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ON THE INTEGRAL OF MEAN CURVATURE OF A FLATTENED CONVEX BODY IN SPACE FORMS

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Abstract. Under the assumptions that E_λ^n is an n-dimensional, simply connected Riemannian manifold of constant sectional curvature λ and L_λ^r is an r-dimensional, totally geodesic submanifold of E_λ^n, the paper investigates the q-th integral of the mean curvature M_q^n of a convex body K^r in E_λ^n and gives the expression of M_q^n in the terms of M_p^r, where M_p^r is the p-th integral of the mean curvature of K^r in L_λ^r. A result of L. A. Santaló holds in particular.

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1. Introduction

Let n be a natural number, let $0 \le r < n$, $0 \le p \le r-1$ and let $0 \le q \le n-1$. Further, let E_λ^n be an n-dimensional, simply connected Riemannian manifold of constant sectional curvature λ, i.e. the sphere space S^n for $\lambda > 0$, the hyperbolic space H^n for $\lambda < 0$ and the Euclidean space E^n for $\lambda = 0$. Besides, let L_λ^r be an r-dimensional, totally geodesic subspace of E_λ^n and let $K^r \subset L_\lambda^r$ be a convex body. Then, the boundary ∂K^r of K^r is an $(r-1)$-dimensional hypersurface in L_λ^r. Assuming that P is a point of ∂K^r, we choose $e_1, \dots, e_{r-1}$ to be the principal curvature directions at the point P. Further, we suppose that $\kappa_1, \dots, \kappa_{r-1}$ are the principal curvatures at the point P, which correspond to the principal curvature directions.

Consider the Gauss map $G : P \to N(P)$, whose differential

$$dG_P(e_i) = x'(t) \to N'(t) \quad (x(0) = P)$$

satisfies the Rodrigues' equations

$$dG_P(e_i) = -\kappa_i e_i, \quad i = 1, \cdots, r-1.$$

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Then we have the mean curvature

$$H = \frac{1}{r-1}(\kappa_1 + \cdots + \kappa_{r-1}) = -\frac{1}{r-1}trace(dG_P),$$

along with the Gauss-Kronecker curvature

$$K = \kappa_1 \cdots \kappa_{r-1} = (-1)^{r-1}det(dG_P).$$

The p-th order mean curvatures are the p-th order elementary symmetric functions of the principal curvatures. By H_q we denote the p-th order mean curvature normalized such that

$$\prod_{p=1}^{r-1}(1+t\kappa_p) = \sum_{q=0}^{r-1}\binom{r-1}{p} H_p t^p.$$

Thus, $H_1 = H$ is the mean curvature and H_{r-1} is the Gauss-Kronecker curvature K. The p-th $(0 < p \le r-1)$ integral of the mean curvature M_p^r of ∂K^r at P is defined by

$$M_p^r(\partial K^r) = \int_{\partial K^r} H_p d\sigma_{r-1} = \binom{r-1}{p}^{-1}\int_{\partial K^r}\left\{\kappa_{i_1}, \cdots, \kappa_{i_p}\right\} d\sigma_{r-1},$$

where $\{\kappa_{i_1}, \ldots, \kappa_{i_p}\}$ denotes the p-th elementary symmetric function of the principal curvatures and $d\sigma_{r-1}$ is the area element of ∂K^r. As a particular case, let $M_0^r = \sigma_{r-1}$ be the area of ∂K^r, for completeness. Moreover, we have $M_{r-1}^r = O_{r-1}$, where O_{r-1} denotes the area of the $(r-1)$-dimensional unit sphere and its value is given by the formula

$$O_{r-1} = \frac{2\pi^{r/2}}{\Gamma(r/2)}.$$

For instance, if $\lambda = 0$ and $r = 2$, and K^2 is a plane convex figure in E^2, then $M_0^2 = \sigma_1$ and $M_1^2 = 2\pi$. If $r = 3$ and K^3 is a convex body in E^3, then $M_0^3 = \sigma_2$ and M_1^3 is the integral of mean curvature of ∂K^3. For more details, see [3, 4].

If $K^r \subset L_\lambda^r$ is a convex body, then it can be considered as a flattened convex body of E_λ^n $(n \ge r)$. In order to define the q-th integral of the mean curvature M_q^n $(q = 1, 2, \ldots, n-1)$ of ∂K^r in E_λ^n, we consider the outer parallel convex body $K_\varepsilon^r := \{x \in E^n : d(K^r, x) \le \varepsilon\}$ of K^r in E_λ^n, where the $d(K^r, \cdot)$ denotes the geodesic distance from K^r in E_λ^n, and then we pass to the limit as $\varepsilon \to +0$.

In this paper, we investigate the q-th integral of the mean curvature M_q^n of ∂K^r in E_λ^n, where K^r is considered as a flattened convex body in E_λ^n. We obtain an expression of M_q^n in terms of the integral of the mean curvature M_p^r. Besides, we prove the following theorem.

Theorem 1.1. *Let E^n_λ be a simply connected Riemannian manifold of the sectional curvature λ, let L^r_λ be the totally geodesic submanifold of E^n_λ, and let $K^r \subset L^r_\lambda$ be a convex body of dimension r with C^2 boundary.*

Then the q-th integral of the mean curvatures M^n_q of K^r, where K^r is considered as a flattened convex body in E^n_λ, satisfy the following conditions, where the quantity $\overline{s}(p, q)$ is that defined later, by (3.1):

1) *If $q > n - r - 1$, then*

$$M^n_q(\partial K^r) = \sum_{m=1}^{n-r} \frac{\binom{r-1}{q-m}}{\binom{n-1}{q}} \overline{s}(m,\, n-r) \frac{O_{n-r+q-m}}{O_{q-m}} M^r_{q-m}(\partial K^r). \tag{1.1}$$

2) *If $q = n - r - 1$, then*

$$\begin{aligned} M^n_q(\partial K^r) &= \binom{n-1}{q}^{-1} \overline{s}(q,\, q) O_q \sigma_r(K^r) \\ &+ \sum_{m=1}^{n-r-1} \frac{\binom{r-1}{q-m}}{\binom{n-1}{q}} \overline{s}(m,\, n-r) \frac{O_{n-r+q-m}}{O_{q-m}} M^r_{q-m}(\partial K^r). \end{aligned} \tag{1.2}$$

3) *If $q < n - r - 1$, then*

$$\begin{aligned} M^n_q(\partial K^r) &= \frac{\binom{n-r-1}{q}}{\binom{n-1}{q}} \overline{s}(q,\, n-r-1) O_{n-r-1} \sigma_r(K^r) \\ &+ \sum_{m=1}^{q} \frac{\binom{r-1}{q-m}}{\binom{n-1}{q}} \overline{s}(m,\, n-r) \frac{O_{n-r+q-m}}{O_{q-m}} M^r_{q-m}(\partial K^r). \end{aligned} \tag{1.3}$$

Especially, when $\lambda = 0$, that is, the convex body in Euclidean space E^n. In this case, if $m < q$, then $\overline{s}(m, q) = \varepsilon^{q-m} = 0$ as $\varepsilon \to 0$. Theorem 1.1 reduces to the below corollary proved by L. A. Santaló in 1957 (see [2, 3]). Note that the results of [2, 3] play an important role in integral geometry and differential geometry and are widely used (see [1, 3, 4, 5]).

Corollary 1.1. *Let E^n denote the Euclidean space, L_r denote the r-dimensional linear subspace in E^n, let K^r be a convex body of the dimension r in L^r. Then K^r can be considered both as a convex body in L^r and as a flattened convex body in E^n, and the q-th integral of mean of the curvature M^n_q satisfies the conditions*

1) *If* $q > n-r-1$, *then*

$$M_q^n(\partial K^r) = \frac{\binom{r-1}{q-n+r}}{\binom{n-1}{q}} \frac{O_q}{O_{q-n+r}} M_{q-n+r}^r(\partial K^r),$$

2) *If* $q = n-r-1$, *then*

$$M_q^n(\partial K^r) = \binom{n-1}{n-r-1}^{-1} O_{n-r-1}\sigma_r(K^r),$$

3) *If* $q < n-r-1$, *then*

$$M_q^n(\partial K^r) = 0.$$

2. Preliminaries

Let $K^r \subset L_\lambda^r$ and let L_λ^r be a totally geodesic submanifold of E_λ^n. Assuming that ∂K_ε^r is a twice differentiable hypersurface with a well defined normal at each point $P' \in \partial K_\varepsilon^r$, by $N(P')$ we denote the normal vector at the point P'. Let P be the intersection point of the normal $N(P')$ with K^r. Then, we say that P' belongs to the region (A) of ∂K_ε^r if P belongs to K^r, besides, we say that P' belongs to the region (B) of ∂K_ε^r if P belongs to ∂K^r.

In later sections we use the following notation:

$$s_\lambda(r) = \begin{cases} \lambda^{-1/2}\sin(r\sqrt{\lambda}), & \lambda > 0, \\ r, & \lambda = 0, \\ |\lambda|^{-1/2}\sinh(r\sqrt{|\lambda|}), & \lambda < 0. \end{cases}$$

By the moving frames introduced in the book of L. A. Santaló [3], the area elements at a point $P' \in \partial K_\varepsilon^r$ with respect to the regions (A) and (B), respectively, are

$$d\sigma_{n-1} = s_\lambda(\varepsilon)^{n-r-1} du_{n-r-1} \wedge d\sigma_r,$$

$$d\sigma_{n-1} = s_\lambda(\varepsilon)^{n-r} du_{n-r} \wedge d\sigma_{r-1},$$

where du_{n-r} denotes the area element of the $(n-r)$-dimensional unit sphere, $d\sigma_r$ and $d\sigma_{r-1}$ denote the volume element of K^r and the element of area of ∂K^r, respectively. Then the q-th integral of the mean curvature of ∂K_ε^r is given by the formula

$$\begin{aligned} (2.1) \qquad M_q^n(\partial K_\varepsilon^r) &= \binom{n-1}{q}^{-1} \int_{\partial K_\varepsilon^r} \left\{\kappa_{i_1}, \cdots, \kappa_{i_q}\right\} d\sigma_{n-1} \\ &= \binom{n-1}{q}^{-1} \left[\int_{K^r} \left\{\kappa_{i_1}, \cdots, \kappa_{i_q}\right\} s_\lambda(\varepsilon)^{n-r-1} du_{n-r-1} \wedge d\sigma_r \right. \\ &\quad \left. + \int_{\partial K^r} \left\{\kappa_{i_1}, \cdots, \kappa_{i_q}\right\} s_\lambda(\varepsilon)^{n-r} du_{n-r} \wedge d\sigma_{r-1} \right]. \end{aligned}$$

Let R_i $(i = 1, 2, \ldots, n-1)$ be the principle radii of the curvature of ∂K^r_ε corresponding to $e_1, \ldots, e_{n-1}$, that is $R_i = 1/\kappa_i$. Then (2.1) can be rewritten as

$$M^n_q(\partial K^r_\varepsilon) = \binom{n-1}{q}^{-1} \left[\int_{K^r} \left\{ \frac{1}{R_{i_1}}, \cdots, \frac{1}{R_{i_q}} \right\} s_\lambda(\varepsilon)^{n-r-1} du_{n-r-1} \wedge d\sigma_r + \int_{\partial K^r} \left\{ \frac{1}{R_{i_1}}, \cdots, \frac{1}{R_{i_q}} \right\} s_\lambda(\varepsilon)^{n-r} du_{n-r} \wedge d\sigma_{r-1} \right]. \tag{2.2}$$

Note that the following statements are true for the quantifies R_i $(i = 1, \ldots, n-1)$.

$a)$ For the points of the region (A), clearly,

$$R_h = \varepsilon \quad \text{for} \quad h = 1, 2, \ldots, n-r-1,$$
$$R_h = \infty \quad \text{for} \quad h = n-r, n-r+1, \ldots, n-1.$$

$b)$ In order to find the values of R_i at the points of (B), suppose $e_1, e_2, \ldots, e_n$ are a frame of n orthogonal unit vectors, such that $e_1, \ldots, e_{n-r}$ are constants independent of $x \in \partial K^r$, orthogonal to L^n_r and $e_{n-r+1}, \ldots, e_{n-1}$ are the principal tangent directions of ∂K^r in L^n_r, while e_n is the normal to ∂K^r in L^n_r. A vector X of ∂K^r_ε can be represented as

$$X = x - \varepsilon N,$$

where $x \in K^r$ and

$$N = cos\theta e_n + \sum_{h=1}^{n-r} cos\theta_h e_h. \tag{2.3}$$

For any x, the vector X describes an $(n-r)$-sphere, and consequently

$$R_h = \varepsilon \quad \text{for} \quad h = 1, 2, \ldots, n-r. \tag{2.4}$$

For $h = n-r+1, \ldots, n-1$, the Rodrigues equations give

$$dN \cdot e_h = -\frac{1}{R_h} dS_h, \tag{2.5}$$

where dS_h denotes the arc element on ∂K^r_ε and

$$dS_h = dX \cdot e_h = ds_h - \varepsilon\, dN \cdot e_h, \tag{2.6}$$

where ds_h is tangent to e_h arc element on ∂K^r.

From (2.3), for $h = 1, 2, \ldots, n-r$, for which the vectors e_h are constant, we get

$$dN \cdot e_h = cos\theta\, de_n \cdot e_h = -\frac{cos\theta}{\rho_{h-n+r}} ds_h \quad (h = n-r+1, \ldots, n-1), \tag{2.7}$$

where $\rho_1, \dots \rho_{r-1}$ are the principal radii of the curvature of ∂K^r. Besides, by (2.5), (2.6) and (2.7) we obtain

$$R_h = \frac{\rho_{h-n+r}}{cos\theta} + \varepsilon \quad \text{for} \quad h = n-r+1, \dots, n-1. \tag{2.8}$$

3. Proof of the main result

Now, we are ready to prove the main result of the present paper, which is given in Theorem 1.1. To this end, first we set

$$\overline{s}(p,\, q) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^p} s_\lambda(\varepsilon)^q \tag{3.1}$$

and note another formula, which will be used later:

$$\int_0^{\pi/2} \cos^{q-m} \theta du_{n-r} = \frac{O_{n-r+q-m}}{O_{q-m}}. \tag{3.2}$$

Proof of Theorem 1.1: 1) If $q > n-r-1$, on region (A), since all the principal radii of curvature $R_{i_1}, R_{i_2}, \dots, R_{i_q}$ cannot be chosen from the normal space of K^r. There essentially exist some R_{i_j} which are selected from the tangent space of K^r. Then, the first integral of (2.2) vanishes and formula (2.2) reduces to

$$M_q^n(\partial K_\varepsilon^r) = \binom{n-1}{q}^{-1} \int_{\partial K^r} \left\{ \frac{1}{R_{i_1}}, \cdots, \frac{1}{R_{i_q}} \right\} s_\lambda(\varepsilon)^{n-r} du_{n-r} \wedge d\sigma_{r-1}.$$

On the region (B), if there are m $(1 \le m \le n-r)$ principal radii of curvature R_{i_j} selected from the normal space, by (2.4) and (2.8) we get

$$\left\{ \frac{1}{R_{i_1}}, \cdots, \frac{1}{R_{i_q}} \right\} = \sum_{m=1}^{n-r} \frac{1}{\varepsilon^m} \left\{ \frac{cos\theta}{\rho_1}, \; \cdots, \frac{cos\theta}{\rho_{q-m}} \right\}.$$

Then, using formula (3.2) and the definition of the q-th integral of the mean curvature M_q^r we obtain

$$\begin{aligned} &M_q^n(\partial K_\varepsilon^r) \\ &= \binom{n-1}{q}^{-1} \sum_{m=1}^{n-r} \frac{1}{\varepsilon^m} \int_{\partial K^r} \left\{ \frac{1}{\rho_1}, \cdots, \frac{1}{\rho_{q-m}} \right\} cos^{q-m}\theta \; s_\lambda(\varepsilon)^{n-r} du_{n-r} \wedge d\sigma_{r-1} \\ &= \sum_{m=1}^{n-r} \frac{\binom{r-1}{q-m}}{\binom{n-1}{q}} \frac{1}{\varepsilon^m} s_\lambda(\varepsilon)^{n-r} \frac{O_{n-r+q-m}}{O_{q-m}} M_{q-m}^r(\partial K^r). \end{aligned}$$

Letting here $\varepsilon \to 0$ we come to formula (1.1).

2) If $q = n - r - 1$, on the region (A), note that $R_h = \infty$ if h changes from $\{n - r, \cdots, n-1\}$. Then all R_{i_h} in $\{\frac{1}{R_{i_1}}, \cdots, \frac{1}{R_{i_q}}\}$ must be $R_{i_1} = R_{i_2} = \ldots = R_{i_q} = \varepsilon$, and the first integral of (2.2) reduces to

$$\int_{K^r} \left\{ \frac{1}{R_{i_1}}, \cdots, \frac{1}{R_{i_q}} \right\} s_\lambda(\varepsilon)^{n-r-1} du_{n-r-1} \wedge d\sigma_r$$
$$= \int_{K^r} \frac{1}{\varepsilon^{n-r-1}} s_\lambda(\varepsilon)^{n-r-1} du_{n-r-1} \wedge d\sigma_r = \frac{1}{\varepsilon^q} s_\lambda(\varepsilon)^q O_q \, \sigma_r(K^r).$$

By the same argument as the case 1), the second integral in (2.2) becomes

$$\int_{\partial K^r} \left\{ \frac{1}{R_{i_1}}, \cdots, \frac{1}{R_{i_q}} \right\} s_\lambda(\varepsilon)^{n-r} du_{n-r} \wedge d\sigma_{r-1}$$
$$= \sum_{m=1}^{n-r-1} \int_{\partial K^r} \frac{1}{\varepsilon^m} \left\{ \frac{1}{\rho_1}, \cdots, \frac{1}{\rho_{q-m}} \right\} cos^{q-m}\theta \, s_\lambda(\varepsilon)^{n-r} du_{n-r} \wedge d\sigma_{r-1}$$
$$= \sum_{m=1}^{n-r-1} \binom{r-1}{q-m} \frac{1}{\varepsilon^m} s_\lambda(\varepsilon)^{n-r} \frac{O_{n-r+q-m}}{O_{q-m}} M^r_{q-m}(\partial K^r).$$

Consequently,

$$M^n_q(\partial K^r_\varepsilon) = \binom{n-1}{q}^{-1} \frac{1}{\varepsilon^q} O_q s_\lambda(\varepsilon)^q \sigma_r(K^r)$$
$$+ \sum_{m=1}^{n-r-1} \frac{\binom{r-1}{q-m}}{\binom{n-1}{q\ cr}} \frac{1}{\varepsilon^m} s_\lambda(\varepsilon)^{n-r} \frac{O_{n-r+q-m}}{O_{q-m}} M^r_{q-m}(\partial K^r).$$

Letting here $\varepsilon \to 0$ we obtain (1.2).

3) If $q < n - r - 1$, then similarly the first integral of (2.2) takes the form

$$\int_{K^r} \left\{ \frac{1}{R_{i_1}}, \cdots, \frac{1}{R_{i_q}} \right\} s_\lambda(\varepsilon)^{n-r-1} du_{n-r-1} \wedge d\sigma_r$$
$$= \binom{n-r-1}{q} O_{n-r-1} \frac{1}{\varepsilon^q} s_\lambda(\varepsilon)^{n-r-1} \sigma_r(K^r),$$

while the second integral in (2.2) becomes

$$\int_{\partial K^r} \left\{ \frac{1}{R_{i_1}}, \cdots, \frac{1}{R_{i_q}} \right\} s_\lambda(\varepsilon)^{n-r} du_{n-r} \wedge d\sigma_{r-1}$$
$$= \sum_{m=1}^{q} \int_{\partial K^r} \frac{1}{\varepsilon^m} \left\{ \frac{1}{\rho_1}, \cdots, \frac{1}{\rho_{q-m}} \right\} cos^{q-m}\theta \, s_\lambda(\varepsilon)^{n-r} du_{n-r} \wedge d\sigma_{r-1}$$
$$= \sum_{m=1}^{q} \binom{r-1}{q-m} \binom{r-1}{q-m} \frac{1}{\varepsilon^m} s_\lambda(\varepsilon)^{n-r} \frac{O_{n-r+q-m}}{O_{q-m}} M^r_{q-m}(\partial K^r).$$

Thus,

$$M_q^n(\partial K_\varepsilon^r) = \frac{\binom{n-r-1}{q}}{\binom{n-1}{q}} \frac{1}{\varepsilon^q} s_\lambda(\varepsilon)^{n-r-1} O_{n-r-1} \sigma_r(K^r)$$

$$+ \sum_{m=1}^{q} \frac{\binom{r-1}{q-m}}{\binom{n-1}{q}} \frac{1}{\varepsilon^m} s_\lambda(\varepsilon)^{n-r} \frac{O_{n-r+q-m}}{O_{q-m}} M_{q-m}^r(\partial K^r).$$

Letting here $\varepsilon \to 0$ we obtain (1.3) and complete the proof of Theorem 1.1. $\square$

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