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POWER SERIES WITH RADIUS OF CONVERGENCE ZERO – OVERCONVERGENCE, ELONGATIONS, SUMMABILITY

W. LUH, A. STEPANYAN

Universität Trier, Trier, Germany
National Academy of Sciences of Armenia, Yerevan, Armenia [1]
E-mails: *Wolfgang.luh@gmx.de*, *arpistepanyan@gmail.com*

Abstract. We deal with overconvergence phenomena of power series with radius of convergence zero. Among others it is shown that the partial sums of such series can be elongated to become Cesàro summable on a set $S \subset \{z : |z| > 0\}$ if and only if the considered power series is overconvergent.

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1. Introduction

We consider a power series

$$(1.1)\qquad \sum_{\nu=0}^{\infty} a_\nu z^\nu \quad \text{with} \quad \overline{\lim_{\nu\to\infty}} |a_\nu|^{1/\nu} = \infty$$

which has radius of convergence zero. Its partial sums

$$(1.2)\qquad s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

are divergent for all $z \neq 0$ and quantitatively we have

$$(1.3)\qquad \overline{\lim_{n\to\infty}} \left|s_n(z)\right|^{1/n} = \infty \qquad \text{for all } z \neq 0.$$

We define C^∞ as the family of all functions $f : \mathbb{R} \to \mathbb{C}$ which are infinitely differentiable on $\mathbb{R}$. Then, according to a classical result of E. Borel (see. e. g. [6], page 191 or [7], page 102) for any sequence $\{\alpha_\nu\}$ of complex numbers there exists a function $\varphi \in C^\infty$

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with the property that its sequence of derivatives satisfies $\varphi^{(\nu)}(0) = \alpha_\nu$ for all $\nu \in \mathbb{N}_0$. Hence, there is a function $g \in C^\infty$ such that the power series (1.1) can be written as

$$\sum_{\nu=0}^{\infty} \frac{g^{(\nu)}(0)}{\nu!} z^\nu$$

and therefore represents the Taylor series of g. Of course, this series has no analytic properties around the origin.

However, among others, we show in this paper that certain power series of the type (1.1) may – by applying simple analytic operations to their partial sums – generate holomorphic (and other) behavior on sets in $|z| > 0$. Our idea is, to modify the sequence of partial sums by „elongation" and to apply a summability method to the new sequence.

We say that a sequence $\{s_n\}$ of complex numbers is being elongated with respect to a sequence $m = \{m_n\}$ of natural numbers if the term s_n is listed m_n-times, i. e. the modified sequence

$$\underbrace{s_0, \ldots, s_0}_{m_0-\text{times}}, \underbrace{s_1, \ldots, s_1}_{m_1-\text{times}}, \ldots, \underbrace{s_k, \ldots, s_k}_{m_k-\text{times}}, \ldots$$

is considered. Occasionally we denote this new sequence as the m-elongation of $\{s_n\}$. It is clear that $\{s_n\}$ converges if and only if any m-elongation converges. This may change drastically when instead of convergence, summability properties are considered.

We mainly concentrate ourselves to the C_1 means (arithmetical means) which transform the partial sums (1.2) into

$$\sigma_n(z) = \frac{1}{n+1} \cdot \sum_{\nu=0}^{n} s_\nu(z). \tag{1.4}$$

This sequence again diverges for all $z \neq 0$ and simple estimations show that (in analogy to the behavior of partial sums) the following holds:

$$\overline{\lim_{n\to\infty}} |\sigma_n(z)|^{1/n} = \infty \qquad \text{for all } z \neq 0.$$

It is the object of this paper:

- To demonstrate in part 2 that certain power series of type (1.1) may show overconvergence phenomena with respect to certain notions of convergence.
- To prove in the main parts 3 and 4 that special elongations of the partial sums of those series may be C_1 summable in sets, contained in $|z| > 0$ and

that this property occurs if and only if the power series under consideration is overconvergent.

2. Overconvergence

A power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence zero is called overconvergent, if there exist a set $S \subset \{z : |z| > 0\}$ and a subsequence $\{s_{p_k}(z)\}$ of its partial sums which converges on S. Such a sequence is called an overconvergent subsequence of the consideres series. (There are other definitions of overconvergence also.)
In this part of our paper we construct power series which have different overconvergence behavior with respect to certain convergence concepts.
For the construction of those series the following Lemma is useful.

Lemma 2.1. *Let K be a compact set with connected complement, $0 \notin K$, a number $q \in \mathbb{N}$, a holomorphic function F on K and $\varepsilon > 0$. Then there exists a polynomial of the type*

$$P(z) = \sum_{\nu=q}^{Q} a_\nu z^\nu$$

with

$$\max_K |F(z) - P(z)| < \varepsilon.$$

Proof. The function $F(z)/z^q$ is holomorphic on K and by Runge's approximation theorem there exists a polynomial $Q(z) = \sum_{\nu=0}^{m} a_\nu z^\nu$ with

$$\max_K \left| Q(z) - \frac{F(z)}{z^q} \right| < \frac{\varepsilon}{\max_K |z^q|}.$$

Then $P(z) := z^q Q(z)$ has the required form and we get

$$\max_K |P(z) - F(z)| < \varepsilon.$$

Lemma 2.1 is proved. □

In the following results we describe various overconvergence phenomena which deal as basis for the main results in parts 3 and 4.

Theorem 2.1. *Let an open set $\mathcal{O}$ with simply connected components be given, $0 \notin \mathcal{O}$ and a function f which is holomorphic on $\mathcal{O}$. Then there exists a power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence zero and a sequence $\{p_k\}$, such that the subsequence $\{s_{p_k}(z)\}$ of partial sums converges compactly to $f(z)$ on $\mathcal{O}$.*

Proof. 1. We have $\mathcal{O} = \bigcup_{j \in J} G_j$ where the set $J = \{1, 2, \ldots\}$ is finite or countable and the G_j are pairwise disjoint simply connected domains. For each $j \in J$ we can choose a sequence of Jordan domains $\{G_j^{(k)}\}$ exhausting G_j, which means that $G_j^{(k)} \subset \overline{G_j^{(k)}} \subset G_j^{(k+1)} \subset G_j$ for all $k \in \mathbb{N}$ and that a compact set $B \subset G_j$ is contained in some $G_j^{(k_0)}$.

We consider $K_k := \bigcup_{\substack{j \in J \\ j \leq k}} G_j^{(k)}$ and a point $z_k \notin K_k$ with $0 < |z_k| < \frac{1}{k}$. Then $K_k \cup \{z_k\}$ is compact with connected complement and does not contain the origin.

2. Now we construct a sequence of polynomials and a strictly increasing sequence of integers by induction. Suppose that $P_0(z) = z, q_0 = 0$ and that for a $k \in \mathbb{N}$ the polynomials

$$P_0(z) = z, \ldots, P_{k-1}(z) = \sum_{\nu = q_{k-1}+1}^{q_k} a_\nu z^\nu$$

are already determined. Then, by Lemma 2.1 there exists a polynomials P_k of the form

$$P_k(z) = \sum_{\nu = q_k + 1}^{q_{k+1}} a_\nu z^\nu$$

with

$$\max_{K_k} \Big| P_k(z) - \{f(z) - \sum_{\nu=0}^{k-1} P_\nu(z)\} \Big| < \frac{1}{k} \tag{2.1}$$

and

$$\big| P_k(z_k) - 2 \big| < 1. \tag{2.2}$$

3. We consider the power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ and with the abbreviation $p_k = q_{k+1}$ we get (observe that there are no overlappings of the coefficients in the polynomials):

$$s_{p_k}(z) := \sum_{\nu=0}^{p_k} a_\nu z^\nu = \sum_{\nu=0}^{k} P_\nu(z).$$

Let be given any compact set $K \subset \mathcal{O}$, then there exists a k_0 with $K \subset K_k$ for all $k \geq k_0$. Then, for $k \geq k_0$, we obtain by (2.1)

$$\max_K \big| s_{p_k}(z) - f(z) \big| \leq \max_{K_k} \big| s_{p_k}(z) - f(z) \big| = \max_{K_k} \Big| \sum_{\nu=0}^{k} P_\nu(z) - f(z) \Big| < \frac{1}{k}.$$

Since $K \subset \mathcal{O}$ was arbitrary, we obtain that $\{s_{p_k}(z)\}$ converges on $\mathcal{O}$ compactly to $f(z)$.

4. The constructed power series has radius of convergence zero. Otherwise there would exist an $r > 0$ such that $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ is uniformly convergent on $|z| \leq r$. We have $0 < |z_k| < r$ for sufficiently great k and it follows from (2.2)

$$\Big|\sum_{\nu=q_k+1}^{q_{k+1}} a_\nu z_k^\nu\Big| = \big|P_k(z_k)\big| > 1,$$

which contradicts Cauchy's criterion for uniform convergence. □

Remark 2.1. *Theorem 2.1 can be derived from a result due to Seleznev [5] (see also Große-Erdmann [1, Theorem 17]) on universal elements. We prefered to give an elementary and direct constructive proof.*

In the next result we deal with pointwise or uniform overconvergence.

Theorem 2.2. *There exists a power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence zero and a sequence $\{p_k\}$, such that the subsequence $\{s_{p_k}(z)\}$ of partial sums converges to zero*

(a) *pointwise for all $z \in \mathbb{C}$,*

(b) *uniformly on each compact set which does not intersect $[0, \infty)$,*

(c) *uniformly on each compact set contained in $(0, \infty)$.*

Proof. We consider the compact set

$$K_k := \Big\{z : \frac{1}{k} \leq |z| \leq k\Big\} \setminus \Big\{z : \mathrm{Re}\,(z) > 0, 0 < \mathrm{Im}\,(z) < \frac{1}{k}\Big\}$$

which has connected complement and construct a sequence of polynomials $\{P_k\}$ and a strictly increasing sequence $\{q_k\}$ of integers by induction. Suppose that $P_0(z) = z, q_0 = 0$, and that for $k \in \mathbb{N}$

$$P_0(z) = z, \ldots, P_{k-1}(z) = \sum_{\nu=q_{k-1}+1}^{q_k} a_\nu z^\nu$$

have already been determined. Then, by Lemma 2.1 we find a polynomial

$$P_k(z) = \sum_{\nu=q_k+1}^{q_{k+1}} a_\nu z^\nu$$

with

$$\max_{K_k}\Big|P_k(z) + \sum_{\nu=0}^{k-1} P_\nu(z)\Big| < \frac{1}{k}, \tag{2.3}$$

and

$$\left|P_k\left(\frac{1}{2k}\right) - 2\right| < 1. \tag{2.4}$$

We consider the power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$. If we abbreviate $p_k = q_{k+1}$ then we have

$$s_{p_k}(z) = \sum_{\nu=0}^{p_k} a_\nu z^\nu = \sum_{\nu=0}^{k} P_\nu(z).$$

Let a point $z \in \mathbb{C}$ be given. If $z = 0$ then $s_{p_k}(0) = 0$ for all k and for $z \neq 0$ there exists a k_0 with $z \in K_k$ for all $k \geq k_0$; it follows from (2.3) that $\{s_{p_k}(z)\}$ converges to zero.

If K is a compact set with $K \cap [0, \infty) = \emptyset$ or $K \subset (0, \infty)$, then in both cases there is a k_1 with $K \subset K_k$ for all $k \geq k_1$ and it follows again from (2.3) that $\{s_{p_k}(z)\}$ converges on K uniformly to zero.

As in step 4 in the proof of Theorem 2.1 it follows from (2.4) that the constructed power series cannot have a positive radius of convergence. □

Remark 2.2. *If f is an entire function with the power series $\sum_{\nu=0}^{\infty} f_\nu z^\nu$, then $\sum_{\nu=0}^{\infty} (a_\nu + f_\nu) z^\nu$ still has radius of convergence zero and its sequence of partial sums $\left\{\sum_{\nu=0}^{p_k} (a_\nu + f_\nu) z^\nu\right\}$ converges to $f(z)$ for all $z \in \mathbb{C}$ and uniformly on each compact set K with $K \cap [0, \infty) = \emptyset$ or $K \subset (0, \infty)$.*

A result, similar to Theorem 2.1 also holds for power series with positive radius of convergence which we mention without proof.

Theorem 2.3. *There exists a power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence 1 and a sequence $\{p_k\}$ such that the subsequence $\{s_{p_k}(z)\}$ of partial sums converges to zero*

(a) *pointwise for all $z, |z| \geq 1$,*
(b) *uniformly on each compact set $K \subset \{z : |z| \geq 1\}$ which does not intersect $[1, \infty)$,*
(c) *uniformly on each compact set $K \subset (1, \infty)$.*

There also exists a result for measurable functions on measurable sets which easily follows from a theorem of Tomm und Trautner [7], where even much more has been proved.

Theorem 2.4. *Let $M \subset \mathbb{C}$ be a (Lebesgue-)measurable set and let the function $f : M \to \mathbb{C}$ be measurable on M. Then there exists a power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence zero and a sequence $\{p_k\}$ such that the subsequence $\{s_{p_k}(z)\}$ of partial sums converges to $f(z)$ almost everywhere on M.*

To see this, consider the function

$$g(z) := \begin{cases} f(z), & \text{if } z \in M \\ 0, & \text{if } z \notin M \end{cases}$$

which is measurable on $\mathbb{C}$ and it suffices to construct a power series with radius of convergence zero with an almost everywhere convergent subsequence to the limit $g(z)$. But the existence of such a series follows from Theorem 1 in the above mentioned paper [7].

3. Elongations

We now show that overconvergence phenomena permit the elongation of partial sums to become C_1 summable throughout the set of overconvergence. Then, in part 4 it is shown – as a converse result – that such an elongation only can exist if the considered power series is overconvergent.

Theorem 3.1. *Let $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence zero and partial sums $s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$. Suppose that there exists an overconvergent subsequence $\{s_{p_k}(z)\}$ which is on a set $S \subset \{z : |z| > 0\}$*

$$\left.\begin{array}{ll} & \textit{compactly convergent} \\ \textit{or} & \textit{uniformly convergent} \\ \textit{or} & \textit{pointwise convergent} \\ \textit{or} & \textit{almost everywhere convergent} \end{array}\right\} \textit{ to a limit } f(z).$$

Then there exists an elongation of $\{s_n(z)\}$ such that its C_1 transforms have the same convergence properties as $\{s_{p_k}(z)\}$ on S.

Proof. Let be given such a set S where $\{s_{p_k}(z)\}$ satisfies one of the assumptions.
1. We consider the sequence $\{p_k\}$ and for every $k \geq 1$ we choose

$$m_k := k \cdot (p_{k+1} + 1) \cdot \max_{0 \leq \nu \leq p_{k+1}} \left\{ \max_{|z|=k} |s_\nu(z)| \right\}. \tag{3.1}$$

we elongate the sequence $\{s_n(z)\}$ to the sequence $\{\tilde{s_n}(z)\}$ where the partial sums $s_{p_k}(z)$ for $k \geq 1$ are listed $m_k + 1$ times, i. e. we consider the sequence

$$s_0, \ldots, s_{p_1-1}, s_{p_1}, \underbrace{s_{p_1}, \ldots, s_{p_1}}_{m_1-\text{times}}, s_{p_1+1}, \ldots, s_{p_k-1}, s_{p_k}, \underbrace{s_{p_k}, \ldots, s_{p_k}}_{m_k-\text{times}}, \ldots .$$

Any natural $n \geq p_1$ has a representation of the form

$$n = p_k + \sum_{\nu=1}^{k} m_\nu + \vartheta, \quad \text{where } 1 \leq \vartheta \leq p_{k+1} - p_k, \tag{3.2}$$

or

$$n = p_{k+1} + \sum_{\nu=1}^{k} m_\nu + \gamma, \quad \text{where } 1 \leq \gamma < m_{k+1}. \tag{3.3}$$

2. If n has the form (3.2), then we get

$$\begin{aligned} \sigma_n(z) &= \frac{1}{n+1} \cdot \sum_{\nu=0}^{n} \tilde{s}_\nu(z) = \\ &= \frac{1}{n+1} \cdot \sum_{\nu=0}^{p_k+\vartheta} s_\nu(z) - \frac{1}{n+1}(p_k + 1 + \vartheta)\, s_{p_{k+1}}(z) + \\ &+ \frac{1}{n+1} \cdot \Big\{ \sum_{\nu=1}^{k} m_\nu\, s_{p_\nu}(z) + (p_k + 1 + \vartheta)\, s_{p_{k+1}}(z) \Big\}. \end{aligned}$$

The last term of this identity is exactly the C_1 mean of a sequence elongated along $\{s_{p_k}(z)\}$ and therefore has (by the regularity of the C_1 transform) the same convergence properties as $\{s_{p_k}(z)\}$.
By (3.1) we get for the remaining terms and all sufficiently great k

$$\max_{|z| \leq k} \Big| \frac{1}{n+1}(p_k + 1 + \vartheta)\, s_{p_{k+1}}(z) \Big| \leq$$

$$\leq \frac{p_k + 1 + \vartheta}{p_k + \sum_{\nu=1}^{k} m_\nu + \vartheta + 1} \cdot \max_{|z| \leq k} |s_{p_{k+1}}(z)| \leq$$

$$\leq \frac{p_{k+1} + 1}{m_k} \max_{|z| \leq k} |s_{p_{k+1}}(z)| \leq \frac{1}{k}$$

and

$$\max_{|z|\le k}\left|\frac{1}{n+1}\sum_{\nu=0}^{p_k+\vartheta} s_\nu(z)\right| \le$$

$$\le \frac{p_k+\vartheta+1}{p_k+\sum_{\nu=1}^{k} m_\nu+\vartheta+1}\cdot \max_{\nu\le p_k+\vartheta}\left\{\max_{|z|\le k}|s_\nu(z)|\right\} \le$$

$$\le \frac{p_{k+1}+1}{m_k}\cdot \max_{\nu\le p_{k+1}}\left\{\max_{|z|\le k}|s_\nu(z)|\right\} = \frac{1}{k}.$$

Therefore these terms are compactly convergent to zero on $\mathbb{C}$.

3. If n has the form (3.3) then we get

$$\begin{aligned}\sigma_n(z) &= \frac{1}{n+1}\cdot\sum_{\nu=0}^{n}\tilde{s}_\nu(z) = \\ &= \frac{1}{n+1}\sum_{\nu=0}^{p_{k+1}} s_\nu(z) - \frac{1+p_{k+1}}{n+1}\cdot s_{p_{k+1}}(z) + \\ &+ \frac{1}{n+1}\cdot\left\{\sum_{\nu=1}^{k} m_\nu s_{p_\nu}(z) + \left(n+1-\sum_{\nu=1}^{k} m_\nu\right) s_{p_{k+1}}(z)\right\}.\end{aligned}$$

Again, the last term has the same convergence properties as $\{s_{p_k}(z)\}$ and the remaining terms are estimated in a similar way as in step 2. □

Remark 3.1. *The Cesàro means C_α of order $\alpha \ge 1$ of the partial sums (1.2) are given by*

$$\sigma_n^{(\alpha)}(z) = \frac{1}{\binom{n+\alpha}{n}}\cdot\sum_{\nu=0}^{n}\binom{n-\nu+\alpha-1}{n-\nu} s_\nu(z).$$

For $\alpha = 1$ we obtain the above considered arithmetical means. It is well known that for $1 \le \alpha_1 < \alpha_2$ the C_{α_2} transform is stronger than C_{α_1}. So, Theorem 3.1 holds also for all Cesàro means C_α of order $\alpha \ge 1$ and in addition for all Hölder means H_α of order $\alpha \ge 1$ since H_α is equivalent to C_α. Moreover, Theorem 3.1 remains true for all summability methods which are stronger than the C_1 means. For a number of those examples see for instance [3, chapter 2] or [4, chapter II, 6.1].

4. THE CONVERSE RESULT

We now show, that elongations as in Theorem 3.1 are possible only in the case that the power series under consideration is overconvergent. For the proof the following Lemma is essential.

Lemma 4.1. *Let $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ be a power series with radius of convergence zero. Suppose that there exists an m-elongation of the partial sums $s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ such that their C_1 transforms are bounded at a point z_0 with $|z_0| > 1$. Then $\left\{\frac{M_n}{m_n}\right\}$ has a bounded subsequence, where $M_n := \sum_{\nu=0}^{n} m_\nu$.*

Proof. 1. Let $\{t_n\}$ be a sequence with the property that its C_1 transforms

$$\tau_n := \frac{1}{n+1}\sum_{\nu=0}^{n} t_\nu$$

are bounded. We have $t_n = (n+1)\,\tau_n - n\,\tau_{n-1}$ and it follows that $\left\{\frac{t_n}{n}\right\}$ is bounded.
2. Let the m-elongation $\{\widetilde{s}_k(z)\}$ of $\{s_n(z)\}$ have the form

$$\underbrace{s_0(z),\ldots,s_0(z)}_{m_0-\text{times}},\ldots,\underbrace{s_n(z),\ldots,s_n(z)}_{m_n-\text{times}},\ldots.$$

Then by the assumption, the C_1 transforms of $\{\widetilde{s}_k(z)\}$ are bounded at z_0. This implies by step 1 that $\left\{\frac{\widetilde{s}_k(z_0)}{k}\right\}$ is bounded. Since $\widetilde{s}_{M_n}(z_0) = s_n(z_0)$ we obtain with a positive constant C:

$$\frac{|s_n(z_0)|}{M_n} \le C \qquad \text{or} \qquad |s_n(z_0)| \le C\cdot M_n\,.$$

3. Now choose any fixed $R > 1$. If $\left\{\frac{M_n}{m_n}\right\}$ would not have a bonded subsequence, then $\frac{M_n}{m_n} \to \infty$ and there exists an n_0 such that $M_n \ge R\cdot m_n$ for all $n \ge n_0$ and it follows for those n

$$M_n - M_{n-1} = m_n \le \frac{1}{R}\,M_n$$

or

$$M_n \le S\cdot M_{n-1}, \quad \text{where } S = \frac{R}{R-1} > 1.$$

It follows

$$M_n \le S\cdot M_{n-1} \le S^2\cdot M_{n-2} \le \ldots \le S^{n-n_0}\cdot M_{n_0}.$$

We therefore have $|s_n(z_0)| \le C \cdot S^{n-n_0} \cdot M_{n_0}$ which implies

$$\varlimsup_{n\to\infty} |s_n(z_0)|^{1/n} \le S$$

in contradiction to (1.3). □

As a corollary one obtains now easily:

Theorem 4.1. *Let* $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ *be a power series with radius of convergence zero and suppose that there exists an m-elongation of the partial sum* $s_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *whose* C_1 *transforms are on a set* $S \subset \{z : |z| > 0\}$*:*

$$\left.\begin{array}{ll} & \textit{compactly convergent} \\ \textit{or} & \textit{uniformly convergent} \\ \textit{or} & \textit{pointwise convergent} \\ \textit{or} & \textit{almost everywhere convergent} \end{array}\right\} \textit{to a limit } f(z).$$

Then there exists a subsequence $\{s_{p_k}(z)\}$ *of the partial sums having the same convergence properties with the same limit on* S.

Proof. According to Lemma 4.1 there exists a sequence $\{p_k\} \subset \mathbb{N}$ such that $\left\{\frac{M_{p_k}}{m_{p_k}}\right\}$ is bounded.

We consider the following subsequence of the C_1 transforms:

$$\rho_n(z) = \frac{1}{M_n} \cdot \sum_{\nu=0}^{n} m_\nu s_\nu(z)$$

(of course, with the corresponding convergence properties on the set S) and obtain

$$M_n \cdot \rho_n(z) = \sum_{\nu=0}^{n} m_\nu s_\nu(z),$$

$$M_{n-1} \cdot \rho_{n-1}(z) = \sum_{\nu=0}^{n-1} m_\nu s_\nu(z),$$

which gives us

$$\begin{aligned} m_n \cdot s_n(z) &= M_n \cdot \rho_n(z) - M_{n-1} \cdot \rho_{n-1}(z) = \\ &= M_n \cdot \{\rho_n(z) - \rho_{n-1}(z)\} + m_n \cdot \rho_{n-1}(z), \\ s_n(z) &= \frac{M_n}{m_n} \cdot \{\rho_n(z) - \rho_{n-1}(z)\} + \rho_{n-1}(z). \end{aligned}$$

Setting now $n = p_k$ we find that $\{s_{p_k}(z)\}$ has on S the same convergence properties as $\{\rho_n(z)\}$ which proves the theorem. □

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