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HOMOCLINIC ORBITS FOR SECOND ORDER NONLINEAR p-LAPLACIAN DIFFERENCE EQUATIONS

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Abstract. The paper proves the existence of nontrivial homoclinic orbits for second order nonlinear p-Laplacian difference equations without on assumptions on pereodicity using the critical point theory. Moreover, if the nonlinearity is an odd function, the existence of an unbounded sequence of nontrivial homoclinic orbits is proved.

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1. Introduction

Below **N**, **Z** and **R** denote the sets of all naturals, integers and real numbers respectively. For any $a,\ b \in \mathbf{Z}$, we denote $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \le b$. Besides, l^p denotes the space of all real functions whose pth powers are summable over **Z**.

The present paper considers the existence of a nontrivial homoclinic orbit for the following p-Laplacian difference equation

$$(1.1)\quad \Delta(\varphi_p(\Delta u(t-1))) - \varphi_p(u(t)) = \lambda(t) f(t, u(t+1), u(t), u(t-1)),\ t \in \mathbf{Z},$$

where Δ is the forward difference operator $\Delta u(t) = u(t+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$, $\varphi_p(s)$ is the p-Laplacian operator $\varphi_p(s) = |s|^{p-2}s (p \ge 2)$, $\lambda \in C(\mathbf{Z}, \mathbf{R})$ and $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$.

Equation (1.1) can be considered as a discrete analog of the second order p-Laplacian functional differential equation

$$(1.2)\qquad [\varphi_p(u')]' - \varphi_p(u) = \lambda(s) f(s, u(s+1), u(s), u(s-1)),\ s \in \mathbf{R},$$

where $f \in C(\mathbf{R}^4, \mathbf{R})$. Equation (1.2) includes the following equation

$$(1.3)\qquad c^2 u''(s) = V'(u(s+1) - u(s)) - V'(u(s) - u(s-1)),\ s \in \mathbf{R}.$$

Equations similar to (1.3) have been studied by many researchers. For example, using a version of the Mountain Pass Theorem, Smets and Willem [27] have proved the existence of solitary waves with prescribed speed on infinite lattices of particles with nearest neighbor interaction for (1.3).

Recently, the theory of nonlinear difference equations has been widely used in the study of discrete models appearing in many fields such as computer science, economics, neural network, ecology, cybernetics see [1, 14, 15, 17, 20]. For example, the simple logistic equation $u_{n+1} = ru_n$ is a formula for approximating the evolution of an animal population over time, where u_n is the number of animals this year, u_{n+1} is the number in the next year and r is the growth rate or fecundity. The price-demand curve of cobweb phenomenon

$$D_n = -m_d p_n + b_d, \ m_d > 0, \ b_d > 0$$

is the economics application of difference equations, where D_n is the number of units demanded in period n, p_n is the price per unit in period n and m_d represents the sensitivity of consumers to price.

In the theory of differential equations, a trajectory which is asymptotic to a constant state as $|s| \to \infty$ (s denotes the time variable) is called a homoclinic orbit. Such orbits have been found in various models of continuous dynamical systems and frequently have tremendous effects on the dynamics of such nonlinear systems. So the homoclinic orbits have been extensively studied since the time of Poincaré, see [4, 6, 9, 13, 16, 21, 23, 25, 28] and the references therein. Similarly, we give the following definition: *if $\bar{x}$ is a solution of a discrete system, another solution z will be called a homoclinic orbit emanating from $\bar{x}$ if $|z(t) - \bar{x}| \to 0$ when $t \to \pm\infty$.*

Homoclinic orbits of dynamical systems are important in applications for a number of reasons. They may be "organising centres" for the dynamics in their neighbourhood. From their existence one can under certain conditions, infer the existence of chaos nearby or the bifurcation behaviour of periodic orbits. In the past two decades many authors studied homoclinic orbits for dynamical systems via critical point theory. Here we only mention [6, 9, 16, 21, 28]. In particular, the second order systems were considered in [6, 16].

It is well-known that critical point theory is powerful tool to deal with the problems for differential equations [18, 19, 24, 29]. Only since 2003, the critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. In particular, Yu, Shi, Chen and their collaborators considered the existence of periodic solutions of second order nonlinear difference equations [8, 10 - 12, 26, 30, 31]. In the recent paper of Cabada, Li and Tersian [7] the existence of homoclinic solutions for semilinear p-Laplacian difference equations with periodic coefficients is studied. However, to our best knowledge, the results on the homoclinic orbits of p-Laplacian difference equations for (1.1) obtained in the literature are very scarce. Since f in (1.1) depends on $u(t+1)$ and $u(t-1)$, the traditional ways of establishing the functional in [2, 3, 10 - 12, 22, 30, 31] are inapplicable to our case. The main purpose of this paper is to give some sufficient conditions for the existence of a nontrivial homoclinic orbit. What is more, if $f(t,\cdot)$ is an odd function for any $t \in \mathbf{Z}$, the existence of an unbounded sequence of nontrivial homoclinic orbits is obtained. The main approach used in our paper is variational techniques and the notable Mountain Pass Lemma introduced by Ambrosetti and Rabinowitz [5, 24]. Now we state the main results of this paper.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*
(λ) $\lambda(t) > 0$ *for all* $t \in \boldsymbol{Z}$ *and* $\sum\limits_{t=-\infty}^{+\infty} \lambda^q(t) < +\infty$, *where* $\frac{1}{p} + \frac{1}{q} = 1$;
(f_1) *there exists a functional* $g(v_1, v_2, v_3) \in C(\boldsymbol{R}^3, \boldsymbol{R})$ *such that*

$$\lim_{r\to 0} \frac{g(v_1,v_2,v_3)}{v_2} = 0, \ r = \sqrt{v_1^2 + v_2^2 + v_3^2},$$

$$|f(t,v_1,v_2,v_3)| \le |g(v_1,v_2,v_3)|, \quad \text{for all} \quad t \in \boldsymbol{Z};$$

(F_1) *there exists a functional* $F(t, v_1, v_2) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *with* $F(t, v_1, v_2) \le 0$ *and it satisfies the conditions*

$$\frac{\partial F(t-1,v_2,v_3)}{\partial v_2} + \frac{\partial F(t,v_1,v_2)}{\partial v_2} = f(t,v_1,v_2,v_3),$$

$$\lim_{\rho\to 0} \frac{F(t,v_1,v_2)}{\rho^p} = 0$$

uniformly for $t \in \boldsymbol{Z}$, $\rho = \sqrt{v_1^2 + v_2^2}$;
(F_2) *there exists a constant* $\beta > p$ *such that*

$$\frac{\partial F(t,v_1,v_2)}{\partial v_1} v_1 + \frac{\partial F(t,v_1,v_2)}{\partial v_2} v_2 \le \beta F(t,v_1,v_2) < 0,$$

for all $(t, v_1, v_2) \in \boldsymbol{Z} \times \boldsymbol{R}^2 \setminus \{(0,0)\}$.
Then equation (1.1) has a nontrivial homoclinic orbit.

Remark 1.1. *The above hypotheses imply that* $u(t) \equiv 0$ *is a trivial solution of (1.1).*

Remark 1.2. *The assumption* (F_2) *implies that for any nonempty finite integer interval* $I \subset \boldsymbol{Z}$ *there exist constants* $a > 0$ *and* $R > 0$ *such that*
(F_2') $F(t,v_1,v_2) \le -a\left(\sqrt{v_1^2+v_2^2}\right)^\beta$, *for all* $t \in I$ *and* $v_1^2 + v_2^2 \ge R^2$.

If $f(t,\cdot)$ is an odd function, we show that (1.1) has an unbounded sequence of nontrivial homoclinic orbits.

Theorem 1.2. *Suppose that the conditions* (λ), (f_1), (F_1) *and* (F_2) *are satisfied. If*

(f_2) $f(t, -v_1, -v_2, -v_3) = -f(t, v_1, v_2, v_3)$,
for all $(t, v_1, v_2, v_3) \in \boldsymbol{Z} \times \boldsymbol{R}^3$, *then equation (1.1) has an unbounded sequence of nontrivial homoclinic orbits.*

2. Variational structure and some lemmas

In order to apply the critical point theory, we establish variational framework corresponding to (1.1) and give some lemmas which will be of fundamental importance in proving our main results. We start by some basic notation.

Let S be the vector space of all real sequences of the form

$$u = \{u(t)\}_{t\in\mathbf{Z}} = (\cdots, u(-t), \cdots, u(-1), u(0), u(1), \cdots, u(t), \cdots),$$

namely

$$S = \{\{u(t)\}|u(t) \in \mathbf{R},\ t \in \mathbf{Z}\}.$$

Define

$$E = \left\{ u \in S| \sum_{t=-\infty}^{+\infty} [|\Delta u(t-1)|^p + |u(t)|^p] < +\infty,\ t \in \mathbf{Z} \right\}.$$

Note that E is a Banach space with the norm

$$\|u\|_E = \left\{ \sum_{t=-\infty}^{+\infty} [|\Delta u(t-1)|^p + |u(t)|^p] \right\}^{\frac{1}{p}} < +\infty,\ \ u \in E. \tag{2.1}$$

For all $u \in E$ define the functional J as follows:

$$J(u) := \sum_{t=-\infty}^{+\infty} \left[\frac{1}{p}|\Delta u(t-1)|^p + \frac{1}{p}|u(t)|^p + \lambda(t)F\left(t, u(t+1), u(t)\right) \right]$$

$$= \frac{1}{p}\|u\|_E^p + \sum_{t=-\infty}^{+\infty} \lambda(t)F(t, u(t+1), u(t)), \tag{2.2}$$

where

$$\frac{\partial F(t-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(t, v_1, v_2)}{\partial v_2} = f(t, v_1, v_2, v_3).$$

The functional J is well-defined C^1 functional on E and Equation (1.1) is easily recognized as the corresponding Euler-Lagrange equation for J. Therefore, we are to look for nonzero critical points of J.

Five lemmas should be stated, which will be used in the proof of our main results. Firstly, let us recall the Palais-Smale condition.

Let E be a real Banach space and $J \in C^1(E, \mathbf{R})$, that is J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition

(P.S. condition for short) if any sequence $\{u(t)\} \subset E$ for which $\{J(u(t))\}$ is bounded and $J'(u(t)) \to 0 (t \to \infty)$ possesses a convergent subsequence in E.
Let B_ρ denote the open ball in E with radius ρ and centered at 0 and let ∂B_ρ denote its boundary.

Lemma 2.1. *(Mountain Pass Lemma [5, 24]). Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*

(J_1) *there exist constants $\rho,\ \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$,*

(J_2) *there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$,*

then J possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} J(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}.$$

Lemma 2.2. *(Symmetric Mountain Pass Lemma [24]). Let E be an infinite dimensional Banach space and let $J \in C^1(E, \boldsymbol{R})$ be even, satisfy the P.S. condition and $J(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and J satisfies the following conditions*

(J_3) *there exist constants $\rho,\ \alpha > 0$ such that $J|_{\partial B_\rho \cap X} \geq \alpha$,*

(J_4) *for each finite dimensional subspace $\tilde{E} \subset E$, there is a $\gamma = \gamma(\tilde{E})$ such that $J \leq 0$ on $\tilde{E} \backslash B_\gamma$,*

then J possesses an unbounded sequence of critical values.

Lemma 2.3. *The following inequalities are true:*

$$\|u\|_{l^p} \leq \|u\|_E, \tag{2.3}$$

$$\|u\|_\infty \leq \|u\|_E, \tag{2.4}$$

where

$$\|u\|_{l^p} = \left[\sum_{t=-\infty}^{+\infty} |u(t)|^p\right]^{\frac{1}{p}} \quad \textit{and} \quad \|u\|_\infty = \sup_{t \in \boldsymbol{Z}} |u(t)|.$$

Proof. By the definition of $\|\cdot\|_{l^p}$, $\|\cdot\|_\infty$ and $\|\cdot\|_E$, inequalities (2.3) and (2.4) follow immediately. □

Lemma 2.4. *For any $x > 0$, the following inequality holds*

$$|x^p - 1| \geq |x - 1|^p, \quad \textit{where} \quad p \geq 1.$$

Proof is obvious. □

Lemma 2.5. *If the conditions (λ), (f_1), (F_1) and (F_2) are satisfied, then J satisfies the P.S. condition.*

Proof. Let $\{u_k\} \subset E$ be such that $\{J(u_k)\}$ is bounded and $J'(u_k) \to 0$ as $k \to \infty$. Then there exists a positive constant K such that $|J(u_k)| \leq K$. Thus by (2.3) and (F_2), we obtain that for k large enough

$$K + \|u_k\|_E \geq J(u_k) - \frac{1}{\beta}\langle J'(u_k), u_k\rangle$$

$$= \left(\frac{1}{p} - \frac{1}{\beta}\right) \| u_k \|_E^p + \sum_{t=-\infty}^{+\infty} \lambda(t) F(t, u_k(t+1), u_k(t))$$

$$- \frac{1}{\beta} \sum_{t=-\infty}^{+\infty} \left[\lambda(t-1) \frac{\partial F(t-1, u_k(t), u_k(t-1))}{\partial v_2} u_k(t) + \lambda(t) \frac{\partial F(t, u_k(t+1), u_k(t))}{\partial v_2} u_k(t)\right]$$

$$= \left(\frac{1}{p} - \frac{1}{\beta}\right) \| u_k \|_E^p + \sum_{t=-\infty}^{+\infty} \lambda(t) F(t, u_k(t+1), u_k(t))$$

$$- \frac{1}{\beta} \sum_{t=-\infty}^{+\infty} \lambda(t) \left[\frac{\partial F(t, u_k(t+1), u_k(t))}{\partial v_1} u_k(t+1) + \frac{\partial F(t, u_k(t+1), u_k(t))}{\partial v_2} u_k(t)\right]$$

$$\geq \left(\frac{1}{p} - \frac{1}{\beta}\right) \| u_k \|_E^p .$$

Since $\beta > p$, it is not difficult to conclude that $\{u_k\}$ is a bounded sequence in E, i.e. there exists a constant $C_1 > 0$ such that $\|u_k\|_E \leq C_1$. So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E. Moreover, since

$$\|u_0\|_E = \sup_{0 \neq h \in E'} \frac{|h(u_0)|}{\|h\|_E} = \sup_{0 \neq h \in E'} \frac{\left|\liminf\limits_{k\to\infty} h(u_k)\right|}{\|h\|_E} \leq \liminf_{k\to\infty} \|u_k\|_E, \tag{2.5}$$

we conclude that $\|u_0\|_E$ is bounded and $\|u_0\|_E \leq C_1$. By (2.5) we have $|u_k(t)| \leq C_1$ and $|u_0(t)| \leq C_1$, for all $t \in \mathbf{Z}$, and by (f_1), there exists a constant $C_2 > 0$ such that

$$|f(t, u_k(t+1), u_k(t), u_k(t-1)) - f(t, u_0(t+1), u_0(t), u_0(t-1))| \leq C_2, \quad t \in \mathbf{Z}.$$

Since $\sum\limits_{t=-\infty}^{+\infty} \lambda^q(t) < +\infty$ for all $\epsilon > 0$ there exists $D \in \mathbf{N}(D > 1)$ such that

$$\left(\sum_{|t|>D} \lambda^q(t)\right)^{\frac{1}{q}} < \epsilon.$$

For this D, we define a $2(D+1)$ dimensional subspace of E:

$$E_{D+1} = \{u = \{u(t)\} \in E : \ u(t) = 0, \ |t| > D+1\}.$$

Let

$$v_k(t) = \begin{cases} u_k(t), & \text{if} \quad |t| \leq D+1, \\ 0, & \text{if} \quad |t| > D+1, \end{cases}$$

$$v_0(t)=\begin{cases} u_0(t), & \text{if } |t|\le D+1,\\ 0, & \text{if } |t|>D+1.\end{cases}$$

Obviously, the above assumptions imply $v_k=\{v_k(t)\}\in E_{D+1}$ and $v_0=\{v_0(t)\}\in E_{D+1}$.

Since $u_k \rightharpoonup u_0,\ k\to\infty$ in E, we get

$$\langle u_k,h\rangle \to \langle u_0,h\rangle\,,\quad h\in E. \tag{2.6}$$

Therefore, for any $h\in E_{D+1}$ we have $\langle v_k,h\rangle\to\langle v_0,h\rangle$, i.e. $v_k\rightharpoonup v_0,\ k\to\infty$ in E_{D+1}. Consequently, we arrive at $v_k\to v_0,\ k\to\infty$ in E_{D+1} which implies $v_k\to v_0,\ k\to\infty$ in E. So for k large enough, we have

$$|u_k(t)-u_0(t)|=|v_k(t)-v_0(t)|\le\epsilon,\ \ |t|\le D+1. \tag{2.7}$$

Thus, we have

$$|f(t,u_k(t+1),u_k(t),u_k(t-1))-f(t,u_0(t+1),u_0(t),u_0(t-1))|\to 0,\ \ k\to\infty,$$

for any t such that $|t|\le D$. On the other hand, by (2.7) and Hölder's inequality, for k large enough, we get

$$\sum_{t=-\infty}^{+\infty}\lambda(t)\left[f(t,u_k(t+1),u_k(t),u_k(t-1))-f(t,u_0(t+1),u_0(t),u_0(t-1))\right](u_k(t)-u_0(t))$$

$$=\sum_{|t|\le D}\lambda(t)\left[f(t,u_k(t+1),u_k(t),u_k(t-1))-f(t,u_0(t+1),u_0(t),u_0(t-1))\right](u_k(t)-u_0(t))$$

$$+\sum_{|t|>D}\lambda(t)\left[f(t,u_k(t+1),u_k(t),u_k(t-1))-f(t,u_0(t+1),u_0(t),u_0(t-1))\right](u_k(t)-u_0(t))$$

$$\le C_2\sum_{|t|\le D}\lambda(t)|u_k(t)-u_0(t)|+C_2\sum_{|t|>D}\lambda(t)|u_k(t)-u_0(t)|$$

$$\le C_2\bar\lambda\sum_{|t|\le D}|u_k(t)-u_0(t)|+C_2\left(\sum_{|t|>D}\lambda^q(t)\right)^{\frac1q}\left(\sum_{|t|>D}|u_k(t)-u_0(t)|^p\right)^{\frac1p}$$

$$\le 2C_2\bar\lambda D\cdot\epsilon+C_2\left\|u_k-u_0\right\|_E\cdot\epsilon=\left(2C_2\bar\lambda D+C_2\left\|u_k-u_0\right\|_E\right)\cdot\epsilon,$$

where $\bar\lambda=\sup\limits_{t\in\mathbf{Z}}\lambda(t)$. Since ϵ is arbitrary, we have

$$\sum_{t=-\infty}^{+\infty}\lambda(t)[f(t,u_k(t+1),u_k(t),u_k(t-1))-$$

$$-f(t,u_0(t+1),u_0(t),u_0(t-1))](u_k(t)-u_0(t))\to 0\quad\text{as}\quad k\to\infty. \tag{2.8}$$

Similarly, we get

$$\sum_{t=-\infty}^{+\infty}\lambda(t)[f(t,u_k(t+1),u_k(t),u_k(t-1))-f(t,u_0(t+1),u_0(t),u_0(t-1))]h(t)$$

$$\leq (\bar{\lambda} + C_2)\|h\|_E \cdot \epsilon, \quad h \in E.$$

Combining with (2.2) and (2.6), we obtain that for any $h \in E$

$$\langle J'(u_k) - J'(u_0), h\rangle \to 0, \quad \text{as} \quad k \to +\infty.$$

Therefore, $\|J'(u_0)\|_E \leq \liminf_{k\to\infty} \|J'(u_k)\|_E = 0$. So we have

$$\langle J'(u_k) - J'(u_0), u_k - u_0\rangle = \langle J'(u_k), u_k - u_0\rangle$$

$$\leq \|J'(u_k)\|_E \left(\|u_k\|_E + \|u_0\|_E\right) \leq 2C_1 \left\|J'(u_k)\right\|_E \to 0 \quad \text{as} \quad k \to +\infty. \tag{2.9}$$

For k large enough, we have

$$\|u_k - u_0\|_E^p \leq \langle J'(u_k) - J'(u_0), u_k - u_0\rangle$$

$$- \sum_{t=-\infty}^{+\infty} \lambda(t)[f(t, u_k(t+1), u_k(t), u_k(t-1)) - \tag{2.10}$$

$$-f(t, u_0(t+1), u_0(t), u_0(t-1))](u_k(t) - u_0(t)).$$

By the definition of J

$$\langle J'(u_k) - J'(u_0), u_k - u_0\rangle$$

$$= \sum_{t=-\infty}^{+\infty} \left[\varphi_p\left(\Delta u_k(t-1)\right) - \varphi_p\left(\Delta u_0(t-1)\right)\right]\left(\Delta u_k(t-1) - \Delta u_0(t-1)\right)$$

$$+ \sum_{t=-\infty}^{+\infty} \left[\varphi_p\left(u_k(t-1)\right) - \varphi_p\left(u_0(t-1)\right)\right]\left(u_k(t-1) - u_0(t-1)\right)$$

$$+ \sum_{t=-\infty}^{+\infty} \lambda(t)\left[f\left(t, u_k(t+1), u_k(t), u_k(t-1)\right) - \tag{2.11}$$

$$-f\left(t, u_0(t+1), u_0(t), u_0(t-1)\right)\right]\left(u_k(t) - u_0(t)\right).$$

If $\Delta u_0(t-1) \neq 0$, we set

$$x = \frac{\Delta u_k(t-1)}{\Delta u_0(t-1)}.$$

Since u_k possesses point-wise limit $u_0(x)$ at any $x > 0$, by Lemma 2.4, for all $t \in \mathbf{Z}$, we easily obtain

$$\left[\varphi_p\left(\Delta u_k(t-1)\right) - \varphi_p\left(\Delta u_0(t-1)\right)\right]\left(\Delta u_k(t-1) - \Delta u_0(t-1)\right) \geq \tag{2.12}$$

$$\geq \left|\Delta u_k(t-1) - \Delta u_0(t-1)\right|^p$$

and

$$\left[\varphi_p\left(u_k(t-1)\right) - \varphi_p\left(u_0(t-1)\right)\right]\left(u_k(t-1) - u_0(t-1)\right) \geq \tag{2.13}$$

$$\geq \left|u_k(t-1) - u_0(t-1)\right|^p.$$

If $\Delta u_0(t-1)=0$, then (2.12) and (2.13) are obviously true. From the definition of $\|\cdot\|_E$, formulas (2.11), (2.12) and (2.13), (2.10) holds. Therefore, (2.9), (2.10) and (2.11) imply that $u_k \to u_0$ in E. The proof of Lemma 2.5 is complete. □

3. Proofs of the main results

In this section, we firstly prove the existence of a nontrivial homoclinic orbit of equation (1.1). Next, if $f(t,\cdot)$ is an odd function for any $t\in\mathbf{Z}$, we prove the existence of an unbounded sequence of nontrivial homoclinic orbits of equation (1.1).

3.1. **Proof of Theorem 1.1.** To prove the existence of a homoclinic orbit to (1.1) recall that as we know that $J\in C^1(E,\mathbf{R})$, $J(0)=0$ and J satisfies P.S. condition. Hence, it suffices to prove that J satisfies the conditions (J_1) and (J_2). By (F_1), there exists $\delta>0$ such that

$$|F(t,v_1,v_2)| \le \frac{1}{4\bar{\lambda}p}\left(\sqrt{v_1^2+v_2^2}\right)^p \quad \text{for} \quad t\in\mathbf{Z} \quad \text{and} \quad v_1^2+v_2^2\le\delta^2.$$

Let $\rho=\frac{1}{\sqrt{2}}\delta$, for any $u\in E$ and $\|u\|_E\le\rho$, we have

$$|u(t)|\le\|u\|_{l^p}\le\|u\|_E\le\rho=\frac{1}{\sqrt{2}}\delta, \quad t\in\mathbf{Z}.$$

Thus, we have $u^2(t+1)+u^2(t)\le\delta^2$ for all $t\in\mathbf{Z}$, which implies

$$|F(t,u(t+1),u(t))| \le \frac{1}{2^{\frac{p}{2}+2}\bar{\lambda}p}\left(\sqrt{u^2(t)+u^2(t+1)}\right)^p \le \tag{3.1}$$

$$\le \frac{1}{4\bar{\lambda}p}\left[u^p(t)+u^p(t+1)\right], \quad t\in\mathbf{Z}.$$

Summing inequalities (3.1) over $\mathbf{Z}$, we get

$$\sum_{t=-\infty}^{+\infty}|F(t,u(t+1),u(t))| \le \frac{1}{4\bar{\lambda}p}\sum_{t=-\infty}^{+\infty}\left[u^p(t)+u^p(t+1)\right] \le \frac{1}{2\bar{\lambda}p}\|u\|_E^p.$$

Thus, if $\|u\|_E=\rho$, then

$$J(u)=\frac{1}{p}\|u\|_E^p+\sum_{t=-\infty}^{+\infty}\lambda(t)F(t,u(t+1),u(t)) \ge \frac{1}{p}\|u\|_E^p-\frac{1}{2\bar{\lambda}p}\bar{\lambda}\|u\|_E^p = \frac{1}{2p}\rho^p,$$

that is $J(u)\ge\alpha>0$, where $\alpha=\frac{1}{2p}\rho^p$. To verify condition (J_2), for all $\tau\in\mathbf{R}$ and any given $w\in E\setminus\{0\}$, we consider the quantity

$$J(\tau w)=\frac{1}{p}\tau^p\|w\|_E+\sum_{t=-\infty}^{+\infty}\lambda(t)F(t,\tau w(t+1),\tau w(t)).$$

Let $\bar{w} \in E$ be such that $\bar{w}^2(t) + \bar{w}^2(t+1) \geq R^2$ on a nonempty finite, integer interval $I \subset \mathbf{Z}$. Then by (F_2') we conclude that for any $\tau \geq R$

$$J(\tau\bar{w}) \leq \frac{1}{p}\tau^p \|\bar{w}\|_E + \sum_{t\in I} \lambda(t) F(t, \tau\bar{w}(t+1), \tau\bar{w}(t))$$

$$\leq \frac{1}{p}\tau^p \|\bar{w}\|_E - a|\tau|^\beta \underline{\lambda} \sum_{t\in I} \left[\sqrt{\bar{w}^2(t+1) + \bar{w}^2(t)}\right]^\beta,$$

where $\underline{\lambda} = \min_{t\in I} \lambda(t) > 0$.

Since $\beta > p$, we can choose τ large enough to ensure that $J(\tau\bar{w}) \leq 0$. Thus, both conditions (J_1) and (J_2) are satisfied. Theorem 1.1 is proved. □

3.2. **Proof of Theorem 1.2.** The condition (f_2) implies that J is even. As we already know $J \in C^1(E, \mathbf{R})$, $J(0) = 0$ and J satisfies P.S. condition. In order to prove Theorem 1.2 by using the Symmetric Mountain Pass Lemma, we prove conditions (J_3) and (J_4). From the proof of Theorem 1.1, condition (J_1) is valid, so (J_3) is also valid. To prove (J_4), suppose $\tilde{E} \subset E$ is a finite-dimensional subspace and consider $u \in \tilde{E}$ with $u \neq 0$. By (F_2'), there exist some constants $R > 1$ and $a > 0$ such that

$$F(t, v_1, v_2) \leq -a \left(\sqrt{v_1^2 + v_2^2}\right)^p,$$

$t \in \mathbf{Z}$ and $v_1^2 + v_2^2 \geq R^2$. For all $u \in \tilde{E}$, we have $\|u\|_E^p \leq c\|u\|_\infty^p$, where $c = c(\tilde{E})$. Choosing u such that $\|u\|_E \geq \sqrt[p]{c}R$, we define $I = \{t \mid |u(t)| \geq R\}$. Hence,

$$\sum_{t\in I} \lambda(t) F(t, u(t+1), u(t)) \geq -a \sum_{t\in I} \lambda(t) \left[\sqrt{(u(t+1))^2 + (u(t))^2}\right]^\beta.$$

Thus, we have

$$J(u) = \frac{1}{p}\|u\|_E^p + \sum_{t=-\infty}^{+\infty} \lambda(t) F(t, u(t+1), u(t))$$

$$\leq \frac{1}{p}\|u\|_E^p - a \sum_{t=-\infty}^{+\infty} \lambda(t) \left[\sqrt{(u(t+1))^2 + (u(t))^2}\right]^\beta$$

$$\leq \frac{1}{p} c\|u\|_\infty^p - a \sum_{t\in I} \lambda(t) \left[\sqrt{(u(t+1))^2 + (u(t))^2}\right]^\beta$$

$$\leq \frac{1}{p} c\|u\|_\infty^p - a\|u\|_\infty^\beta \sum_{t\in I} \lambda(t).$$

Since $\beta > p$ there exists $\gamma = \gamma(\tilde{E})(\gamma \geq R)$ such that $J(u) \leq 0$ whenever $\|u\|_\infty > \gamma$. By Lemma 2.2 J possesses an unbounded sequence of critical values c_j with $c_j = J(u_j)$, $j \in \mathbf{N}$. Hence, by (F_1) we have

$$c_j = \frac{1}{p}\|u_j\|_E^p + \sum_{t=-\infty}^{+\infty} \lambda(t) F(t, u_j(t+1), u_j(t)) \leq \frac{1}{p}\|u_j\|_E^p. \tag{3.2}$$

Since $c_j \to \infty$ as $j \to \infty$, (3.2) implies that $\{u_j\}$ is unbounded in E. Therefore, the existence of an unbounded sequence homoclinic orbits is obtained. □

Remark 3.1. *As an application of Theorems 1.1 and 1.2, we give an example illustrating our results.*

For all $t \in \mathbf{Z}$, assume that

$$\Delta(\varphi_p(\Delta u(t-1))) - \varphi_p(u(t)) =$$
$$= -\beta e^{-t^2} u(t) \left[\left(1+\cos^2 2\pi t\right) \left((u(t+1))^2 + (u(t))^2\right)^{\frac{\beta}{2}-1} \right.$$
$$\left. + \left(1+\cos^2 2\pi(t-1)\right) \left((u(t))^2 + (u(t-1))^2\right)^{\frac{\beta}{2}-1} \right], \tag{3.3}$$

where $\beta > p$. Then, we have $\lambda(t) = e^{-t^2}$ and

$$f(t, v_1, v_2, v_3) = -\beta v_2 \left[\left(1+\cos^2 2\pi t\right) \left(v_1^2+v_2^2\right)^{\frac{\beta}{2}-1} + \right.$$
$$\left. + \left(1+\cos^2 2\pi(t-1)\right) \left(v_2^2+v_3^2\right)^{\frac{\beta}{2}-1} \right],$$
$$g(v_1, v_2, v_3) = -4\beta v_2 \left[\left(v_1^2+v_2^2+v_3^2\right)^{\frac{\beta}{2}-1} \right],$$
$$F(t, v_1, v_2) = -\left(1+\cos^2 2\pi t\right) \left(v_1^2+v_2^2\right)^{\frac{\beta}{2}}.$$

Then

$$\frac{\partial F(t-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(t, v_1, v_2)}{\partial v_2}$$
$$= \beta v_2 \left[\left(1+\cos^2 2\pi t\right) \left(v_1^2+v_2^2\right)^{\frac{\beta}{2}-1} + \left(1+\cos^2 2\pi(t-1)\right) \left(v_2^2+v_3^2\right)^{\frac{\beta}{2}-1} \right].$$

It is easy to verify that all assumptions of Theorems 1.1 and 1.2 are satisfied. Consequently, equation (3.3) has an unbounded sequence of nontrivial homoclinic orbits u_j, $j \in \mathbf{N}$.

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