Известия НАН Армении. Математика, том 46, н. 2, 2011, стр. 27-36.

DESCRIPTION OF RANDOM FIELDS BY MEANS OF ONE-POINT FINITE-CONDITIONAL DISTRIBUTIONS

A. S. DALALYAN AND B. S. NAHAPETIAN

Ecole des Ponts ParisTech, Université Paris-Est, France
Institute of Mathematics, National Academy of Sciences of Armenia
E-mails: *nahapet@instmath.sci.am*, *dalalyan@certis.enpc.fr*

Аннотация. The aim of this note is to investigate the relationship between strictly positive random fields on a lattice $\mathbb{Z}^\nu$ and the conditional probability measures at one point given the values on a finite subset of the lattice $\mathbb{Z}^\nu$. We exhibit necessary and sufficient conditions for a one-point finite-conditional system to correspond to a unique strictly positive probability measure. It is noteworthy that the construction of the aforementioned probability measure is done explicitly by some simple procedure. Finally, we introduce a condition on the one-point finite conditional system that is sufficient for ensuring the mixing of the underlying random field.

MSC2000 number: 12E10

Ключевые слова: Random field; one-point conditional distribution; mixing properties.

1. INTRODUCTION

The mathematical theory of random fields is an active area of research aimed at study of the probabilistic properties of systems of interacting particles. In recent years, random fields have been successfully applied to the analysis of biological sequences, text and image processing, as well as to many areas of computer vision and artificial intelligence. In most of these applications, a random field is defined by its finite-dimensional, conditional distributions and is therefore often termed conditional random field. The reconstruction of distributions of random fields from such conditional probabilities is the subject of the present paper.

The description of a random field by means of its conditional distributions is an old problem, most important contributions to which date back to Dobrushin [5, 6]. In his seminal paper [6], Dobrushin considered some systems of conditional distributions on finite sets under the condition that the values of the field are known outside that sets and he proved that, under some assumptions, there exists a random field with the given conditional distributions. This line of research has been developed in recent papers [4, 2, 1, 3, 8], while the present paper is complementing the mentioned works by an exhaustive description of one-point finite-conditional distributions that give rise to positive random fields.

To be more precise, we consider a random field $\boldsymbol{X}$ on the ν-dimensional regular grid $\mathbb{Z}^\nu$ and with values in a finite set $\mathcal{X}$. Given the distribution of $\boldsymbol{X}$, the conditional probabilities $\boldsymbol{Q}_t^{X_\Lambda}(x) = \mathbf{P}(X_t = x|X_\Lambda)$ can be easily computed for every $x \in \mathcal{X}$ and for every finite set $\Lambda \subset \mathbb{Z}^\nu$. In some situations, however, the random field may be unavailable, and only a set of conditional distributions $\{\boldsymbol{Q}_t^{\boldsymbol{x}_\Lambda}\}$ can be defined. In image segmentation, for instance, it is more convenient [11, 9] to define a random field $\boldsymbol{X}$ by specifying the conditional distribution of $\boldsymbol{X}$ at any lattice point t given its values on the neighboring points. Then, the segmentation is obtained by assigning to each point t the most likely value taken by X_t (cf. Figure 1, an example). In such a situation, it is relevant to raise the question of the existence of a random field corresponding to a set of conditional distributions $\{\boldsymbol{Q}_t^{\boldsymbol{x}_\Lambda}\}$. This is the main issue studied in this work. We prove that under some consistency assumptions on the collection $\{\boldsymbol{Q}_t^{\boldsymbol{x}_\Lambda}\}$ there exists a random field corresponding to the mentioned collection. Furthermore, the distribution of this random field is uniquely determined by the collection $\{\boldsymbol{Q}_t^{\boldsymbol{x}_\Lambda}\}$.

The rest of the paper is organized as follows. In Section 2, we introduce the main notation used throughout the paper and present the mathematical formulation of the questions which are of our interest. The main result of the paper is stated and proved in Section 3. We briefly discuss the mixing properties in Section 4 and summarize the main results of the paper in Section 5.

2. NOTATION AND PROBLEM STATEMENT

Let $\boldsymbol{X}$ be a random field on $\mathbb{Z}^\nu$ with a finite state space $\mathcal{X}$ drawn from a probability distribution $\mathbf{P}$ on $(\mathcal{X}^{\mathbb{Z}^\nu}, \mathcal{A})$, where the σ-algebra $\mathcal{A}$ is defined as $(2^{\mathcal{X}})^{\mathbb{Z}^\nu}$, with $2^{\mathcal{X}}$ being the set of all subsets of $\mathcal{X}$. Note that $\mathbf{P}$ is a probability measure acting on an infinite-dimensional space. A classical way of characterizing such probability measures passes through the collection of its finite-dimensional distributions

$$\big\{\mathbf{P}_\Lambda,\ \Lambda \subset \mathbb{Z}^\nu \text{ and } \mathrm{Card}(\Lambda) < \infty\big\}.$$

The famous result of Kolmogorov states that a collection of finite-dimensional distributions corresponds to a unique probability measure on $(\mathcal{X}^{\mathbb{Z}^\nu}, \mathcal{A})$ if and only if it satisfies Kolmogorov's consistency condition.

In this paper, we focus our attention on strictly positive random fields, i.e. random

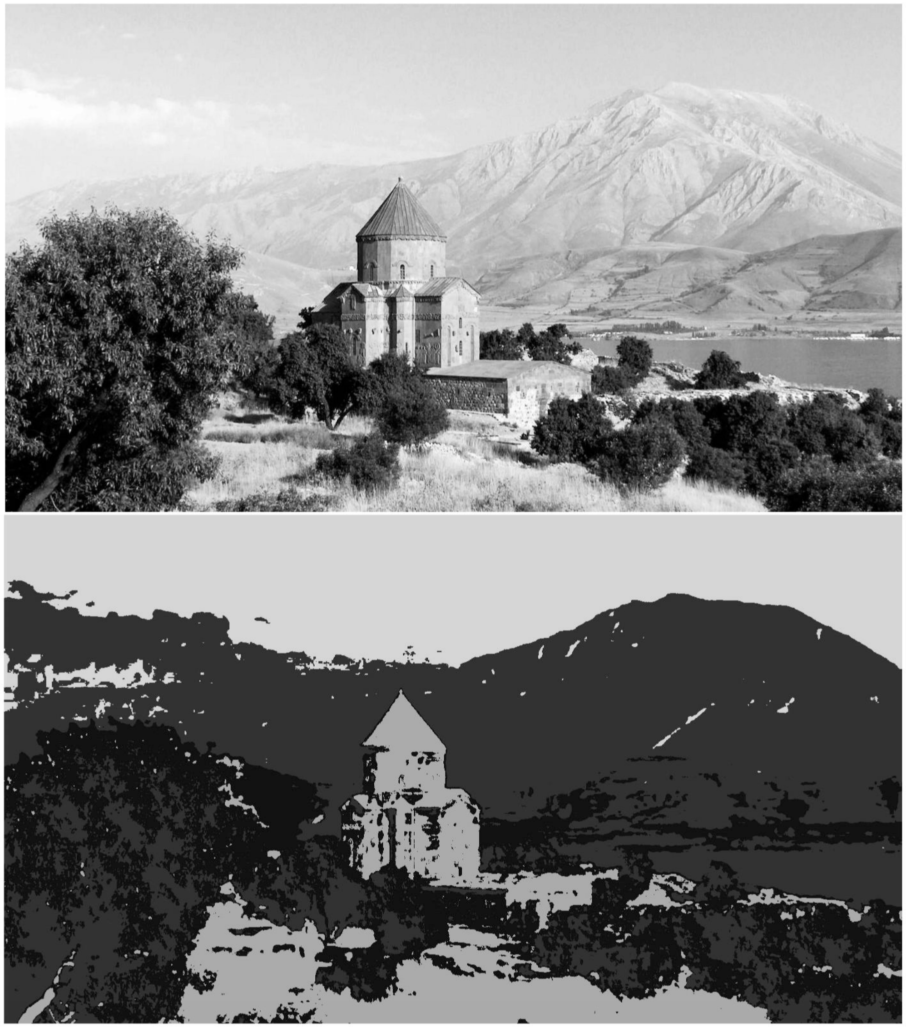

Fig. 1. *A natural image and its segmentation into 4 regions. The segmented image is obtained as the most likely configuration with respect to a probability measure corresponding to a random field on* $\mathbb{Z}^2$ *with state space* $\{1, 2, 3, 4\}$.

fields $\boldsymbol{X}$ satisfying $\mathbf{P}(X_\Lambda = x_\Lambda) > 0$ for all non-empty, finite sets $\Lambda \subset \mathbb{Z}^\nu$ and for all $x_\Lambda \in \mathcal{X}^\Lambda$. For such a random field, the one-point finite-conditional probabilities are defined as follows. For any $\Lambda \subset \mathbb{Z}^\nu$, $\widetilde{\mathcal{X}^\Lambda}$ stands for the set of all functions $\widetilde{\boldsymbol{x}}$ defined on some non-empty, finite subset J of $\mathbb{Z}^\nu \setminus \Lambda$ and take values in $\mathcal{X} : \widetilde{\boldsymbol{x}} : J \to \mathcal{X}$. We will refer to J as the support of $\widetilde{\boldsymbol{x}} \in \widetilde{\mathcal{X}^\Lambda}$. For every $t \in \mathbb{Z}^\nu$ and for every $\widetilde{\boldsymbol{x}} \in \widetilde{\mathcal{X}^t}$ the following conditional probability measure can be defined on $\mathcal{X}$:

$$\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(\cdot) = \mathbf{P}(X_t = \cdot \,|\, X_J = \widetilde{\boldsymbol{x}}), \qquad \text{where } J = \mathrm{supp}(\widetilde{\boldsymbol{x}}).$$

The set $\boldsymbol{q}(\mathbf{P}) = \{\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}},\ t \in \mathbb{Z}^\nu \text{ and } \widetilde{\boldsymbol{x}} \in \widetilde{\mathcal{X}^t}\}$ is called one-point finite-conditional distribution of $\mathbf{P}$. The problems, we are interested in, can be formulated as those related to the inversion of the operator $\boldsymbol{q}$.

To be more precise, let $\widetilde{\boldsymbol{q}} = \{\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}},\ t \in \mathbb{Z}^\nu \text{ and } \widetilde{\boldsymbol{x}} \in \widetilde{\mathcal{X}^t}\}$ be a system of probability distributions on $\mathcal{X}$, such that $\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x) > 0$ for all $x \in \mathcal{X}$. We define $\mathcal{P}(\widetilde{\boldsymbol{q}})$ the set of all strictly positive probability measures $\mathbf{P}$, such that $\boldsymbol{q}(\mathbf{P}) = \widetilde{\boldsymbol{q}}$, or in other terms, $\mathcal{P}(\widetilde{\boldsymbol{q}}) = \boldsymbol{q}^{-1}(\widetilde{\boldsymbol{q}})$. The main goal of the present paper is to accomplish the following tasks:

(a) Determine necessary and sufficient conditions on $\widetilde{\boldsymbol{q}}$, under which the set $\mathcal{P}(\widetilde{\boldsymbol{q}})$ is non-empty.

(b) Prove that if $\mathcal{P}(\widetilde{\boldsymbol{q}})$ is non-empty, then it is a singleton, which means that there is a unique random field having $\widetilde{\boldsymbol{q}}$ as a one-point finite-conditional distribution.

(c) Describe some conditions on $\widetilde{\boldsymbol{q}}$, entailing that the corresponding random field, if exists, possesses mixing properties.

3. NECESSARY AND SUFFICIENT CONDITION FOR EXISTENCE AND UNIQUENESS

It is quite clear that not every one-point finite-conditional distribution $\widetilde{\boldsymbol{q}}$ corresponds to a random field. For instance, it is obvious that any random field with strictly positive probability distribution the following property

(3.1)
$$\mathbf{P}\big((X_t, X_s) = (x,y) \,|\, X_J = \widetilde{\boldsymbol{x}}\big) = \mathbf{P}\big(X_t = x \,|\, X_J = \widetilde{\boldsymbol{x}}\big)\mathbf{P}\big(X_s = y \,|\, X_J = \widetilde{\boldsymbol{x}}, X_t = x\big)$$

should be satisfied for any $t, s \in \mathbb{Z}^\nu$, $(x, y) \in \mathcal{X}^{\{t,s\}}$ and for all $\widetilde{\boldsymbol{x}} \in \widetilde{\mathcal{X}^{\{t,s\}}}$. Therefore, if for $\widetilde{\boldsymbol{q}}$ the condition $\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x)\boldsymbol{Q}_s^{\widetilde{\boldsymbol{x}}x}(y) = \boldsymbol{Q}_s^{\widetilde{\boldsymbol{x}}}(y)\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}y}(x)$ fails for some $(s, t, x, y, \widetilde{\boldsymbol{x}})$, then there is no random field having $\widetilde{\boldsymbol{q}}$ as its one-point finite-conditional distribution. The next theorem provides a precise characterization of systems $\widetilde{\boldsymbol{q}}$ that can be extended to strictly positive random fields. Moreover, it shows that the corresponding random field is unique and can be constructed by a simple procedure.

Theorem 3.1. *Let $\widetilde{\boldsymbol{q}} = \{\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}},\ t \in \mathbb{Z}^\nu \text{ and } \widetilde{\boldsymbol{x}} \in \widetilde{\mathcal{X}^t}\}$ be a one-dimensional finite-conditional distribution on $\mathcal{X}$. Then, there exists a strictly positive random field having*

$\widetilde{\boldsymbol{q}}$ as its system of conditional probabilities, i.e. $\mathcal{P}(\widetilde{\boldsymbol{q}}) \neq \varnothing$, if and only if the following conditions are fulfilled:

[C1] $\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x) > 0$ *for all* $t \in \mathbb{Z}^\nu$, $x \in \mathcal{X}$ *and* $\widetilde{\boldsymbol{x}} \in \widetilde{\mathcal{X}^t}$.

[C2] *For all* $t, s \in Z^\nu$, $x \in \mathcal{X}$, $y \in \mathcal{X}$ *and* $\widetilde{\boldsymbol{x}} \in \widetilde{\mathcal{X}^{\{t,s\}}}$ *it holds that*

$$\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x)\boldsymbol{Q}_s^{\widetilde{\boldsymbol{x}}x}(y) = \boldsymbol{Q}_s^{\widetilde{\boldsymbol{x}}}(y)\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}y}(x).$$

[C3] *For all* $t, s \in Z^\nu$, $x, x' \in \mathcal{X}^t$ *and* $y, y' \in \mathcal{X}^s$ *it holds that*

$$\boldsymbol{Q}_t^{y}(x)\boldsymbol{Q}_s^{x'}(y)\boldsymbol{Q}_t^{y'}(x')\boldsymbol{Q}_s^{x}(y') = \boldsymbol{Q}_t^{y'}(x)\boldsymbol{Q}_s^{x'}(y')\boldsymbol{Q}_t^{y}(x')\boldsymbol{Q}_s^{x}(y). \tag{3.2}$$

Under these conditions all random fields, having $\widetilde{\boldsymbol{q}}$ as conditional distributions, possess the same distribution.

Proof. We start by proving that all three conditions are necessary. This is obvious for the first condition. For the second one, the necessity follows from the property (3.1) of random fields. Further, note that if P is a probability measure and A, B, C, D are any events, then

$$P(A|B)P(B|C)P(C|D)P(D|A) = P(A|D)P(D|C)P(C|B)P(B|A). \tag{3.3}$$

Applying this identity to the random field $\boldsymbol{X}$ drawn from $\mathbf{P}$ and to the events $A = \{X_t = x\}$, $B = \{X_s = y\}$, $C = \{X_t = x'\}$ and $D = \{X_s = y'\}$, we get the necessity of condition (3.2).

To prove that all three conditions [C1-C3] are sufficient, assume that $\widetilde{\boldsymbol{q}}$ is a one-point finite-conditional system satisfying these conditions. Then for any $t \in \mathbb{Z}^\nu$, we choose $s \in \mathbb{Z}^\nu \setminus \{t\}$, $y \in \mathcal{X}^s$ and set

$$\mathbf{P}_t(x) = \frac{\boldsymbol{Q}_t^{y}(x)}{\boldsymbol{Q}_s^{x}(y)} \left[\sum_{u \in \mathcal{X}^t} \frac{\boldsymbol{Q}_t^{y}(u)}{\boldsymbol{Q}_s^{u}(y)} \right]^{-1}, \quad x \in \mathcal{X}^t. \tag{3.4}$$

Under condition [C2] and [C3], $P_t(x)$ defined as above is independent of s and of y. Indeed, it follows from [C2] that for any distinct points $t, s, r \in \mathbb{Z}^\nu$ and any $x \in \mathcal{X}^t$, $y \in \mathcal{X}^s$, $z \in \mathcal{X}^r$

$$\boldsymbol{Q}_t^{z}(x)\boldsymbol{Q}_s^{zx}(y) = \boldsymbol{Q}_s^{z}(y)\boldsymbol{Q}_t^{zy}(x),$$
$$\boldsymbol{Q}_r^{y}(z)\boldsymbol{Q}_t^{zy}(x) = \boldsymbol{Q}_t^{y}(x)\boldsymbol{Q}_r^{yx}(z),$$
$$\boldsymbol{Q}_s^{x}(y)\boldsymbol{Q}_r^{yx}(z) = \boldsymbol{Q}_r^{x}(z)\boldsymbol{Q}_s^{zx}(y).$$

Multiplying these equations, one can see that many terms vanish, and the result is the identity

$$\boldsymbol{Q}_t^{z}(x)\boldsymbol{Q}_s^{x}(y)\boldsymbol{Q}_r^{y}(z) = \boldsymbol{Q}_s^{z}(y)\boldsymbol{Q}_r^{x}(z)\boldsymbol{Q}_t^{y}(x). \tag{3.5}$$

Since (3.5) holds for any $x \in \mathcal{X}^t$, we have also

$$\boldsymbol{Q}_t^z(u)\boldsymbol{Q}_s^u(y)\boldsymbol{Q}_r^y(z) = \boldsymbol{Q}_s^z(y)\boldsymbol{Q}_r^u(z)\boldsymbol{Q}_t^y(u), \quad u \in \mathcal{X}^t. \tag{3.6}$$

Dividing (3.5) by (3.6), we come to the equality

$$\frac{\boldsymbol{Q}_t^z(x)\boldsymbol{Q}_s^x(y)}{\boldsymbol{Q}_t^z(u)\boldsymbol{Q}_s^u(y)} = \frac{\boldsymbol{Q}_r^x(z)\boldsymbol{Q}_t^y(x)}{\boldsymbol{Q}_r^u(z)\boldsymbol{Q}_t^y(u)}. \tag{3.7}$$

Rearranging the terms, we come to the equality

$$\frac{\boldsymbol{Q}_t^z(x)\boldsymbol{Q}_t^y(u)}{\boldsymbol{Q}_r^x(z)\boldsymbol{Q}_s^u(y)} = \frac{\boldsymbol{Q}_t^y(x)\boldsymbol{Q}_t^z(u)}{\boldsymbol{Q}_s^x(y)\boldsymbol{Q}_r^u(z)} \tag{3.8}$$

which implies

$$\frac{\boldsymbol{Q}_t^z(x)}{\boldsymbol{Q}_r^x(z)}\sum_u \frac{\boldsymbol{Q}_t^y(u)}{\boldsymbol{Q}_s^u(y)} = \frac{\boldsymbol{Q}_t^y(x)}{\boldsymbol{Q}_s^x(y)}\sum_u \frac{\boldsymbol{Q}_t^z(u)}{\boldsymbol{Q}_r^u(z)} \tag{3.9}$$

after summing with respect to $u \in \mathcal{X}^t$. In other terms,

$$\frac{\boldsymbol{Q}_t^z(x)}{\boldsymbol{Q}_r^x(z)}\left[\sum_u \frac{\boldsymbol{Q}_t^z(u)}{\boldsymbol{Q}_r^u(z)}\right]^{-1} = \frac{\boldsymbol{Q}_t^y(x)}{\boldsymbol{Q}_s^x(y)}\left[\sum_u \frac{\boldsymbol{Q}_t^y(u)}{\boldsymbol{Q}_s^u(y)}\right]^{-1}. \tag{3.10}$$

This equality proves that the definition of $\mathbf{P}_t$ given by (3.4) does not depend on the choice of s and y, i.e. taking $r \neq s$ instead of s and z instead of y does not affect the result. The independence of the right-hand side of (3.4) of y follows directly from [C3].

So far, we proved that for any one-point finite-conditional distribution $\widetilde{\boldsymbol{q}}$ satisfying [C1-C3] one can uniquely determine one-point unconditional distributions. Let us now look at what happens with the remaining finite-dimensional unconditional distributions. To this end, let Λ be a finite subset of $\mathbb{Z}^d$, the elements of which are somehow enumerated as $\Lambda = \{t_1, \dots, t_n\}$. Then, for every $\boldsymbol{x}_\Lambda \in \mathcal{X}^\Lambda$ we define

$$\mathbf{P}_\Lambda(\boldsymbol{x}_\Lambda) = \mathbf{P}_{t_1}(x_{t_1})\boldsymbol{Q}_{t_2}^{x_{t_1}}(x_{t_2}) \cdot \ldots \cdot \boldsymbol{Q}_{t_n}^{x_{t_1}\ldots x_{t_{n-1}}}(x_{t_n}), \tag{3.11}$$

where $\mathbf{P}_{t_1}(x_{t_1})$ is well defined by (3.4). We prove below that this definition is independent of the enumeration of elements of Λ and that the family $\{\mathbf{P}_\Lambda : \Lambda \text{ is a finite subset of } \mathbb{Z}^\nu\}$ is a collection of probability measures that are consistent in the Kolmogorov sense.

To prove that the definition of $\mathbf{P}_\Lambda$ is invariant w.r.t. the order on the elements of Λ, we use the fact that any permutation of $t_1, \dots, t_n$ can be obtained as a composition of a finite set of permutations of two successive elements. Therefore, it is sufficient to prove that the replacement of $\{t_1, t_2, \dots, t_{k-1}, t_k, \dots, t_n\}$ by the ordered set

$\{t_1, t_2, \ldots, t_k, t_{k-1}, \ldots, t_n\}$ does not affect the definition of $\mathbf{P}_\Lambda$. Thus, we aim at showing that

(3.12)
$$\boldsymbol{Q}_{t_{k-1}}^{x_{t_1}\ldots x_{t_{k-2}}}(x_{t_{k-1}})\boldsymbol{Q}_{t_k}^{x_{t_1}\ldots x_{t_{k-1}}}(x_{t_k}) = \boldsymbol{Q}_{t_k}^{x_{t_1}\ldots x_{t_{k-2}}}(x_{t_k})\boldsymbol{Q}_{t_{k-1}}^{x_{t_1}\ldots x_{t_{k-2}}, x_{t_k}}(x_{t_{k-1}}), \quad k > 2,$$

and that

$$\mathbf{P}_{t_1}(x_{t_1})\boldsymbol{Q}_{t_2}^{x_{t_1}}(x_{t_2}) = \mathbf{P}_{t_2}(x_{t_2})\boldsymbol{Q}_{t_1}^{x_{t_2}}(x_{t_1}). \tag{3.13}$$

Observe that (3.12) is reduced to the condition [C2] by setting $t = t_{k-1}$, $s = t_k$, $\widetilde{\boldsymbol{x}} = \{x_{t_1}, \ldots, x_{t_{k-2}}\}$, $x = x_{t_{k-1}}$ and $y = x_{t_k}$. The case of (3.13) is a bit more delicate and requires the use of [C3]. To simplify notation, we set $t = t_1$, $s = t_2$, $x_{t_1} = x$ and $x_{t_2} = y$ and intend to show that $\mathbf{P}_t(x)\boldsymbol{Q}_s^x(y) = \mathbf{P}_s(y)\boldsymbol{Q}_t^y(x)$, which amounts to

$$\boldsymbol{Q}_t^y(x)\Bigg[\sum_{x' \in \mathfrak{X}^t} \frac{\boldsymbol{Q}_t^y(x')}{\boldsymbol{Q}_s^{x'}(y)}\Bigg]^{-1} = \boldsymbol{Q}_s^x(y)\Bigg[\sum_{y' \in \mathfrak{X}^s} \frac{\boldsymbol{Q}_s^x(y')}{\boldsymbol{Q}_t^{y'}(x)}\Bigg]^{-1}.$$

This can be equivalently written as

$$\sum_{y' \in \mathfrak{X}^s} \frac{\boldsymbol{Q}_s^x(y')\boldsymbol{Q}_t^y(x)}{\boldsymbol{Q}_t^{y'}(x)} = \sum_{x' \in \mathfrak{X}^t} \frac{\boldsymbol{Q}_t^y(x')\boldsymbol{Q}_s^x(y)}{\boldsymbol{Q}_s^{x'}(y)}. \tag{3.14}$$

Using the equality $\sum_{x'} \boldsymbol{Q}_t^{y'}(x') = \sum_{y'} \boldsymbol{Q}_s^{x'}(y') = 1$, one can rewrite (3.14) as follows:

(3.15)
$$\sum_{x' \in \mathfrak{X}^t}\sum_{y' \in \mathfrak{X}^s} \frac{\boldsymbol{Q}_s^x(y')\boldsymbol{Q}_t^y(x)\boldsymbol{Q}_s^{x'}(y)\boldsymbol{Q}_t^{y'}(x')}{\boldsymbol{Q}_t^{y'}(x)\boldsymbol{Q}_s^{x'}(y)} = \sum_{y' \in \mathfrak{X}^s}\sum_{x' \in \mathfrak{X}^t} \frac{\boldsymbol{Q}_t^y(x')\boldsymbol{Q}_s^x(y)\boldsymbol{Q}_t^{y'}(x)\boldsymbol{Q}_s^{x'}(y')}{\boldsymbol{Q}_t^{y'}(x)\boldsymbol{Q}_s^{x'}(y)}.$$

Now, it follows from [C3] that the equality (3.15) is true.Thus, if conditions [C2] and [C3] are fulfilled, then the distribution $\mathbf{P}_\Lambda$ remains the same for any enumeration.

In order to prove the consistency in Kolmogorov's sense, we use the fact that $\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(\cdot)$ is a probability measure on $\mathfrak{X}$. This provides the equality $\sum_{x \in \mathfrak{X}^t} \boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x) = 1$ which implies that

$$\sum_{x \in \mathfrak{X}^t} \mathbf{P}_{\Lambda \cup \{t\}}(\widetilde{\boldsymbol{x}}x) = \sum_{x \in \mathfrak{X}^t} \mathbf{P}_\Lambda(\widetilde{\boldsymbol{x}})\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}}(x) = \mathbf{P}_\Lambda(\widetilde{\boldsymbol{x}})$$

for any finite $\Lambda \subset \mathbb{Z}^\nu$, any $t \in \mathbb{Z}^\nu \setminus \Lambda$ and any $\widetilde{\boldsymbol{x}} \in \mathfrak{X}^\Lambda$. This concludes the proof.

4. SUFFICIENT CONDITION FOR MIXING

The study of the mixing properties [7] of random fields is of primary interest in probability theory, since they characterize the behavior of additive functionals of the random field by means of central limit theorems [10]. The aim of this section is to describe a simple condition on a one-point finite-conditional distribution satisfying conditions [C1-C3], that allows to evaluate the mixing properties of the underlying random field. To this end we introduce the notation

$$\rho_{s,t} = \sup_{\substack{\Lambda: s\in\Lambda, t\notin\Lambda, |\Lambda|<\infty \\ \widetilde{\boldsymbol{x}}:\mathrm{supp}(\widetilde{\boldsymbol{x}})=\Lambda\setminus\{s\}}} \max_{y,z\in\mathcal{X}^s} \sum_{x\in\mathcal{X}^t} |\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}y}(x) - \boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}z}(x)| \tag{4.1}$$

for every pair of distinct points $t, s \in \mathbb{Z}^\nu$ and prove the following theorem.

Theorem 4.1. *Let $\widetilde{\boldsymbol{q}} = \{\boldsymbol{Q}_t^{\widetilde{\boldsymbol{x}}},\ t \in \mathbb{Z}^\nu$ and $\widetilde{\boldsymbol{x}} \in \widetilde{\mathcal{X}^t}\}$ be a one-dimensional finite-conditional distribution on $\mathcal{X}$, which satisfies the conditions of Theorem 3.1. For every pair of disjoint finite subsets I and V from $\mathbb{Z}^\nu$ the reconstructed from $\widetilde{\boldsymbol{q}}$ random field $\mathbf{P}$ satisfies the inequality*

$$\max_{\substack{\boldsymbol{x}\in\mathcal{X}^V \\ \boldsymbol{y}\in\mathcal{X}^I}} \big|\mathbf{P}_V(\boldsymbol{x}) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\big| \le \sum_{s\in I}\sum_{t\in V} \rho_{s,t}. \tag{4.2}$$

Proof. Denoting the cardinalities of I and V by m and n respectively, we perform induction on $m+n$. First suppose that $m+n=2$. Then $m=n=1$ and hence $V=\{t\}$ and $I=\{s\}$. Therefore,

$$\begin{aligned}
\big|\mathbf{P}_t(x) - \mathbf{P}_{t|s}(x|y)\big| &= \Big|\sum_{z\in\mathcal{X}^s} \boldsymbol{Q}_t^z(x)\mathbf{P}_s(z) - \boldsymbol{Q}_t^y(x)\Big| \\
&\le \sum_{z\in\mathcal{X}^s} \mathbf{P}_s(z)\big|\boldsymbol{Q}_t^z(x) - \boldsymbol{Q}_t^y(x)\big| \\
&\le \max_{z\in\mathcal{X}^s} \big|\boldsymbol{Q}_t^z(x) - \boldsymbol{Q}_t^y(x)\big| \le \rho_{s,t},
\end{aligned} \tag{4.3}$$

which proves our statement for $m+n=2$.

Now, we suppose our statement is true for every pair of strictly positive integers (m,n), such that $m+n\le k$ and prove (4.2) for $m+n=k+1$. To this end, we choose an arbitrarily point u in I and set $J = I\setminus\{u\}$. Then

$$\begin{aligned}
\big|\mathbf{P}_V(\boldsymbol{x}) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\big| &\le \big|\mathbf{P}_V(\boldsymbol{x}) - \mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J)\big| + \big|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\big| \\
&\le \sum_{s\in J}\sum_{t\in V}\rho_{s,t} + \big|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\big|,
\end{aligned} \tag{4.4}$$

since our statement is assumed to be true for the pair (V,J). One can easily verify that

$$\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) = \boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,\boldsymbol{x}_{V\setminus\{t'\}}}(x_{t'})\mathbf{P}_{V\setminus\{t'\}|J}(\boldsymbol{x}_{V\setminus\{t'\}}|\boldsymbol{y}_J)$$
$$\mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y}) = \boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus\{t'\}}}(x_{t'})\mathbf{P}_{V\setminus\{t'\}|I}(\boldsymbol{x}_{V\setminus\{t'\}}|\boldsymbol{y})$$

for every $t' \in V$, and hence

$$\left|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right| \leq \left|\boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,\boldsymbol{x}_{V\setminus\{t'\}}}(x_{t'}) - \boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus\{t'\}}}(x_{t'})\right| + \left|\mathbf{P}_{V\setminus t'|J}(\boldsymbol{x}_{V\setminus t'}|\boldsymbol{y}_J) - \mathbf{P}_{V\setminus t'|I}(\boldsymbol{x}_{V\setminus t'}|\boldsymbol{y})\right|. \tag{4.5}$$

It follows from the total probabilities formula that

$$\begin{aligned}\left|\boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,\boldsymbol{x}_{V\setminus t'}}(x_{t'}) - \boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus t'}}(x_{t'})\right| &= \Big|\sum_{z\in\mathfrak{X}^u}\Big(\boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,z,\boldsymbol{x}_{V\setminus t'}}(x_{t'}) - \boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus t'}}(x_{t'})\Big)\mathbf{P}(z|\boldsymbol{y}_J,\boldsymbol{x}_{V\setminus t'})\Big| \\ &\leq \max_{z\in\mathfrak{X}^u}\left|\boldsymbol{Q}_{t'}^{\boldsymbol{y}_J,z,\boldsymbol{x}_{V\setminus t'}}(x_{t'}) - \boldsymbol{Q}_{t'}^{\boldsymbol{y},\boldsymbol{x}_{V\setminus t'}}(x_{t'})\right| \leq \rho_{u,t'}.\end{aligned}$$

Combining with (4.5), we get

$$\left|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right| \leq \rho_{u,t'} + \left|\mathbf{P}_{V\setminus t'|J}(\boldsymbol{x}_{V\setminus t'}|\boldsymbol{y}_J) - \mathbf{P}_{V\setminus t'|I}(\boldsymbol{x}_{V\setminus t'}|\boldsymbol{y})\right|.$$

Repeating the same argument, we obtain

$$\left|\mathbf{P}_{V|J}(\boldsymbol{x}|\boldsymbol{y}_J) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})\right| \leq \sum_{t'\in V}\rho_{u,t'}.$$

Joining this estimate with (4.4), we complete the proof.

As an application of Theorem 4.1, let us consider the one-dimensional case $\nu = 1$. Assume that there exist $\rho_* < 1$ and $d_* > 0$, such that $\rho_{s,t} \leq \rho_*^{|t-s|}$ as soon as $|t-s| \geq d_*$. Then, one can easily verify that

$$\sum_{s\in I}\sum_{t\in V}\rho_{s,t} \leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\rho_*^{d+i+j} = \sum_{k=0}^{\infty}(k+1)\rho_*^{d+k} = \frac{\rho_*^d}{(1-\rho_*)^2}$$

for every pair of finite intervals $V, I \subset \mathbb{Z}$ such that

$$d = d(V,I) = \min_{t\in V, s\in I}|t-s| \geq d_*.$$

This short computation shows that the quantity $|\mathbf{P}_V(\boldsymbol{x}) - \mathbf{P}_{V|I}(\boldsymbol{x}|\boldsymbol{y})|$ exponentially decreases to zero when the distance between the intervals V and I tends to infinity.

5. CONCLUSION

In the present paper, we have introduced the notion of the one-point finite-conditional distributions and established necessary and sufficient conditions (cf. [C1-C3] in Theorem 3.1) for which such a system to be the set of conditional probabilities of a strictly positive random field on $\mathbb{Z}^\nu$ and with finite state-space. The conditions [C2-C3], which are the most important ones, can be observed as consistency conditions in the same spirit as the Kolmogorov consistency conditions for finite-dimensional distributions of random processes indexed by infinite sets. It is demonstrated that it is possible to assess the rate of mixing of a random field by evaluating some characteristics of one-point finite-conditional distributions, without resorting to the computation of the unconditional distributions of the random field.

The relaxation of the assumption of strong positiveness, *e.g.* by introducing a notion of weakly positive random fields in the spirit of [2], is an essential open problem.

We wish to thank Sergueї Dashyan for his careful reading of the initial version of the manuscript and his numerous valuable suggestions.

Список литературы

[1] S. Yu. Dashyan, B. S. Nahapetian, "Description of random fields by means of one-point conditional distributions and some applications", Markov Process. Related Fields, **7**(2), 193-214 (2001).

[2] S. Yu. Dashyan, B. S. Nahapetian, "Description of specifications by means of probability distributions in small volumes under condition of very weak positivity", J. Statist. Phys., **117** (1-2), 281-300 (2004).

[3] S. Yu. Dashyan, B. S. Nahapetian, "On Gibbsianness of random fields. Markov Process. Related Fields, **15** (1), 81-104 (2009).

[4] S. Yu. Dashyan, B. S. Nahapetian, "Inclusion-exclusion description of random fields", Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences), **30** (6), 50-61 (1995).

[5] R. L. Dobrushin, "The problem of uniquenessof a Gibbsian random field and the problem of phase transitions", Funktsional. Anal. i Prilojen., **2** (4), 44-57 (1968).

[6] R. L. Dobrushin, "Description of a random field by means of conditional probabilities and conditions for its regularity", Teor. Verojatnost. i Primenen., **13**, 201-229 (1968).

[7] P. Doukhan, *Mixing. Properties and examples*, Lecture Notes in Statistics, **85** (Springer-Verlag, New York, 1994).

[8] R. Fernández, R., G. Maillard, "Construction of a specification from its singleton part", ALEA Lat. Am. J. Probab. Math. Stat., **2**, 297-315 (2006).

[9] S. Geman, D. Geman, "Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images", IEEE Transl. Pattern Anal. Mach. Intell, **6**, 721-741 (1984).

[10] B. Nahapetian, *Limit theorems and some applications in statistical physics*, Teubner-Texte zur Mathematik **123** (B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1991).

[11] N. Shental, A. Zomet, T. Hertz, Y. Weiss, "Learning and inferring image segmentations using the GBP typical cut algorithm", in **2** (Proceedings of ICCV, October 13-16, 2003).

Поступила 19 октября 2010