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BIVARIATE MEAN VALUE INTERPOLATION ON CIRCLES

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Abstract. In this paper we prove that the bivariate mean-value interpolation problem, where part of the interpolation parameters are integrals over concentric circles is not poised.

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1. Introduction

Denote by $\Pi_n = \Pi_n(\mathbb{R}^2)$ the space of bivariate polynomials of total degree not exceeding n

$$\Pi_n = \left\{ p(x,y) = \sum_{i+j \le n} a_{ij} x^i y^j : i, j \in Z_+ \right\}.$$

Set

$$N := dim\Pi_{\rm n} = \binom{\rm n+2}{2}.$$

Let

$$\mathbb{D} := \mathbb{D}^{(n)} = \{D_k : k = 1, \dots, N\}$$

be a collection of Lebesgue measurable sets of finite nonzero measure.

In this paper the following mean-value interpolation problem is considered. Find a unique polynomial $p \in \Pi_n$ whose mean-values over the sets D_k are equal to given numbers c_k, $k = 1, \dots, N$

$$\frac{1}{\mu(D_k)} \int\int_{D_k} p(x,y)dxdy = c_k, \quad k = 1, \dots, N, \tag{1.1}$$

where μ is the Lebesgue measure. Denote this interpolation problem by $(\Pi_n, \mathbb{D})$. More precisely we consider the mean-value interpolation problem (1.1), where part of sets of $\mathbb{D}$ are concentric circles.

Definition 1.1. *The mean-value interpolation problem* $(\Pi_n, \mathbb{D})$ *is poised if for any real values* $c_k, k = 1, \dots, N$, *there exists a unique polynomial* $p \in \Pi_n$ *satisfying the conditions* (1.1).

In the sequel we will use the following well known

Proposition 1.1. *The mean-value interpolation problem* $(\Pi_n, \mathbb{D})$ *is poised if and only if*

$$p \in \Pi_n, \int\int_{D_k} p(x,y)dxdy = 0, \quad k = 1, \ldots, N \quad \textit{implies} \quad p = 0.$$

It is worth mentioning that in the case when $\mathbb{D}$ is a set of circles with the same radius, i.e.

$$\mathbb{D} = \mathbb{B} := \{B_{\mathbf{a}_i,r} : i = 1, \ldots, N\}$$

where $B_{\mathbf{a}_i,r}$, is the circle of radius $r \in \mathbb{R}_+$ centered at $\mathbf{a}_i \in \mathbb{R}^2$ then the following

Theorem 1.1. *(see [4]) The mean-value interpolation problem* $(\Pi_n, \mathbb{B})$ *is poised if and only if the pointwise Lagrange interpolation with* Π_n *and the centers of circles of* $\mathbb{B}$ *is poised.*

In the next section we use the following

Theorem 1.2. *(see [2]) The mean-value interpolation problem* $(\Pi_1, \mathbb{D}^{(1)})$ *is poised if and only if the centroids of sets of* $\mathbb{D}^{(1)}$ *are not collinear.*

For some other versions of mean-value interpolation problem we refer to [1 - 4].

2. The result

Denote by $[x]$ the greatest integer not exceeding x.

Theorem 2.1. *Suppose that among the regions of interpolation problem* $(\Pi_n, \mathbb{D})$ *there are* $[\frac{n}{2}]+2$ *concentric circles, where* $n \geq 1$. *Then the mean-value interpolation problem* $(\Pi_n, \mathbb{D})$ *is not poised.*

To prove Theorem 2.1 we use the following lemma.

Lemma 2.1. *If Theorem 2.1 is true for* $n = 2k$, *then it is true also for* $n = 2k + 1$.

Proof. Assuming that the parameters related to the concentric circles are linearly dependent in the case $n = 2k$, i.e.,

$$\sum_{l=1}^{k+2} c_l \int\int_{D_l} p\, dx\, dy = 0, \quad \text{for any} \quad p \in \Pi_{2k}, \tag{2.1}$$

where not all c_l are zero, we establish a similar dependence in the case $n = 2k+1$. Without loss of generality we assume that the concentric circles are centered at the origin. For any $p \in \Pi_{2k+1}$ we set

$$p(x,y) = \sum_{i+j=2k+1} a_{ij} x^i y^j + q(x,y),$$

where $q \in \Pi_{2k}$. Then we have

$$\sum_{l=1}^{k+2} c_l \int\int_{D_l} p(x,y)dxdy = \sum_{l=1}^{k+2} c_l \int\int_{D_l} a_{2k+1,0} x^{2k+1} dxdy + \cdots +$$
$$+ \sum_{l=1}^{k+2} c_l \int\int_{D_l} a_{0,2k+1} y^{2k+1} dxdy + \sum_{l=1}^{k+2} c_l \int\int_{D_l} q(x,y)dxdy =$$
$$= 0 + \cdots + 0 + \sum_{l=1}^{k+2} c_l \int\int_{D_l} q(x,y)dxdy = 0.$$

In the last equality we used (2.1). □

Proof of Theorem 2.1. First let $n = 1$. In this case, the interpolation is not poised since according to Theorem 1.2, the center of two concentric circles along with the center of third circle are on a straight line.

Now in view of Lemma 2.1, it is sufficient to assume that $n = 2k$ where $n > 1$. According to Proposition 1.1 it is enough to show that there exists some $p \in \Pi_n$ with

$$\int\int_{D_l} p(x,y)dxdy = 0, \quad l = 1, \ldots, N, \quad p \neq 0. \tag{2.2}$$

Here we assume that $D_l := B_{\mathbf{0},r_l}$, $l = 1, \ldots, k+2$, are the concentric circles with radii r_l centered at $(0,0)$, while the remaining $N-(k+2)$ regions are arbitrary.

Let $p(x,y) = \sum_{i+j\le n} a_{ij} x^i y^j$ then we have

$$\int\int_{D_l} p(x,y)dxdy = r_l^2 \int\int_{D:x^2+y^2\le 1} p(r_l x, r_l y)dxdy =$$
$$= r_l^2 \int\int_{D} \sum_{i+j\le n} a_{ij} x^i y^j r_l^{i+j} dxdy = r_l^2 \sum_{i+j\le n} r_l^{i+j} a_{ij} \int\int_D x^i y^j dxdy.$$

Using the fact that the integral of a monomial $x^i y^j$, $i,j \in Z_+$ over circles D_l, $l = 1, \ldots, k+2$ vanish if i or j is odd, we get

$$\int\int_{D_l} p(x,y)dxdy = \pi r_l^2 [a_{0,0}] r_l^0 + \frac{\pi r_l^2}{4} . r_l^2 [a_{2,0} + a_{0,2}] + ... \tag{2.3}$$
$$+ r_l^{2k+2} \left[a_{2k,0} \int\int_D x^{2k} dxdy + a_{2k-2,2} \int\int_D x^{2k-2} y^2 dxdy + ... + a_{0,2k} \int\int_D y^{2k} dxdy \right].$$

Now consider the following homogeneous linear system with left-hand sides coincide with the above expressions in brackets:

$a_{0,0} = 0$
$a_{2,0} + a_{0,2} = 0$
.
.
.
$a_{2k-2,0} \int\int_D x^{2k-2} dxdy + \cdots + a_{0,2k-2} \int\int_D y^{2k-2} dxdy = 0$
$a_{2k,0} \int\int_D x^{2k} dxdy + \cdots + a_{0,2k} \int\int_D y^{2k} dxdy = 0.$

Next we add to this system the homogeneous conditions over the remaining $N-(k+2)$ arbitrary regions:

$$\int\int_{D_l} p(x,y) dxdy = \sum_{i+j\le n} a_{ij} \int\int_{D_l} x^i y^j = 0, \quad l = k+3, \ldots, N.$$

The resulting system has $N-1$ equations and N unknowns which are the coefficients of p. Therefore it has a non-trivial solution. It is easily seen that the polynomial with these coefficients satisfies (2.3). Hence the problem is not poised. $\square$

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