papangelou processes. If π is *simple* in the sense that π is a kernel from $\mathcal{M}_\cdot(X)$ to X satisfying the condition

$$\pi(\eta, supp\, \eta) = 0 \text{ for any } \eta \in \mathcal{M}_\cdot,$$

then a Papangelou process for the kernel π is simple too. (See [7] e.g..) It follows that a Papangelou process for the above kernel $\tilde{\gamma}$ then is simple too. Thus we have a method which allows to construct a large class of simple Papangelou processes.

The above theorem induces several other problems and questions. Under which conditions on π the associated Papangelou process is of first order or uniquely determined? Under which additional condition on π the above construction leads to some Papangelou process for π and not for $\tilde{\pi}$? These questions can been seen also as follows: We explained already in the introduction that the collection of Papangelou processes for a given kernel π coincides with the collection of abstract Gibbs states (in the sense of Preston [11]) for the local specification $\mathcal{V}_\pi$ induced by π in a natural way. And for this collection some of these questions have been analyzed in [11]. As has been remarked already here one can find also a general construction method for Gibbs states specified by $\mathcal{V}_\pi$ and thereby for Papangelou processes with kernel π.

The above construction shows that Papangelou processes are processes in *spacetime* which have an infinitely long memory. The underlying symmetry given by the cocycle condition is much stronger then reversibility in time. Thus there is a new underlying spacetime structure here which has to be developped further.

4. Examples

We discuss some applications of the theorem. We don't repeat here the example of Gibbs processes for classical interacting systems which can be found in [16]. Instead we discuss the analog constructions of interacting Bosons and Fermions which seem to be new. In particular the Polya difference process or ideal Fermi process is considered here for the first time.

Cox processes. We include this well known class of processes to make explicit the appearance of the parameter ϱ in the above setting. Cox processes P are of the form $P = \int_{\mathcal{M}} P_\kappa W(d\kappa)$, where W is a random measure in X, i.e. a probability on $\mathcal{M}$.

It is well known (see [15]) that any Cox process is a Papangelou process for a kernel having the structure $\pi(\mu, dx) = P^{\mu}(d\varrho)\,\varrho(dx)$. Here $(P^{\mu})_{\mu\in\mathcal{M}^{\cdot\cdot}}$ is a Markovian kernel from $\mathcal{M}^{\cdot\cdot}$ to $\mathcal{M}$, i.e. a measurable family of random measures on X indexed by configurations μ. Thus $\pi(\mu, .)$ is given by the intensity measure of P^{μ} which we assume to be always Radonian. A deep result of Wakolbinger [15] is that the following converse is true: If P is a Papangelou process whose kernel is tail-field measurable then P can be represented as a Cox process. Thus we see that a tail-field measurable Papangelou kernel π can always be represented as $\pi(\mu, .) = P^{\mu}(d\varrho)\varrho(.)$ with a tail-field measurable family of random measures $P^{\mu}, \mu \in \mathcal{M}^{\cdot\cdot}$.

In this case the above theorem yields a construction of Cox processes which in fact is well known. To be more precise: Consider the kernel $\pi(\varrho, \eta; .)$, which is tail-field measurable with respect to η, together with a random measure $P_0(d\varrho)$. Then the normalizing constants $\Xi_n(\varrho)$ do not depend on η, and it is obvious that the conditions of the theorem are satisfied, so that, for a given ϱ, P^{ϱ}_{π} is a Papangelou process with kernel $\pi(\varrho, 0)$ and therefore by Mecke's characterization a Poisson process with intensity measure $\pi(\varrho, 0)$. As a consequence $P_{\pi} = \int P_{\pi(\varrho,0)}\, P_0(d\varrho)$ is a Cox process with random intensity measure $\pi(\varrho, 0)$.

Polya sum processes and the ideal Bose process. Here we consider Polya sum processes, which had been introduced in [16]. For the convenience of the reader we repeat here their construction and some of their properties. They describe the ideal Bose process of quantum mechanics (cf. [1]).

Let $0 < z < 1$ and ϱ be some fixed Radon measure on X. Consider the kernel $\pi_+(\varrho, \eta; .) = z(\varrho + \eta)$. (For simplicity we skip ϱ in the sequel.) In this case local integrability holds true with a normalizing constant independent of η:

$$\Xi_n(\varrho) = \exp(\varrho(X_n)\kappa(z)) = \frac{1}{(1-z)^{\varrho(X_n)}},$$

where $\kappa(z) = \sum_{j\geq 1} \frac{z^j}{j}$. Here we used the fact that

$$\sum_{m\geq 0} \frac{1}{m!}\varrho(X_n)^{[m]} z^m = \exp(\varrho(X_n)\kappa(z)) = \frac{1}{(1-z)^{\varrho(X_n)}}$$

with $a^{[m]} = a(a+1)\cdots(a+m-1)$. It is evident that the other assumptions of the theorem are also satisfied. In particular π_+ has range 0 in the strict sense. The corresponding Papangelou process $P_+ = P^{\varrho}_{\pi_+}$ was called in [16] the Polya sum process

specified by (z, ϱ). But from the point of view of quantum mechanics this is the *ideal Bose process specified by* (z, ϱ).

Using that P_+ is a Papangelou process for π_+ we show directly that this process has independent increments, we deduce the distribution of the field variables ξ_B, show that P_+ is of first order (i.e. all ξ_B are integrable) and finally that P_+ is uniquely determined by the kernel π_+.

Let $B \in \mathcal{B}_0$ and $k \geq 1$ and consider a non-negative measurable function φ on the space of configurations which depends only on what happens outside B. Thus φ is measurable with respect to $\mathcal{F}_{\ddot{B}^c}$. Then using its character as a Papangelou process we obtain the recursion

$$\begin{aligned} P_+(1_{\{\xi_B=k\}} \cdot \varphi) &= \frac{1}{k} \int 1_B(x) 1_{\{\xi_B=k\}}(\eta) \varphi(\eta) \eta(dx) P_+(d\eta) \\ &= \frac{z}{k} \int 1_B(x) 1_{\{\xi_B=k-1\}}(\eta) \varphi(\eta) (\varrho + \eta)(dx) P_+(d\eta) \\ &= \frac{z}{k} \big[\varrho(B) + (k-1)\big] P_+(1_{\{\xi_B=k-1\}} \cdot \varphi). \end{aligned}$$

Iterating this process until B is void of particles yields

$$P_+(1_{\{\xi_B=k\}} \cdot \varphi) = \frac{z^k}{k!} \varrho(B)^{[k]} P_+(1_{\{\xi_B=0\}} \cdot \varphi).$$

Choosing now for φ the indicator of the event $\{\xi_{B_2} = k_2, \ldots, \xi_{B_n} = k_n\}$, where $k_j \geq 1$ and $B_1 = B, B_2, \ldots, B_n \in \mathcal{B}_0$ are pairwise disjoint, and iterating the above procedure we obtain that for $k_1, \ldots, k_n \geq 0$

$$P_+\big(\xi_{B_1} = k_1, \ldots, \xi_{B_n} = k_n\big) = \prod_{j=1}^{n} \frac{z^{k_j}}{k_j!} \varrho(B_j)^{[k_j]} \cdot P_+\big(\xi_{B_1} = 0, \ldots, \xi_{B_n} = 0\big).$$

This equation determines $P_+\big(\xi_{B_1} = 0, \ldots, \xi_{B_n} = 0\big)$ as a product of terms which depend only on a single k_j. Thus P_+ has independent increments.

The distribution of $\xi_B, B \in \mathcal{B}_0$, amounts to

$$P_+\{\xi_B = k\} = \exp(-\varrho(B)\kappa(z)) \cdot z^k \cdot \frac{\varrho(B)^{[k]}}{k!}, k \geq 0.$$

If $\varrho(B)$ is an integer this is a negative binomial distribution. From this we obtain that P_+ is of first order and

$$P_+(\xi_B) = \frac{z}{1-z} \cdot \varrho(B), B \in \mathcal{B}_0.$$

The above considerations used only the partial integration formula, for which P_+ is a solution. Thus the finite-dimensional distributions of the random field $(\xi_B)_{B \in \mathcal{B}_0}$ and thereby P_+ is completely determined by π_+.

We finally add an important observation of Mathias Rafler that P_+ is infinitely divisible. (See [12], Prop. 6.7 .) We thus see that the Polya sum process resp. the ideal Bose process P_+ for (z, ϱ) has analog properties as the Poisson process and thus has the character of an ideal gas.

The interacting Bose process. Here we propose a theory of interacting Bosons in analogy to the Gibbsian theory of classical interacting particles. The idea is to replace the Poisson process by the Polya sum process and to build by means of a given potential the interacting system as in the classical case . To be more precise, we replace the (conditional) intensity ϱ of the Poisson process by the conditional intensities of the Polya processes.
Let $V(x, \eta)$ be a non-negative, measurable function on $X \times \mathcal{M}^{\cdot\cdot}(X)$ satisfying the conditions (3.3) – (3.6); furthermore, let $0 < z < 1$ and $\varrho \in \mathcal{M}(X)$. Denote by $\pi_+(\eta, .) = z(\varrho + \eta)$ the Polya sum kernel for (z, ϱ) from above. We then consider the kernel

$$\gamma(\eta, dx) = V(x, \eta) \cdot \pi_+(\eta, dx), \eta \in \mathcal{M}^{\cdot\cdot}(X).$$

As a consequence of the considerations above we know the existence of a Papangelou process for the modified kernel $\tilde{\gamma} = p_\varrho \cdot V \cdot \pi_+$ and call it the *Bosonic state for* (ϱ, V). In the case where V (and thereby C) does not depend on η, say $V(x, \eta) \equiv f(x)$, we obtain a Papangelou process for $\gamma(\eta, .) = f \cdot z(\varrho + \eta)$. (Thus the infinite product is trivial here; f is playing the role of the exponential of a self-potential.) This state differs from the above Polya sum process in that the constant z may depend on the position x. We call this process also the *ideal Bose process for* (f, z, ϱ). It has the same properties as the former Polya sum process. A quantum mechanical derivation of this model can be found in [1].

The Polya difference process and the ideal Fermi process. Here we study the Papangelou process which describes Fermions and which we call the Polya difference process or ideal Fermi process. This notion has been foreshadowed already in [1].
For simplicity we assume first that the underlying space X is countably infinite or even finite. The measure ϱ is the counting measure on X. Thus $\varrho \equiv 1$. Again $0 < z < 1$ is a given parameter. We then consider the *Polya difference kernel for* z defined by

$$\pi_-(\eta, .) = z \cdot (\varrho - \eta), \eta \in \mathcal{M}^{\cdot}(X).$$

Note that this kernel is a kernel from $\mathcal{M}^{\cdot}(X)$, the collection of simple point measures, to X, and thus well-defined as a Radon measure in the second variable. Moreover, $\pi_{-}(\eta, supp\, \eta) = 0$, and thus a Papangelou process for π is simple. Again local integrability holds true with a normalizing constant

$$\Xi_n(\varrho) = \sum_{m \geq 0} \frac{1}{m!} \varrho(X_n)_{[m]} z^m = \big(1+z\big)^{\varrho(X_n)},$$

which again does not depend on η. Note that here appears now the symbol $a_{[m]} = a(a-1)\cdots(a-m+1), a \in \mathbb{R}$.

Again we are in the situation of the theorem. We call the corresponding Papangelou process $P_{-} = P_{\pi_{-}}$ the *Polya difference process for* z. From the point of view of quantum mechanics it is the *ideal Fermi process for* z. (For a quantum mechanical explication we refer again to [1].) By construction this process is simple, i.e. respects the Pauli exclusion property, and has independent increments. Exactly as above, by using only its character as a Papangelou process for π_{-}, the field variables $\xi_B, B \in \mathcal{B}_0$, have the following distribution:

$$P_{-}\{\xi_B = k\} = \frac{1}{Z_{-}(B)} \cdot z^k \cdot \frac{\varrho(B)_{[k]}}{k!}, k \geq 0,$$

where $Z_{-}(B)$ denotes the normalizing constant. In case that $\varrho(B)$ is an integer this is a binomial distribution.

From this we obtain that P_{-} is of first order and

$$P_{-}(\xi_B) = \frac{z}{1+z} \cdot \varrho(B), B \in \mathcal{B}_0.$$

Again P_{-} is completely determined by π_{-}. Therefore also P_{-} has the character of an ideal gas. We indicate shortly how one can define difference kernels on abstract spaces X and how to develop the corresponding interacting theory.

Let X denote a Polish space and fix some Radon measure ϱ on it. Consider a (random) element ζ in $\mathcal{M}^{\cdot\cdot}(X)$, i.e. a point process in X. Given some Radon measure ϱ and a parameter $z > 0$ consider the following kernel called again Polya difference kernel

$$\pi_{-}(\zeta, \eta; .) = z \cdot \varrho + (\zeta - \eta) \cdot 1_{\mathcal{M}^{\cdot\cdot}(\zeta)}(\eta),$$

where $\mathcal{M}^{\cdot\cdot}(\zeta) = \{\eta \in \mathcal{M}^{\cdot\cdot}(X) | \eta \leq \zeta\}$. Here $\eta \leq \zeta$ means that η is a subconfiguration of ζ and therefore the difference $\zeta - \eta$ a well defined Radon point measure. The corresponding Papangelou process, called again Polya difference process, exists and is uniquely determined by π_{-} and the distribution of ζ. For a deterministic ζ it has

independent increments and is of first order. This model seems to be interesting not only for quantum mechanical applications.

We finally remark that in the case of simple Papangelou processes the structure of the *discrete part* of the Papangelou kernel π has been analysed already in complete generality by Kallenberg [4] in Theorem 3.1 . This discrete part π_a has the following structure:

$$\pi_a(\mu, dx) = V_a(x,\mu)\cdot\big(1-\mu\big)(dx), x \in X, \mu \in \mathcal{M}_\cdot(X), \text{where} \tag{4.1}$$

$V_a(x,\mu) = \frac{\tau(x,\mu)}{1-\tau(x,\mu)}$ and τ is intuitively given by $\tau(x,.) = P\big(\xi_x = 1|\xi_{\{x\}^c}\big)(.)$. (For more details we refer to the following scholion.)

The atomic part of the kernel for an *ideal* Fermi gas has the form where V_a does not depend on μ; and in this case the corresponding process $(\xi_x)_x$ has independent increments. Important examples are given by $V_a = \varrho$ with ϱ a probability on X which is not a Dirac measure; or $V_a = \frac{\varrho}{1-\varrho}$. A very special modification of the second case which takes into account also interactions with respect to the particles in μ then leads to so called determinantal processes. For details we refer to Shirai/Takahashi [14].

Scholion: The probabilistic structure of Papangelou kernels. For the convenience of the reader we add a rigorous derivation of the probabilistic structure of a Papangelou kernel in the discrete setting. The general theory can be found in [3, 4, 9]. The situation now is elementary: X is finite and $\pi(\mu, x), x \in X, \mu \in \mathcal{M}^{\cdot\cdot}$, is a kernel. We consider *Papangelou processes* P *for* π, i.e. point processes P in X solving the equations

$$\mathbb{C}_P(h) = \sum_x \sum_\mu h(x, \mu+\delta_x)\pi(\mu, x)P(\mu), h \in F_+,$$

and ask for the meaning of this condition in terms of the corresponding random field $(\xi_x)_{x\in X}$. Recall that the Campbell measure $\mathbb{C}_P$ is concentrated on $C = \{(x,\mu)|\mu(x) \geq 1\}$.

The above equation is equivalent to

$$\mu(x)P(\mu) = \pi(\mu-\delta_x, x)P(\mu-\delta_x) \text{ for any } x, \mu,\ \mu(x) \geq 1,\ i.e.\ \mu \in \mathcal{M}^{\cdot\cdot}. \tag{4.2}$$

This is equivalent to saying

$$(\mu+\delta_x)(x)P(\mu+\delta_x) = \pi(\mu, x)P(\mu), x \in X, \mu \in \mathcal{M}^{\cdot\cdot}. \tag{4.3}$$

Note that $\pi(\mu, x) > 0$ iff $\big(P(\mu + \delta_x) > 0 \Longleftrightarrow P(\mu) > 0\big)$. We observe that in the case of simple point processes P condition (4.2) is equivalent to saying

$$(1 - \mu)(x)P(\mu + \delta_x) = \pi(\mu, x)P(\mu), x \in X, \mu \in \mathcal{M}^{\cdot}. \tag{4.4}$$

In this case π can always be chosen as a kernel $\pi : X \times \mathcal{M}^{\cdot} \longrightarrow \bar{\mathbb{R}}_+$ satisfying $\pi(\mu, x) = 0$ if $\mu(x) \geq 1$. Thus $\pi(\mu, .)$ is supported by the complement of the support of μ. (4.2) means that π is a *local specification of the point process* P in the sense that

$$\pi(\mu, x) = (1 + \mu)(x) \cdot \frac{P(\mu + \delta_x)}{P(\mu)} \text{ for all } x\, P - a.s.[\mu].$$

Note that one can choose always a version V for the quotient on the right hand side which is defined everywhere. Thus the kernel π has the following structure:

$$\pi(\mu, x) = (1 + \mu)(x) \cdot V(x, \mu), x \in X, \mu \in \mathcal{M}^{\cdot\cdot}, \tag{4.5}$$

where $V : X \times \mathcal{M}^{\cdot\cdot} \longrightarrow \bar{\mathbb{R}}_+$.

Interpretation of π in terms of the random field $\xi_x, x \in X$: Condition (4.2) can be expressed as follows: For all x and all μ

$$(1 + \mu)(x)P\big(\xi_x = 1 + \mu(x), \xi_{\{x\}^c} = \mu_{\{x\}^c}\big) = \pi(\mu, x) \cdot P\big(\xi_x = \mu(x), \xi_{\{x\}^c} = \mu_{\{x\}^c}\big).$$

In the case of simple point processes P this condition reduces to: For all x, μ

$$(1 - \mu)(x)P\big(\xi_x = 1, \xi_{\{x\}^c} = \mu_{\{x\}^c}\big) = \pi(\mu, x) \cdot P\big(\xi_x = 0, \xi_{\{x\}^c} = \mu_{\{x\}^c}\big);$$

or equivalently for all x, μ with $P\big(\xi_{\{x\}^c} = \mu_{\{x\}^c}\big) > 0$

$$(1 - \mu)(x)P\big(\xi_x = 1 | \xi_{\{x\}^c} = \mu_{\{x\}^c}\big) = \pi(\mu, x) \cdot P\big(\xi_x = 0 | \xi_{\{x\}^c} = \mu_{\{x\}^c}\big).$$

Setting $q_x = P\big(\xi_x = 1 | \xi_{\{x\}^c}\big)$ this means that the process P satisfies the condition

$$\big(\Sigma\big) \qquad \big(q_x > 0 \Longrightarrow 1 - q_x > 0\big)\, P - a.s., x \in X,$$

and thereby $P\big(\xi_{\{x\}^c} = (.)\big) > 0$, and that π is given by

$$\pi(\mu, x) = \big(1 - \mu\big)(x) \cdot \frac{q_x(\mu)}{1 - q_x(\mu)}, x \in X,\ P - a.s.[\mu]. \tag{4.6}$$

Thus in the simple case we have: P is a Papangelou process for π iff condition (Σ) is true; and in this case π satisfies (4.6). This is a well known representation of the Papangelou kernel by Kallenberg [4], theorem 3.1., for simple processes in this elementary context. Note that for the right hand side of (4.6) one can choose always a version $V(x, \mu)$ which depends only on $\mu_{\{x\}^c}$ if x is given. Thus for simple Papangelou processes P the kernel π has the following structure

$$\pi(\mu, x) = \big(1 - \mu\big)(x) \cdot V(x, \mu), x \in X, \mu \in \mathcal{M}^{\cdot}, \tag{4.7}$$

where $V : X \times \mathcal{M}^{\cdot} \longrightarrow \bar{\mathbb{R}}_+$ is *exvisible*, i.e. is measurable with respect to the $\sigma-$field $\mathcal{Z}$ in $X \times \mathcal{M}^{\cdot}$ generated by all sets of the form $\{x\} \times N$, where $x \in X$ and $N \in \mathcal{F}_{\{x\}^c}$.

To summarize we have the following situation: If P is a Papangelou process for a given kernel $\pi : X \times \mathcal{M}^{\cdot\cdot} \longrightarrow \bar{\mathbb{R}}_+$, then π can be represented as

$$\pi(\mu, x) = \big(1 + \mu\big)(x) \cdot V(x, \mu), x \in X, \mu \in \mathcal{M}^{\cdot\cdot}, \tag{4.8}$$

where $V : X \times \mathcal{M}^{\cdot} \longrightarrow \bar{\mathbb{R}}_+$.

Let P be a simple Papangelou process for a given kernel $\pi : X \times \mathcal{M}^{\cdot} \longrightarrow \bar{\mathbb{R}}_+$. Then π can be represented as

$$\pi(\mu, x) = \big(1 - \mu\big)(x) \cdot V(x, \mu), x \in X, \mu \in \mathcal{M}^{\cdot}, \tag{4.9}$$

where V is exvisible. We observe here that this implies that $\pi(., x)$ is the conditional intensity of ξ in the sense that for any x

$$\pi(., x) = P\big(\xi_x | \xi_{\{x\}^c}\big) \ Pa.s..$$

We use this terminology also in the non-simple case though this interpretation is no longer valid.

Thus we now know the general structure of a Papangelou kernel in the discrete case which remains true in the general case (cf. [4]).

If the random point field $(\xi_x)_x$ is independent (or has independent increments) then V can be chosen in such a way that it depends only on x; if the field is even identically distributed then V is a constant. In these cases one can speak of an ideal gas.

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