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HOMOGENEOUS GEODESICS AND THE CRITICAL POINTS OF THE RESTRICTED FINSLER FUNCTION

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Abstract. In this paper we study the set of homogeneous geodesics of a left-invariant Finsler metric on Lie groups. We first give a simple criterion that characterizes geodesic vectors. We extend J. Szenthe's result on homogeneous geodesics to left-invariant Finsler metrics. This result gives a relation between geodesic vectors and restricted Minkowski norm in Finsler setting. We show that if a compact connected and semisimple Lie group has rank greater than 1, then for every left-invariant Finsler metric there are infinitely many homogeneous geodesics through the identity element.

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1. Introduction

A connected Finsler space (M, F) is called homogeneous if it admits a transitive connected Lie group G of isometries. Then M can be viewed as a coset space $\frac{G}{H}$ with a $G-$invariant Finsler metric, where H is the isotropy subgroup of some point in M. A geodesic $\gamma(t)$ through the origin O of $M = \frac{G}{H}$ is called homogeneous if it is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tZ)(O), \qquad t \in R,$$

where Z is a nonzero vector in the Lie algebra $\mathfrak{g}$ of G.

Homogeneous geodesics on homogeneous Riemannian manifolds have been studied by many authors. For results on homogeneous geodesics in homogeneous Riemannian manifolds we refer for example to [4], [9], [12], [13], [17]. Homogeneous geodesics in a Lie group were studied by V. V. Kajzer in [11] where he proved that a Lie group G with a left-invariant metric has at least one homogeneous geodesic through the identity. In [20] J. Szenthe proved that if a compact connected and semisimple Lie group has rank greater than 1, then for every left-invariant Riemannian metric there are infinitely many homogeneous geodesics through the identity element. A generalization of Kajzer's result was obtained by O. Kowalski and J. Szenthe [12] who proved that every Riemannian homogeneous manifold admits at least one homogeneous geodesic through each point.

Homogeneous geodesics have important applications to mechanics. For example, the equation of motion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold M. Homogeneous geodesic of M correspond to "relative equilibriums" of the corresponding system [2]. Geodesics of left-invariant Riemannian metrics on Lie groups were studied by V. I. Arnold extending Euler's theory of rigid-body motion [1].

Homogeneous geodesics are interesting also in pseudo-Riemannian geometry. For results on homogeneous geodesics in homogeneous pseudo-Riemannian manifolds we refer for example to [5] — [8], [18], [19]. The existence of a homogeneous geodesic in every homogeneous pseudo-Riemannian manifold (not necessarily reductive) was proved by Z. Dušek in [7], the proof is not using any advanced algebra but rather elementary facts from differential topology. In Physics, Penrose limits along null homogeneous geodesics are studied in [8] and [19]. In [19], it is shown that the Penrose limit of a Lorentzian spacetime along a homogeneous geodesic is a homogeneous plane wave and the Penrose limit of a reductive homogeneous spacetime along a homogeneous geodesic is a reductive homogeneous plane wave. Null homogeneous geodesics on Lorentzian homogeneous spaces are also studied in [18].

In [14] the second author studied homogenous geodesics in homogeneous Finsler spaces. In this paper we consider Lie groups with invariant Finsler metrics and give an extension of J. Szenthe's result on homogeneous geodesics of left-invariant Finsler metrics. This result gives a relation between geodesic vectors and critical points of restricted Minkowski norms in Finsler setting.

2. Preliminaries

In this section, we recall some well-known facts about Finsler geometry. See [3] for more details. Let M be an n-dimensional smooth manifold and TM denotes its tangent bundle. A Finsler structure on a manifold M, is a continuous map $F : TM \longrightarrow [0, \infty)$ which has the following properties:

(1) F is smooth on $TM \setminus \{0\}$;
(2) $F(tv) = tF(v)$, for all $t > 0$, $v \in T_xM$ i.e. F is positively homogeneous of degree one;
(3) For each $y \in T_xM \setminus \{0\}$, the induced symmetric bilinear form g_y on T_xM is positive definite, where

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(x, y + su + tv)]|_{s=t=0}, \qquad u, v \in T_xM.$$

The Chern connection on a Finsler manifold is a linear connection on the pull-back bundle π^*TM. This connection is almost $g-$compatible and has no torsion. Let $\sigma : [0, r] \longrightarrow M$ be a smooth curve with velocity field $T = T(t) = \dot{\sigma}(t)$. Suppose that U and W are vector fields defined along σ. We define the covariant derivative D_TU with *reference vector* W as

$$D_TU = \left[\frac{dU^i}{dt} + U^jT^k(\Gamma^i_{jk})_{(\sigma,W)}\right] \frac{\partial}{\partial x^i}\bigg|_{\sigma(t)},$$

where Γ^i_{jk} are the coefficients of Chern connection.
A curve $\sigma : [0, r] \longrightarrow M$, with velocity $T = \dot{\sigma}$ is a Finslerian *geodesic* if $D_T\left[\frac{T}{F(T)}\right] = 0$, with reference vector T.
We assume that all our geodesics $\sigma(t)$ have been parameterized to have constant Finslerian speed. That is, the length $F(T)$ is constant. These geodesics are characterized by the equation $D_TT = 0$, with reference vector T.
Since $T = \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i}$, this equation says that

$$\frac{d^2\sigma^i}{dt^2} + \frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt}(\Gamma^i_{jk})_{(\sigma,T)} = 0.$$

If U, V and W are vector fields along a curve σ, which has velocity $T = \dot{\sigma}$, we have the derivative rule

$$\frac{d}{dt}g_W(U, V) = g_W(D_TU, V) + g_W(U, D_TV)$$

whenever D_TU and D_TV are with reference vector W and one of the following conditions holds:

i): U or V is proportional to W, or
ii): W=T and σ is a geodesic.

3. Left-invariant Finsler metrics over Lie groups and their homogeneous geodesics

Let G be a connected Lie group with Lie algebra $\mathfrak{g} = T_eG$. We may identify the tangent bundle TG with $G \times \mathfrak{g}$ by means of the diffeomorphism that sends (g, X) to $T_eL_gX \in T_gG$.

Definition 3.1. *A Finsler function $F : TG \longrightarrow R_+$ is called G-invariant if F is constant on all G-orbits in $TG = G \times \mathfrak{g}$; that is $F(g, X) = F(e, X)$ for all $g \in G$ and $X \in \mathfrak{g}$.*

The G-invariant Finsler functions on TG may be identified with the Minkowski norms on $\mathfrak{g}$. If $F : TG \longrightarrow R_+$ is an G-invariant Finsler function, then we may define $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$ by $\widetilde{F}(X) = F(e, X)$, where e denotes the identity in G. Conversely, if we are given a Minkowski norm $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$, then $\widetilde{F}$ arises from an G-invariant Finsler function $F : TG \longrightarrow R_+$ given by $F(g, X) = \widetilde{F}(X)$ for all $(g, X) \in G \times \mathfrak{g}$.
Let G be a connected Lie group, $L : G \times G \longrightarrow G$ the action being defined by the left-translations $L_g : G \longrightarrow G$, $g \in G$ and $TL : G \times TG \longrightarrow TG$ the action given by the tangent linear maps $TL_g : TG \longrightarrow TG$, $g \in G$ of the left-translations.
A smooth vector field $X : TG - \{0\} \longrightarrow TTG$ is said to be left-invariant if

$$TTL_g \circ X \circ TL_g^{-1} = X \quad \text{for all} \quad g \in G.$$

By a classical argument of calculus of variation we have the following proposition.

Proposition 3.1. *If $F : TG \longrightarrow R_+$ is a left-invariant Finsler metric then its geodesic spray X is left-invariant as well.*

Definition 3.2. *Let G be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra identified with the tangent space at the identity element, $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$ a Minkowski norm and F the left-invariant Finsler metric induced by $\widetilde{F}$ on G. A geodesic $\gamma : R \longrightarrow G$ is said to be* homogeneous *if there is a $Z \in \mathfrak{g}$ such that $\gamma(t) = \exp(tZ)\gamma(0)$, $t \in R$ holds. A tangent vector $X \in T_eG - \{0\}$ is said to be a* geodesic vector *if the 1-parameter subgroup $t \longrightarrow \exp(tX)$, $t \in R$, is a geodesic of F.*

The geodesic defined by a geodesic vector is obviously a homogeneous one. Conversely, let γ be a geodesic with $\gamma(0) = g$ which is homogeneous with respect to a 1-parameter group of left-translations, namely

$$\gamma(t) = \exp(tY)g, \quad t \in R,$$

then a homogeneous geodesic $\widetilde{\gamma}$ is given by

$$\begin{aligned} \widetilde{\gamma}(t) &= L_g^{-1} \circ \gamma(t) = L_g^{-1} \circ R_g \circ \exp(tY) \\ &= \exp(Ad(g^{-1})tY).e = \exp(Ad(g^{-1})tY)\widetilde{\gamma}(0), \end{aligned}$$

which means that $X = Ad(g^{-1})Y$ is a geodesic vector.

For results on homogeneous geodesics in homogeneous Finsler manifolds we refer to [14] — [16]. The basic formula characterizing geodesic vector in the Finslerian case was derived in [14], Theorem 3.1. For Lie groups with left invariant metrics we have the following theorem.

Theorem 3.1. [14] *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let F be a left-invariant Finsler metric on G. Then $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if*

$$g_X(X, [X, Z]) = 0$$

holds for every $Z \in \mathfrak{g}$.

4. Homogeneous geodesics and the critical points of the restricted Finsler function

Let G be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra, $Ad : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$ the adjoint action, $G(X) = \{Ad(g)X \mid g \in G\} \subset \mathfrak{g}$ the orbit of an element $X \in \mathfrak{g}$ and $G_X < G$ the isometry subgroup at X. The set $\frac{G}{G_X}$ of left-cosets of G_X endowed with its canonical smooth manifold structure admits the canonical left-action

$$\Lambda : G \times \frac{G}{G_X} \longrightarrow \frac{G}{G_X} \quad (g, aG_X) \longrightarrow gaG_X,$$

which is also smooth. Moreover, a smooth bijection $\rho : \frac{G}{G_X} \longrightarrow G(X)$ is defined by $\rho(aG_X) = Ad(a)X$ which thus yields an injective immersion into $\mathfrak{g}$ which is equivariant with respect to the actions Λ and Ad.

Now consider a Minkowski norm $\widetilde{F} : \mathfrak{g} \longrightarrow R$, then F defines a left-invariant Finsler metric on G by

$$F(x, U) = \widetilde{F}(TL_{x^{-1}}U), \quad U \in T_xG.$$

Let $Q(Z) = \widetilde{F}^2(Z), \quad Z \in \mathfrak{g}$. Using the formula $\widetilde{F}(Z) = \sqrt{g_Z(Z,Z)}$, we have $Q(Z) = g_Z(Z,Z)$. The smooth function $q = Q \circ \rho : \frac{G}{G_X} \longrightarrow R$ is called the restricted Minkowski norm on $\frac{G}{G_X}$.

In the following, we give an extension of results of [20] to left-invariant Finsler metrics.

Theorem 4.1. *Let G be a connected Lie group and $\widetilde{F}$ a Minkowski norm on its Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g} - \{0\}$ let $U \in \mathfrak{g}$ be such that $X \in G(U)$ for the corresponding adjoint orbit and let $gG_U \in \frac{G}{G_U}$ be the unique coset with $\rho(gG_U) = X$. Then X is a geodesic vector if and only if gG_U is a critical point of $q = Q \circ \rho$ the restricted Minkowski norm on $\frac{G}{G_U}$.*

Proof. The coset gG_U is a critical point of q if and only if $vq = 0$ for $v \in T_{gG_U}(\frac{G}{G_U})$. But as $\frac{G}{G_U}$ is homogeneous, for each v there is a $Z \in \mathfrak{g}$ such that $v = \widetilde{Z}(gG_U)$ where $\widetilde{Z} : \frac{G}{G_U} \longrightarrow T(\frac{G}{G_U})$ is the infinitesimal generator of the action Λ corresponding to Z. Consider also the infinitesimal generator $\widehat{Z} : \mathfrak{g} \longrightarrow T\mathfrak{g}$ of the adjoint action corresponding to Z. Since the injective immersion ρ is equivariant with respect to the action Λ and Ad the following holds: $\widehat{Z} \circ \rho = T\rho \circ \widetilde{Z}$. But then the following is valid:

$$\begin{aligned} v(q) = \widetilde{Z}(q)\Big|_{gG_U} &= \widetilde{Z}(Q \circ p)\Big|_{gG_U} = \left(T\rho\widetilde{Z}\right)\Big|_{gG_U} Q = (\widehat{Z} \circ \rho)\Big|_{gG_U} Q = \\ &= \left(\frac{d}{dt}\,|_{t=0}\ (Ad(\exp tZ)X)\right) Q = \frac{d}{dt}\Big|_{t=0} Q(Ad(\exp tZ)X) = \\ &= \frac{d}{dt}\Big|_{t=0} g_{Ad(\exp tZ)X}(Ad(\exp tZ)X, Ad(\exp tZ)X) = \\ = g_X([Z,X],X) &+ g_X(X,[Z,X]) + 2C_X([Z,X],X,X) = 2g_X([Z,X],X), \end{aligned}$$

where C_X is the Cartan tensor of F at X. It follows from the homogeneity of F that $C_X([Z,X],X,X) = 0$. Since the map $\alpha : \mathfrak{g} \longrightarrow T_{gG_U}(\frac{G}{G_U}),\ Z \longrightarrow \widetilde{Z}(gG_U)$ is an epimorphism, the assertion of the theorem follows. $\square$

Corollary 4.1. *Let G be a compact connected semi-simple Lie group and $\widetilde{F}$ a Minkowski norm on its Lie algebra $\mathfrak{g}$. Then each orbit of the adjoint action $Ad : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$ contains at least two geodesic vectors.*

Proof. Consider an orbit $G(X)$ of the adjoint action, the corresponding coset manifold $\frac{G}{G_X}$ and the injective immersion $\rho : \frac{G}{G_X} \longrightarrow \mathfrak{g}$. Since G is compact and semi-simple then the manifold $\frac{G}{G_X}$ becomes compact, and the restricted Minkowski norm $q = Q \circ \rho : \frac{G}{G_X} \longrightarrow R$ has at least two critical points. $\square$

The following corollary is a consequence of the preceding corollary. Two geodesics are considered different if their images are different.

Corollary 4.2. *Let G be compact connected semi-simple Lie group of rank ≥ 2 and $\widetilde{F}$ a Minkowski norm on its Lie algebra. Then the left-invariant Finsler metric F induced by $\widetilde{F}$ on G has infinitely many homogeneous geodesic issuing from the identity element.*

Proof. The proof is similar to the Riemannian case, so we omit it. □

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