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ASYMPTOTIC FORM AND INFINITE PRODUCT REPRESENTATION OF SOLUTION OF SECOND ORDER INITIAL VALUE PROBLEM WITH A COMPLEX PARAMETER AND A FINITE NUMBER OF TURNING POINTS

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Abstract. The paper studies the differential equation

$$y'' + \left(\rho^2\phi^2(x) - q(x)\right) y = 0 \qquad (*)$$

on the interval $I = [0, 1]$, containing a finite number of zeros $0 < x_1 < x_2 < \ldots < < x_m < 1$ of ϕ^2, i.e. so-called turning points. Using asymptotic estimates from [6] for appropriate fundamental systems of solutions of (*) as $|\rho| \to \infty$, it is proved that, if there is an asymptotic solution of the initial value problem generated by (*) in the interval $[0, x_1)$, then the asymptotic solutions in the remaining intervals can be obtained recursively. Furthermore, an infinite product representation of solutions of (*) is studied. The representations are useful in the study of inverse spectral problems for such equations.

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1. Introduction

There are numerous research papers devoted entirely or partially to the study of the equation

$$y'' + \left(\rho^2\phi^2(x) - q(x)\right) y = 0, \quad 0 \le x \le 1, \tag{1.1}$$

where the real valued functions ϕ^2 and q are said to be the coefficients of the problem, ϕ^2 is the weight and q is the potential function. Let I_+ be the set of points $x \in (0, 1)$, where $\phi^2(x) > 0$, I_- be the set of $x \in (0, 1)$, where $\phi^2(x) < 0$, and let I_0 be the set of those $x \in (0, 1)$, where $\phi^2(x) = 0$. If both I_+ and I_- are of positive Lebesgue measure, then the weight function $\phi^2(x)$ is said to be indefinite. The zeros of $\phi^2(x)$, which are assumed to form a discrete set, are called turning points or transition points (TP) of (1.1). It is impossible to obtain exact solutions for the majority of differential equations with variable coefficients, so it is necessary to find the best possible method for approximation of solutions. One of the most useful mathematical methods is the representation of a solution in an asymptotic form. For the existence of a solution of (1.1) depending on the parameter ρ^2, we would like to call the reader's attention to a complete historical review by Mingarelli [19].

The asymptotic techniques for solving differential equations of the form (1.1) play a crucial role in analysis and in the development of methods of modern applied mathematics and theoretical physics. The development of the asymptotic theory for linear differential equations started by a work of Birkhoff [2], based on transformations to first order systems, various diagonal transformation methods were applied by many specialists. The reader can find them in the books by Wasow [26, 27]. The results of Doronidcyn [4], Kazarinoff [11], McKelvey [18], Langer [14], Olver [22], Wazwaz [28], Dyachenko [5], Tumanov [25] and Kheiri, Jodayree & Mingarelli [12] give important innovations in the asymptotic approximation of solutions of the Sturm-Liouville equations with two turning points.

The importance of asymptotic analysis in obtaining information on solutions of the Sturm-Liouville equation with multiple turning points was shown by Leung [15], Olver [20–22], Heading [8], and Eberhard, Freiling & Schneider [6] in the case when the coefficients are smooth, in particular at TP. But the weakness of asymptotic methods is that generally it is impossible to express exact solutions in closed forms, which is necessary for the methods connected with dual equations. On representation of solutions in closed forms, a result of Halvorsen [7] is known, which states that for any $x \in [0, 1]$ a solution $y(x, \lambda)$ of (1.1) satisfying a fixed set of initial conditions is an entire function of the variable λ, whose order does not exceed $1/2$. Thus, it follows from Hadamard's factorization theorem (see, eg. [16]) that the solutions are representable as infinite products, and this gives an alternate description that has not been used for approximation purposes in various applications. Such infinite product representations have been effectively used by Trubowitz [23] and others [3, 13], etc., in some theoretical considerations related to the inverse spectral problem associated with (1.1) in definite cases, namely where the coefficient of the parameter λ in (1.1) is of a fixed sign in $[0,1]$, which is not true in the more general indefinite cases. Nevertheless, there are some works of the same spirit as those of Trubowitz and the mentioned above specialists (see, e.g. [10]), where such techniques is effectively used in the study of inverse spectral problems associated with the indefinite problem (1.1).

This paper considers the Sturm-Liouville equation (1.1) on a finite interval I, say $I=[0,1]$, with a finite number of arbitrary turning points $x_1, x_2, \ldots, x_m$ subject to any initial conditions, say $y(0,\lambda)=0$, $y'(0,\lambda)=1$, and the asymptotic solution in intervals $[x_{v-1}, x_v]$, $v=1,2,\ldots,m+1$, $x_{m+1}=1$, $x_0=0$ is recursively obtained. Then, the infinite product representation of solutions is studied.

2. Notations, fundamental solutions and preliminary results

Let us consider the real, second order differential equation (1.1) where $\rho^2=\lambda$ is the spectral parameter. In addition, we assume that

$$\phi^2(x)=\phi_0(x)\prod_{v=1}^{m}(x-x_v)^{\ell_v}, \tag{2.1}$$

where $0=x_0<x_1<x_2<\ldots<x_m<1=x_{m+1}$, ℓ_v is natural, $\phi_0(x)>0$ for $x\in I=[0,1]$, and ϕ_0 is twice continuously differentiable on I. In other words, $\phi^2(x)$ has m zeros x_v, $v=1,\ldots,m$, of orders l_v in I, which are also called turning points. We also assume that q is bounded and integrable on I.

We distinguish four different types of turning points as follows: the symbol

$$T_v=\begin{cases} I, & \text{if } l_v \text{ is even and } \phi^2(x)(x-x_v)^{-l_v}<0\ , \\ II, & \text{if } l_v \text{ is even and } \phi^2(x)(x-x_v)^{-l_v}>0, \\ III, & \text{if } l_v \text{ is odd and } \phi^2(x)(x-x_v)^{-l_v}<0, \\ IV, & \text{if } l_v \text{ is odd and } \phi^2(x)(x-x_v)^{-l_v}>0. \end{cases}$$

is said to be the type of the turning point x_v, $1\le v\le m$.

Further, for a turning point x_v, $1\le v\le m$, we set

$$I_{v,\varepsilon}=[x_{v-1}+\varepsilon, x_{v+1}-\varepsilon] \quad \text{and} \quad \mu_v=\frac{1}{2+\ell_v},$$

where $\varepsilon > 0$ is assumed to be a sufficiently small, fixed number, and

$$\vartheta_v = \begin{cases} 2, & \text{if} \quad T_v = III, IV \\ 1, & \text{if} \quad T_v = I, II \end{cases} \tag{2.2}$$

$$\delta_v = \begin{cases} 1, & \text{if} \quad T_v = II, IV \\ 2, & \text{if} \quad T_v = I, III \end{cases} \tag{2.3}$$

Also, we define

$$[1] \equiv 1 + O\left(\frac{1}{\lambda}\right) \quad \text{as} \quad \lambda \to \infty,$$

$$[\alpha] \equiv \alpha + O(\rho^{-\sigma_0}), \quad \alpha \in \mathbb{C},$$

$$\sigma_0 = \min\{\mu_1, \mu_2, \dots \mu_m\}.$$

Besides, we set

$$S_{-1} = \left\{\rho : \arg\rho \in \left[-\frac{\pi}{4}, 0\right]\right\}$$

and note that by [6] for each fixed $x \in I_{v,\varepsilon}$, according to its type, there exists a fundamental system of solutions $\left\{Z_{v,1}^{T_v}(x,\rho), Z_{v,2}^{T_v}(x,\rho)\right\}$ of (1.1), which is described by the following formulas.

Turning point of type I:

$$Z_{v,1}^{I}(x,\rho) = \begin{cases} |\phi(x)|^{-1/2} \exp\left\{\rho \int_{x_v}^{x} |\phi(t)| dt\right\} [1], & x_{v-1} < x < x_v, \\ |\phi(x)|^{-1/2} \csc \pi\mu_v \exp\left\{\rho \int_{x_v}^{x} |\phi(t)| dt\right\} [1], & x_v < x < x_{v+1}, \end{cases} \tag{2.4}$$

$$Z_{v,2}^{I}(x,\rho) = \begin{cases} |\phi(x)|^{-1/2} \exp\left\{-\rho \int_{x_v}^{x} |\phi(t)| dt\right\} [1], & x_{v-1} < x < x_v, \\ |\phi(x)|^{-1/2} \sin \pi\mu_v \exp\left\{-\rho \int_{x_v}^{x} |\phi(t)| dt\right\} [1], & x_v < x < x_{v+1}, \end{cases} \tag{2.5}$$

$$Z_{v,1}^{I}(x_v,\rho) = \frac{\sqrt{2\pi}}{2} (\imath\rho)^{1/2-\mu_v} \csc \pi\mu_v e^{\imath\pi(-1/4+\mu_v/2)} \frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)} [1], \tag{2.6}$$

$$Z_{v,2}^{I}(x_v,\rho) = \frac{\sqrt{2\pi}}{2} (\imath\rho)^{1/2-\mu_v} e^{\imath\pi(-1/4+\mu_v/2)} \frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)} [1], \tag{2.7}$$

where

$$\psi(x_1) = \lim_{x \to x_1} \phi^{-1/2}(x) \left\{\int_{x_1}^{x} \phi(t) dt\right\}^{1/2-\mu_1},$$

and

$$W(\rho) = W(Z_{v,1}^{I}(x,\rho), Z_{v,2}^{I}(x,\rho)) = -2\rho[1], \tag{2.8}$$

where $W(f(x), g(x)) := f(x)g'(x) - f'(x)g(x)$ is the Wronskian of f and g.

Turning point of type II:

(2.9)

$$Z^{II}_{v,1}(x,\rho)=\begin{cases} |\phi(x)|^{-1/2}\exp\left\{\imath\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1], & \text{if } x_{v-1}<x<x_v,\\ |\phi(x)|^{-1/2}\csc\pi\mu_v\left[\exp\left\{\imath\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1]\right.\\ \left.+\imath\cos\pi\mu_v\exp\left\{-\imath\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1]\right], & \text{if } x_v<x<x_{v+1}, \end{cases}$$

(2.10)

$$Z^{II}_{v,2}(x,\rho)=\begin{cases} |\phi(x)|^{-1/2}\left[\exp\left\{-\imath\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1]+\right.\\ \left.+\imath\cos\pi\mu_v\exp\left\{\imath\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1]\right], & \text{if} \quad x_{v-1}<x<x_v,\\ |\phi(x)|^{-1/2}\sin\pi\mu_v\exp\left\{-\imath\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1], & \text{if} \quad x_v<x<x_{v+1}, \end{cases}$$

$$Z^{II}_{v,1}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{1/2-\mu_v}\csc\pi\mu_v e^{\imath\pi(1/4-\mu_v/2)}\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \tag{2.11}$$

$$Z^{II}_{v,2}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{1/2-\mu_v}e^{\imath\pi(1/4-\mu_v/2)}\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \tag{2.12}$$

$$W(\rho)=W\left(Z^{II}_{v,1}(x,\rho),Z^{II}_{v,2}(x,\rho)\right)=-2\imath\rho[1]. \tag{2.13}$$

Turning point of type III:

$$Z^{III}_{v,1}(x,\rho)=\begin{cases} |\phi(x)|^{-1/2}\exp\left\{\imath\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1], & \text{if} \quad x_{v-1}<x<x_v,\\ \frac{1}{2}|\phi(x)|^{-1/2}\csc\frac{\pi\mu_v}{2}\\ \qquad\times\exp\left\{\rho\int_{x_v}^{x}|\phi(t)|dt+\frac{\iota\pi}{4}\right\}[1], & \text{if} \quad x_v<x<x_{v+1}, \end{cases} \tag{2.14}$$

$$Z^{III}_{v,2}(x,\rho)=\begin{cases} |\phi(x)|^{-1/2}\left[\exp\left\{-\imath\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1]\right.\\ \qquad\left.+\imath\exp\left\{\imath\rho\int_{x_v}^{x}|\phi(t)|dt\right\}\right][1], & \text{if} \quad x_{v-1}<x<x_v,\\ 2|\phi(x)|^{-1/2}\sin\frac{\pi\mu_v}{2}\\ \qquad\times\exp\left\{-\rho\int_{x_v}^{x}|\phi(t)|dt+\frac{\iota\pi}{4}\right\}[1], & \text{if} \quad x_v<x<x_{v+1}, \end{cases} \tag{2.15}$$

$$Z^{III}_{v,1}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}(\iota\rho)^{1/2-\mu_v}\csc\pi\mu_v\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \tag{2.16}$$

$$Z^{III}_{v,2}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}(\iota\rho)^{1/2-\mu_v}e^{\iota\pi\mu_v/2}\sec\left(\frac{\pi\mu_v}{2}\right)\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1], \tag{2.17}$$

$$W(\rho)=W\left(Z^{III}_{v,1}(x,\rho),Z^{III}_{v,2}(x,\rho)\right)=-2\iota\rho[1]. \tag{2.18}$$

Turning point of type IV:

$$(2.19)\quad Z^{IV}_{v,1}(x,\rho)=\begin{cases}|\phi(x)|^{-1/2}\exp\left\{\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1], & \text{if}\quad x_{v-1}<x<x_v,\\ \frac{1}{2}\csc\frac{\pi\mu_v}{2}|\phi(x)|^{-1/2} & \\ \qquad\times\left[\exp\left\{\imath\rho\int_{x_v}^{x}|\phi(t)|dt-\imath\frac{\pi}{4}\right\}[1]\right. & \\ \qquad\left.+\exp\left\{-\imath\rho\int_{x_v}^{x}|\phi(t)|dt+\imath\frac{\pi}{4}\right\}[1]\right], & \text{if}\quad x_v<x<x_{v+1},\end{cases}$$

$$(2.20)\quad Z^{IV}_{v,2}(x,\rho)=\begin{cases}|\phi(x)|^{-1/2}\exp\left\{-\rho\int_{x_v}^{x}|\phi(t)|dt\right\}[1], & \text{if}\quad x_{v-1}<x<x_v,\\ 2\sin\frac{\pi\mu_v}{2}|\phi(x)|^{-1/2} & \\ \quad\times\left[\exp\left\{-\imath\rho\int_{x_v}^{x}|\phi(t)|dt-\imath\frac{\pi}{4}\right\}[1]\right], & \text{if}\quad x_v<x<x_{v+1},\end{cases}$$

$$(2.21)\quad Z^{IV}_{v,1}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{1/2-\mu_v}\csc\pi\mu_v\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1],$$

$$(2.22)\quad Z^{IV}_{v,2}(x_v,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{1/2-\mu_v}e^{-\imath\pi\mu_v/2}\sec\left(\frac{\pi\mu_v}{2}\right)\frac{2^{\mu_v}\psi(x_v)}{\Gamma(1-\mu_v)}[1],$$

$$(2.23)\quad W(\rho)=W\left(Z^{IV}_{v,1}(x,\rho),Z^{IV}_{v,2}(x,\rho)\right)=-2\rho[1].$$

Halvorsen [7] proved that if $0<c<x<1$ and

$$\int_c^x|\phi(t)|dt\neq 0,$$

then the solution $y(x,\lambda)$ of (1.1), determined by fixed values of y,y' at c, is an entire function of λ of order $1/2$. On the other hand, by Hadamard's theorem an entire function $f(z)$ of a finite order l, can be represented in the following form:

$$f(z)=z^m e^{g(z)}\prod_n\left(1-\frac{z}{a_n}\right)\exp\left\{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots+\frac{1}{h}\left(\frac{z}{a_n}\right)^h\right\},\quad z\in\mathbb{C},$$

where $a_n\neq 0$ are the zeros of $f(z)$, arranged in order of increasing magnitude, $h\leq l$, g(z) is a polynomial whose degree q does not exceed l and m is the multiplicity of the zero of $f(z)$ at the origin. By means of Hadamard's theorem, we can find an infinite expansion for $\sinh z$ and $J'_v(z)$. It is well known that

$$\sinh z=z\prod_{m=1}^{\infty}\left(1+\frac{z^2}{m^2\pi^2}\right),\quad z\in\mathbb{C},$$

and therefore

$$\sinh c\sqrt{z} = c\sqrt{z}\prod_{m=1}^{\infty}\left(1+\frac{zc^2}{m^2\pi^2}\right) = c\sqrt{z}\prod_{m=1}^{\infty}\left(1+\frac{z}{z_m^2}\right), \quad z\in\mathbb{C}, \tag{2.24}$$

where $z_m = m\pi/c$, and the holomorphity domain of the function $f(z) = z^{1/2}$ is the complement of the negative real axis $z \le 0$, while its range is the right half of the z-plane with the imaginary axis excluded.

Also, from [1] we have

$$J_v'(z) = \frac{(z/2)^{v-1}}{2\Gamma(v)}\prod_{m=1}^{\infty}\left(1-\frac{z^2}{\tilde{j}_m^2}\right), \quad v>0, \tag{2.25}$$

where $\tilde{j}_m$, $m = 1, 2, \ldots$, is the sequence of positive zeros of $J_1'(z)$ and

$$\tilde{j}_m^2 = m^2\pi^2 - \frac{m\pi^2}{2} + O(1). \tag{2.26}$$

Putting $z = c\sqrt{\lambda}$ and $\Gamma(1) = 1$ in (2.25), we obtain

$$J_1'\left(c\sqrt{\lambda}\right) = \frac{1}{2}\prod_{m=1}^{\infty}\left(1-\frac{\lambda c^2}{\tilde{j}_m^2}\right), \tag{2.27}$$

and similarly

$$J_1'\left(\iota c\sqrt{\lambda}\right) = \frac{1}{2}\prod_{m=1}^{\infty}\left(1+\frac{\lambda c^2}{\tilde{j}_m^2}\right). \tag{2.28}$$

For completeness, below we give the following well-known theorems which play an important role in estimation of the infinite product and which can be found, for instance, in [24].

Theorem 2.1. *For any sequence of complex numbers p_n , $n = 0, 1, \ldots,$ the product*

$$\prod_{n=0}^{\infty}(1+p_n)$$

converges absolutely if and only if the series $\sum_0^\infty p_n$ converges absolutely.

Theorem 2.2. *If $p_n(z)$, $n = 0, 1, \ldots,$ are analytic functions in a simply connected domain D, and the series*

$$\sum_{n=0}^{\infty}|p_n(z)|$$

uniformly converges in every closed region R of D, then

$$\prod_{n=0}^{\infty}(1+p_n(z))$$

uniformly converges in every such R to a function $f(z)$ analytic in D.

Theorem 2.3. (a) *If a_{mn}, $m, n > 1$ are complex numbers such that*

$$|a_{mn}| = O\left(\frac{1}{|m^2 - n^2|}\right), \quad m \neq n,$$

then for any $n \geq 1$

$$\prod_{m\geq 1,\, m\neq n} (1 + a_{mn}) = 1 + O\left(\frac{\log n}{n}\right).$$

(b) *In addition, if b_n $(n \geq 1)$, is a square summable sequence of complex numbers, then the product*

$$\prod_{m\geq 1,\, m\neq n} (1 + a_{mn} b_n)$$

is convergent.

3. Asymptotic form of the solution

Let us consider the second order differential equation (1.1) on a finite interval I, say $I = [0, 1]$, with any initial conditions, where y and y' are determinated at a fixed point c, for example $y(0, \lambda) = 0$, $y'(0, \lambda) = 1$, also $\phi^2(x)$ is of the form (2.1) meaning that I contains m turning points $x_1, x_2, \ldots, x_m$, which are zeros of ϕ. A solution of the problem in $I_{1,\varepsilon}$ can be obtained by applying the initial conditions to

$$y(x, \rho) = C_1(\rho) Z_{1,1}^{T_1}(x, \rho) + C_2(\rho) Z_{1,2}^{T_1}(x, \rho), \quad x \in I_{1,\varepsilon}.$$

In view of formulas (2.4) - (2.23) for the functions of $\{Z_{1,1}^{T_1}(x, \rho), Z_{1,2}^{T_1}(x, \rho)\}$ and their derivatives, it is clear that this solution can be represented in the following form:

$$y_{(0,x_1)}(x, \rho) := H(x, \rho) \exp\left\{(-1)^{\delta_1 - 1} (\iota)^{\delta_1} \rho \int_0^x |\phi(t)| dt\right\} E_k(x, \rho),$$

$$y(x_1, \rho) := F(x_1, \mu_1, \rho) \csc \pi\mu_1 \exp\left\{(-1)^{\theta_1} (\iota)^{\vartheta_1} \rho \int_0^{x_1} |\phi(t)| dt\right\} E_k(x_1, \rho),$$

as it is done in Example 3.1 below. Here, the functions $H(x, \rho)$ and $F(x_1, \mu_1, \rho)$ depend on the initial conditions and the type of the turning point x_1. Moreover,

$$E_k(x, \rho) = [1] + \sum_{n=1}^{\nu(x)} e^{\rho \alpha_k \beta_{kn}(x)} [b_{kn}(x)],$$

where $\alpha_{-2} = \alpha_1 = -1$, $\alpha_0 = -\alpha_{-1} = \imath$, $\beta_{k\nu(x)}(x) \neq 0$, $0 < \delta \leq \beta_{k1}(x) < \beta_{k2}(x) < \ldots < \beta_{k\nu(x)}(x) \leq 2\,max\{R_+(1), R_-(1)\}$, and the integer-valued functions ν and b_{kn} are constant in all intervals $[0, x_1 - \varepsilon]$ and $[x_1 + \varepsilon, x_2 - \varepsilon]$ for sufficiently small $\varepsilon > 0$, and

$$R_+(x) = \int_0^x \sqrt{max\{0, \phi^2(t)\}}\,dt, \quad R_-(x) = \int_0^x \sqrt{max\{0, -\phi^2(t)\}}\,dt. \tag{3.1}$$

The following theorem proves that if an asymptotic solution of initial value problem (1.1) is obtained in the interval $[0, x_1]$, then it can be obtained recursively in the remaining intervals.

Theorem 3.1. *Let $y_{(x_v,x_{v+1})}(x,\rho)$ be an asymptotic solution of the initial value problem (1.1) in the interval (x_v, x_{v+1}). Then*

$$\begin{aligned} y_{(0,x_1)}(x,\rho) &= H(x,\rho)\exp\left\{(-1)^{\delta_1 - 1}(\imath)^{\delta_1}\rho\int_0^x |\phi(t)|dt\right\}E_k(x,\rho) \\ &= A(x,\rho)E_k(x,\rho), \end{aligned} \tag{3.2}$$

and for $x \in (x_v, x_{v+1})$, $v \geq 1$, we have

$$\begin{aligned} y_{(x_v,x_{v+1})}(x,\rho) = {}& \frac{1}{2^k}A(x_1,\rho)\prod_{x_i \leq x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i - 1}} \\ & \times\exp\left\{(-1)^{\delta_v - 1}(\imath)^{\delta_v}\rho\sum_{x_i < x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt\right. \\ & \left. + \sum_{x_i \leq x_v : T_i = III, IV}(-1)^{\delta_v}\frac{\imath\pi}{4} + (-1)^{\delta_v - 1}(\imath)^{\delta_v}\rho\int_{x_v}^x|\phi(t)|dt\right\}E_k(x,\rho), \end{aligned} \tag{3.3}$$

where k is the number of turning points $x_i \leq x_v$ of III or IV type. Also,

$$y(x_1,\rho) = F(x_1,\mu_1,\rho)\csc\pi\mu_1\exp\left\{(-1)^{\theta_1}(\imath)^{\vartheta_1}\rho\int_0^{x_1}|\phi(t)|dt\right\}E_k(x_1,\rho), \tag{3.4}$$

and for $x_v > x_1$, which are turning points of II or IV type, we have

$$
\begin{aligned}
y(x_v,\rho) = {} & \frac{1}{2^s} F(x_v,\mu_v,\rho) \exp\left\{(2-\vartheta_v)\iota\pi\left(\frac{1}{4}-\frac{\mu_v}{2}\right)\right. \\
& \left. +(-1)^{\theta_1}(\iota)^{\vartheta_1}\rho\int_0^{x_1}|\phi(t)|dt\right\}\csc\pi\mu_v\prod_{x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}} \\
& \times\exp\left\{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt\right. \\
& \left. +\sum_{x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}\right\}E_k(x_v,\rho).
\end{aligned}
\tag{3.5}
$$

Similarly, for $x_v > x_1$, which are turning points of I or III type, we have

$$
\begin{aligned}
y(x_v,\rho) = {} & \frac{1}{2^s} F(x_v,\mu_v,\iota\rho) \exp\left\{(2-\vartheta_v)\iota\pi\left(-\frac{1}{4}+\frac{\mu_v}{2}\right)\right. \\
& \left. +(-1)^{\theta_1}(\iota)^{\vartheta_1}\rho\int_0^{x_1}|\phi(t)|dt\right\}\csc\pi\mu_v\prod_{x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}} \\
& \times\exp\left\{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt\right. \\
& \left. +\sum_{x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}\right\}E_k(x_v,\rho),
\end{aligned}
\tag{3.6}
$$

where s is the number of turning points $x_i < x_v$, which are III or IV type.

Proof: It is clear that in the intervals $(0, x_1) \cup (x_1, x_2)$ the solution can be obtained by the formula

$$y(x,\rho) = C_1(\rho)Z_{1,1}^{T_1}(x,\rho) + C_2(\rho)Z_{1,2}^{T_1}(x,\rho),$$

and the initial conditions. For example, if x_1 is a turning point of IV type and $y(0,\lambda) = 1$, $y'(0,\lambda) = 0$, then using formulas (2.19)-(2.21) and (2.23) we obtain

$$
\begin{aligned}
y_{(0,x_1)}(x,\rho) &= \frac{1}{2}|\phi(x)|^{-1/2}|\phi(0)|^{1/2}\exp\left\{\rho\int_0^x|\phi(t)|dt\right\}E_k(x,\rho) \\
&:= H(x,\rho)\exp\left\{\rho\int_0^x|\phi(t)|dt\right\}E_k(x,\rho) \\
&:= A_{(0,x_1)}(x,\rho)E_k(x,\rho).
\end{aligned}
$$

Moreover, for the interval (x_1, x_2) we get

$$y_{(x_1,x_2)}(x,\rho) = \frac{1}{4}|\phi(x)|^{-1/2}|\phi(0)|^{1/2}\csc\frac{\pi\mu_1}{2} \tag{3.7}$$

$$\times\exp\left\{\rho\int_0^{x_1}|\phi(t)|dt + \imath\rho\int_{x_1}^{x}|\phi(t)|dt - \frac{\iota\pi}{4}\right\}E_k(x,\rho) \tag{3.8}$$

$$= \frac{1}{2}A_{(0,x_1)}(x_1,\rho)\csc\frac{\pi\mu_1}{2}\exp\left\{\imath\rho\int_{x_1}^{x}|\phi(t)|dt - \frac{\iota\pi}{4}\right\}.$$

In order to find the solution in $I_{2,\varepsilon}$, we fix $x \in (x_1, x_2)$ and write the obtained solution in this interval as linear combination of fundamental solutions in $I_{2,\varepsilon}$:

$$y_{(x_2,x_3)}(x,\rho) = A(\rho)Z_{2,1}^{T_2}(x,\rho) + B(\rho)Z_{2,2}^{T_2}(x,\rho). \tag{3.9}$$

Then, by Cramer's rule we can determine the connection coefficients

$$\begin{aligned} A(\rho) &= \frac{y_{(x_1,x_2)}(x,\rho)Z_{2,2}^{T_2'}(x,\rho) - Z_{2,2}^{T_2}(x,\rho)y'_{(x_1,x_2)}(x,\rho)}{W\left(Z_{2,1}^{T_2}(x,\rho), Z_{2,2}^{T_2}(x,\rho)\right)}, \\ B(\rho) &= \frac{y'_{(x_1,x_2)}(x,\rho)Z_{2,1}^{T_2}(x,\rho) - Z_{2,1}^{T_2'}(x,\rho)y_{(x_1,x_2)}(x,\rho)}{W\left(Z_{2,1}^{T_2}(x,\rho), Z_{2,2}^{T_2}(x,\rho)\right)}. \end{aligned} \tag{3.10}$$

Note that $\{Z_{2,1}^{T_2}(x,\rho), Z_{2,2}^{T_2}(x,\rho)\}$ are fundamental system of solutions in the intervals $(x_1, x_2)\cup(x_2, x_3)$, and hence the continuation of the solution $y(x,\lambda)$ to (x_2, x_3) satisfies (3.9), where $A(\rho)$ and $B(\rho)$ are the coefficients obtained in the previous interval, but now the formulas for $\{Z_{2,1}^{T_2}(x,\rho), Z_{2,2}^{T_2}(x,\rho)\}$ are used in $x_2 < x < x_3$. Further, one can suppose that

$$y_{(x_i,x_{i+1})}(x,\rho) = A_{(x_i,x_{i+1})}(x,\rho)E_k(x,\rho)$$

is the asymptotic form of the solution of the initial value problem in the interval (x_i, x_{i+1}), $i = 1, 2, \ldots, m$, with $x_{m+1} = 1$. Without loss of generality we can suppose that x_1 and x_2 are turning points of IV and III types. Then, using formulas (2.14) - (2.16) and those which they imply for the derivatives and taking into account (2.18), we can calculate the connection coefficients $A(\rho)$ and $B(\rho)$ by (3.10). Thus, (3.9) yields

$$y_{(x_2,x_3)}(x,\rho) = \frac{1}{4}H(x,\rho)\csc\frac{\pi\mu_1}{2}\csc\frac{\pi\mu_2}{2}\exp\left\{\rho\int_0^{x_1}|\phi(t)|dt\right\}$$
$$\times\exp\left\{\iota\rho\int_{x_1}^{x_2}|\phi(t)|dt - \frac{\iota\pi}{4} + \rho\int_{x_2}^{x}|\phi(t)|dt + \frac{\iota\pi}{4}\right\}$$
$$= \frac{1}{2}A_{(x_1,x_2)}(x_2,\rho)\csc\frac{\pi\mu_2}{2}\exp\left\{\rho\int_{x_v}^{x}|\phi(t)|dt + \frac{\iota\pi}{4}\right\}.$$

This procedure can be used to calculate the solution in the remaining intervals. For completeness of the proof, note that acting in the same way one can obtain the following representations.

(i) If x_v is a I type turning point, then

$$A_{(x_v,x_{v+1})}(x,\rho) = A_{(x_{v-1},x_v)}(x_v,\rho)\csc\pi\mu_v\exp\left\{\rho\int_{x_v}^{x}|\phi(t)|dt\right\}.$$

(ii) If x_v is a II type turning point, then

$$A_{(x_v,x_{v+1})}(x,\rho) = A_{(x_{v-1},x_v)}(x_v,\rho)\csc\pi\mu_v\exp\left\{\iota\rho\int_{x_v}^{x}|\phi(t)|dt\right\}.$$

(iii) If x_v is a III type turning point, then

$$A_{(x_v,x_{v+1})}(x,\rho) = \frac{1}{2}A_{(x_{v-1},x_v)}(x_v,\rho)\csc\frac{\pi\mu_v}{2}\exp\left\{\rho\int_{x_v}^{x}|\phi(t)|dt + \frac{\iota\pi}{4}\right\}.$$

(iv) If x_v is a IV type turning point, then

$$A_{(x_v,x_{v+1})}(x,\rho) = \frac{1}{2}A_{(x_{v-1},x_v)}(x_v,\rho)\csc\frac{\pi\mu_v}{2}\exp\left\{\iota\rho\int_{x_v}^{x}|\phi(t)|dt - \frac{\iota\pi}{4}\right\}.$$

This completes the proof. □

Example 3.1. Let $y(x,\rho)$ be the solution of 1.1 corresponding the initial conditions

$$y(0,\lambda) = 0, \quad y'(0,\lambda) = 1.$$

Suppose, the first turning point x_1 is of IV type, that is of the order $l_1 = 4m+1$, and the other $m-1$ turning points are arbitrary. Then, using the fundamental system of solutions $\{Z_{1,1}^{IV}(x,\rho), Z_{1,2}^{IV}(x,\rho)\}$ we obtain

$$y(x,\rho) = \frac{1}{-2\rho}\left(Z_{1,1}^{IV}(0,\rho)Z_{1,2}^{IV}(x,\rho) - Z_{1,1}^{IV}(x,\rho)Z_{1,2}^{IV}(0,\rho)\right), \quad x\in(0,x_1). \tag{3.11}$$

Hence, from (2.19) and (2.20) it follows that

$$y(x,\rho) = \frac{|\phi(x)\phi(0)|^{\frac{-1}{2}}}{2\rho}\left[\exp\left\{\rho\int_0^x|\phi(t)dt\right\}[1] - \exp\left\{-\rho\int_0^x|\phi(t)dt\right\}[1]\right] \tag{3.12}$$

for $x \in (0, x_1)$ or

$$y(x,\rho) = \frac{|\phi(x)\phi(0)|^{-1/2}}{2\rho} \exp\left\{\rho \int_0^x |\phi(t)|dt\right\} E_k(x,\rho) \quad \text{for} \quad x \in (0,x_1). \tag{3.13}$$

Besides, in virtue of (2.21) and (2.22) we get

$$\begin{aligned} y(x_1,\rho) = &\frac{|\phi(0)|^{-1/2}\sqrt{2\pi}\rho^{1/2-\mu_1}2^{\mu_1}\psi(x_1)\csc\pi\mu_1}{4\Gamma(1-\mu_1)} \\ &\times \exp\left\{\rho\int_0^{x_1} |\phi(t)|dt\right\} E_k(x,\rho). \end{aligned} \tag{3.14}$$

Further, using Theorem 3.1 we can calculate

$$\begin{aligned} y(x,\rho) = &\frac{|\phi(x)\phi(0)|^{-1/2}}{4\rho} \csc\frac{\pi\mu_1}{2} \\ &\times \exp\left\{\rho\int_0^{x_1} |\phi(t)dt + \iota\rho\int_{x_1}^{x} |\phi(t)dt - \frac{\iota\pi}{4}\right\} E_k(x,\rho), \quad x_1 < x < x_2. \end{aligned} \tag{3.15}$$

We note that (3.15) also can be obtained directly.

Now, for a turning point of IV type we use Theorem 3.1. Then we find the solution in the interval (x_v, x_{v+1}):

$$\begin{aligned} y_{(x_v,x_{v+1})}(x,\rho) = &\frac{1}{2^k}\frac{|\phi(x)\phi(0)|^{-1/2}}{2\rho}\csc\frac{\pi\mu_1}{2}\csc\frac{\pi\mu_v}{2}\exp\left\{\rho\int_0^{x_1}|\phi(t)|dt - \frac{\iota\pi}{4}\right\} \\ &\times \prod_{x_2\le x_i<x_v} \csc\frac{\pi\mu_i}{2^{\vartheta_i-1}} \exp\left\{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt\right. \\ &+ \left.\sum_{x_2\le x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4} + \iota\rho\int_{x_v}^{x}|\phi(t)|dt - \frac{\iota\pi}{4}\right\} E_k(x,\rho). \end{aligned} \tag{3.16}$$

Besides, for a turning point of III type

$$\begin{aligned} y_{(x_v,x_{v+1})}(x,\rho) = &\frac{1}{2^k}\frac{|\phi(x)\phi(0)|^{-1/2}}{2\rho}\csc\frac{\pi\mu_1}{2}\csc\frac{\pi\mu_v}{2}\exp\left\{\rho\int_0^{x_1}|\phi(t)|dt - \frac{\iota\pi}{4}\right\} \\ &\times \prod_{x_2\le x_i<x_v} \csc\frac{\pi\mu_i}{2^{\vartheta_i-1}} \exp\left\{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt\right. \\ &+ \left.\sum_{x_2\le x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4} + \rho\int_{x_v}^{x}|\phi(t)|dt + \frac{\iota\pi}{4}\right\} E_k(x,\rho). \end{aligned} \tag{3.17}$$

Similarly, for a turning point of II type we obtain

$$(3.18)\quad y_{(x_v,x_{v+1})}(x,\rho)=\frac{1}{2^k}\frac{|\phi(x)\phi(0)|^{1-1/2}}{2\rho}\csc\frac{\pi\mu_1}{2}\csc\pi\mu_v\exp\left\{\rho\int_0^{x_1}|\phi(t)|dt-\frac{\iota\pi}{4}\right\}$$
$$\times\prod_{x_2\le x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}\exp\left\{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt\right.$$
$$\left.+\sum_{x_2\le x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}+\iota\rho\int_{x_v}^{x}|\phi(t)|dt\right\}E_k(x,\rho).$$

Further, for a turning point of I type we have

$$(3.19)\quad y_{(x_v,x_{v+1})}(x,\rho)=\frac{1}{2^k}\frac{|\phi(x)\phi(0)|^{-1/2}}{2\rho}\csc\frac{\pi\mu_1}{2}\csc\pi\mu_v\exp\left\{\rho\int_0^{x_1}|\phi(t)|dt-\frac{\iota\pi}{4}\right\}$$
$$\times\prod_{x_2\le x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}\exp\left\{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt\right.$$
$$\left.+\sum_{x_2\le x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}+\rho\int_{x_v}^{x}|\phi(t)|dt\right\}E_k(x,\rho).$$

Indeed, using Theorem 3.1 we can obtain that for a turning point of II or IV type

$$(3.20)\quad y(x_v,\rho)=\frac{1}{2^s}\frac{|\phi(0)|^{-\frac{1}{2}}\sqrt{2\pi}\rho^{\frac{1}{2}-\mu_1}2^{\mu_1}\psi(x_1)\csc\pi\mu_v}{4\Gamma(1-\mu_1)}$$
$$\times\exp\left\{\rho\int_0^{x_1}|\phi(t)|dt+(2-\vartheta_v)\iota\pi\left(\frac{1}{4}-\frac{\mu_v}{2}\right)\right\}\prod_{x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}$$
$$\times\exp\left\{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt\right\},$$

and for a turning point of I or III type

$$(3.21)\quad y(x_v,\rho)=\frac{1}{2^s}\frac{|\phi(0)|^{-1/2}\sqrt{2\pi}(\iota\rho)^{1/2-\mu_1}2^{\mu_1}\psi(x_1)\csc\pi\mu_v}{4\Gamma(1-\mu_1)}$$
$$\times\exp\left\{\rho\int_0^{x_1}|\phi(t)|dt+(2-\vartheta_v)\iota\pi\left(-\frac{1}{4}+\frac{\mu_v}{2}\right)\right\}\prod_{x_i<x_v}\csc\frac{\pi\mu_i}{2^{\vartheta_i-1}}$$
$$\times\exp\left\{(-1)^{\delta_v-1}(\iota)^{\delta_v}\rho\sum_{x_i<x_v}\int_{x_i}^{x_{i+1}}|\phi(t)|dt\right.$$
$$\left.+\sum_{x_i<x_v:T_i=III,IV}(-1)^{\delta_v}\frac{\iota\pi}{4}\right\}E_k(x_v,\rho).$$

4. Infinite product representation

In this section, we consider the real, second order differential equation (1.1) with the initial conditions

$$y(0,\lambda)=0, \quad y'(0,\lambda)=1, \tag{4.1}$$

under the assumption that $\phi^2(x)$ is a real function having m zeros x_v, of the orders ℓ_v, $v=1,\ldots,m$, in I, where ℓ_1 is an odd number and $\ell_2,\ell_3\ldots,\ell_m$ are even. Specially, we suppose that $\ell_1=4k+1$ and $\ell_v=4k$, $v=2,3,\ldots,m$. In the terminology of [6], x_1 is of the type IV while $x_2,x_3,\ldots x_v$ are of the type II. Besides, the function $\phi_0: I\to R-\{0\}$, defined as

$$\phi_0(x)=\phi^2(x)\prod_{v=1}^{m}(x-x_v)^{-\ell_v},$$

is twice continuously differentiable.

Now, let $S(x,\lambda)$ be the solution of the initial value problem (1.1), (4.1). Then by Halvorsen's result, $S(x,\lambda)$ is an entire function of the order $1/2$ for any fixed $x\in(0,1)$, and therefore by Hadamard's theorem [3], $S(x,\lambda)$ is representable in the form

$$S(x,\lambda)=s(x)\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{w_n(x)}\right),$$

where $s(x)$ is a function independent of λ but can depend on x. For any x, the sequence $\{w_n(x)\}_1^\infty$ is the set of zeros of $S(x,\lambda)$, i.e. $S(x,w_n(x))=0$, which corresponds to the eigenvalues of the boundary value problem $L(\phi^2(x),q(x),x)$ defined by the second-order differential equation (1.1) with the boundary conditions

$$y(0,\lambda)=0, \quad y(0,\lambda)=1, \quad y(x,\lambda)=0.$$

We see that for any fixed x each $w_n(x)$, $n=1,2,\ldots$, appears in the denominator and hence must be nonzero. Adding the extra condition $q(x)\geq 0$ for any x, we get $w_n(x)\neq 0$ by Sturm's comparison theorem. Further, for any fixed $x\in(0,x_1)$ the function $S(x,\lambda)$ has the asymptotic representation

$$S(x,\lambda)=\frac{1}{2\rho}|\phi(x)\phi(0)|^{-1/2}\exp\left\{\rho\int_0^x|\phi(t)|dt\right\}E_k(x,\rho). \tag{4.2}$$

For any $x \in [0, x_1)$, the boundary value problem $L(\phi^2(x), q(x), x)$ has infinitely many negative eigenvalues, say $\{\lambda_n^-(x)\}$, and in this case $w_n(x) = \lambda_n^-(x)$. Besides, due to (1.1) the asymptotic representation of each $\lambda_n^-(x)$ is of the form

$$\sqrt{-\lambda_n^-(x)} = n\pi \left(\int_0^x |\phi(t)| dt \right)^{-1} + O\left(\frac{1}{n}\right). \tag{4.3}$$

By Hadamard's theorem, for a fixed $x \in [0, x_1)$ the following representation is true:

$$S(x, \lambda) = s(x) \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n^-(x)}\right),$$

where the function $s(x)$ is independent of λ but may depend on x, and for any x infinitely many negative eigenvalues $\{\lambda_n^-(x)\}_{n=1}^{\infty}$ form the zero set of $S(x, \lambda)$. We rewrite the infinite product as follows:

$$S(x, \lambda) = s(x) \prod_{n \geq 1} \left(1 - \frac{\lambda}{\lambda_n^-(x)}\right) = s_1(x) \prod_{n \geq 1} \frac{\lambda - \lambda_n^-(x)}{z_n^2}, \tag{4.4}$$

where

$$s_1(x) := s(x) \prod_{n \geq 1} \frac{-z_n^2}{\lambda_n^-(x)} \quad \text{and} \quad z_n = \frac{n\pi}{R_-(x)}.$$

Note that

$$\frac{-z_m^2}{\lambda_m^-(x)} = 1 + O\left(\frac{1}{m^2}\right),$$

and hence by Theorem 2.1 the infinite product

$$\prod_{m \geq 1} \frac{-z_m^2}{\lambda_m^-(x)}$$

is absolutely convergent on any compact subinterval of $(0, x_1)$. Besides, the function $\frac{-z_m^2}{\lambda_m^-(x)}$ is continues, and so the O-term is uniformly bounded in x.

The proof of the following theorem is similar to Theorem 6.1 of [9] and therefore omitted.

Theorem 4.1. *Let $S(x, \lambda)$ be the solution of* (1.1) *satisfying the initial conditions $S(0, \lambda) = 0$, $S'(0, \lambda) = 1$. Then*

$$S(x, \lambda) = |\phi(x)\phi(0)|^{-1/2} R_-(x) \prod_{m \geq 1} \frac{\lambda - \lambda_m^-(x)}{z_m^2}, \quad 0 < x < x_1,$$

where

$$z_m = \frac{m\pi}{R_-(x)}, \quad R_-(x) = \int_0^x \sqrt{max\{0, -\phi^2(t)\}}dt,$$

and the sequence $\lambda_m^-(x)$ $(m \geq 1)$ *is that of the negative eigenvalues of the boundary value problem* L *on* $[0, x]$.

Similarly, Theorem 3.1 implies that for any $x_v < x < x_{v+1}$, $v = 1, 2 \ldots m$, $x_{m+1} = 1$, the asymptotic form of the solution of the initial value problem (1.1), (4.1) is

$$\begin{aligned} S(x,\rho) = \frac{1}{4\rho}|\phi(x)\phi(0)|^{-1/2} \csc\frac{\pi\mu_1}{2}\csc\pi\mu_2\cdots\csc\pi\mu_{v-1}\csc\pi\mu_v \\ \times \exp\left\{\rho\int_0^{x_1}|\phi(t)|dt + \iota\rho\int_{x_1}^x|\phi(t)|dt - \frac{\iota\pi}{4}\right\}E_k(x,\rho). \end{aligned} \tag{4.5}$$

Further, the boundary value problem L in $[0, x]$ has a countable set of positive and negative eigenvalues which we denote by $\{\lambda_n(x)\} = \{\lambda_n^+(x)\} \cup \{\lambda_n^-(x)\}$, and the following formulas are true:

$$\begin{aligned} \sqrt{\lambda_n^+(x)} &= \left(n\pi - \frac{\pi}{4}\right)\left(\int_{x_1}^x |\phi(t)|dt\right)^{-1} + O\left(\frac{1}{n}\right), \\ \sqrt{-\lambda_n^-(x)} &= -\left(n\pi - \frac{\pi}{4}\right)\left(\int_0^{x_1} |\phi(t)|dt\right)^{-1} + O\left(\frac{1}{n}\right). \end{aligned} \tag{4.6}$$

By Hadamard's theorem, for $x_v < x < x_{v+1}$ the solution on $[0, x]$ is of the form

$$S(x,\lambda) = g(x)\prod_{n\geq1}\left(1 - \frac{\lambda}{\lambda_n^-(x)}\right)\left(1 - \frac{\lambda}{\lambda_n^+(x)}\right).$$

Let $\tilde{j}_n$ be the sequence of positive zeros of $J_1'(z)$. Then (see [1], § 9.5.11)

$$\begin{aligned} \frac{\tilde{j}_n^2}{R_+^2(x)\lambda_n^+(x)} &= 1 + O\left(\frac{1}{n^2}\right), \\ \frac{-\tilde{j}_n^2}{R_-^2(x)\lambda_n^-(x)} &= 1 + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Consequently, the infinite products

$$\prod_{n\geq1}\frac{\tilde{j}_n^2}{R_+^2(x)\lambda_n^+(x)} \quad \text{and} \quad \prod_{n\geq1}\frac{-\tilde{j}_n^2}{R_-^2(x)\lambda_n^-(x)}$$

are absolutely convergent for each $x \in (x_v, x_{v+1})$. Therefore, we can write

$$S(x,\lambda) = g_v(x)\prod_{n\geq1}\frac{(\lambda - \lambda_n^-(x))R_-^2(x_1)}{\tilde{j}_n^2}\prod_{n\geq1}\frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}, \tag{4.7}$$

where

$$g_v(x) = g(x) \prod_{n\geq 1} \frac{\tilde{j}_n^2}{R_+^2(x)\lambda_n^+(x)} \prod_{n\geq 1} \frac{-\tilde{j}_n^2}{R_-^2(x_1)\lambda_n^-(x)}.$$

Theorem 4.2. *If $x_v < x < x_{v+1}$, $v = 1, 2, \ldots m$ and $x_{m+1} = 1$, then*

$$S(x,\lambda) = \frac{\pi}{8}|\varphi(x)\varphi(0)|^{-1/2}\big(R_-(x)R_+(x)\big)^{1/2} \csc\frac{\pi\mu_1}{2} \csc\pi\mu_2 \cdots \csc\pi\mu_v \times \prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_1)}{\tilde{j}_n^2} \prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}, \tag{4.8}$$

where

$$R_+(x) = \int_0^x \sqrt{max\{0, \phi^2(t)\}}dt, \quad R_-(x) = \int_0^x \sqrt{max\{0, -\phi^2(t)\}}dt,$$

the sequence $\{\lambda_n^+(x)\}$ is that of the positive eigenvalues and $\{\lambda_n^-(x)\}$ is that of the negative eigenvalues of the boundary value problem L in $[0, x]$.

Proof: By Lemmas 2 and 3 of [10], for a fixed x the infinite products

$$\prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_1)}{\tilde{j}_n^2}, \quad \prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}$$

are entire functions of λ, with sequences of zeros $\lambda_n^-(x)$ and $\lambda_n^+(x)$ $(n \geq 1)$ respectively. Moreover,

$$\prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_1)}{\tilde{j}_n^2} \prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2} = \frac{4\exp\left\{R_-(x_1)\sqrt{\lambda}\right\}}{\pi R_-^{1/2}(x_1)R_+^{1/2}(x)\sqrt{\lambda}} \left\{\cos\left(R_+(x)\sqrt{\lambda} - \pi/4\right) + O\left(\frac{1}{\sqrt{\lambda}}\right)\right\}$$

as $\lambda \to \infty$. Consequently, using of the asymptotic expansion (4.5) of $S(x, \lambda)$ we get

$$\begin{aligned} g_v(x) &= S(x,\lambda)\left(\prod_{n\geq 1} \frac{(\lambda - \lambda_n^-(x))R_-^2(x_1)}{\tilde{j}_n^2} \prod_{n\geq 1} \frac{(\lambda_n^+(x) - \lambda)R_+^2(x)}{\tilde{j}_n^2}\right)^{-1} \\ &= \frac{\pi}{8}|\varphi(x)\varphi(0)|^{-1/2}\big(R_-(x_1)R_+(x)\big)^{1/2} \csc\frac{\pi\mu_1}{2} \csc\pi\mu_2 \cdots \csc\pi\mu_v, \end{aligned}$$

which completes the proof. □

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