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INHOMOGENEOUS RANDOM PLANAR TESSELLATIONS GENERATED BY LINES

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Аннотация. Random planar tessellations in bounded convex windows are generated by dividing random cells with random lines. It is suggested that the random STIT tessellations of Nagel and Weiss, if restricted to a bounded convex window, can be interpreted as a special case.

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1. Introduction

The Euclidean plane is represented by $\mathbb{R}^2$ and addressed simply as the "plane". Let W be a non-empty, bounded, convex, polygonal, open subset of R^2, called window. Loosely speaking, our random tessellations in W will be constructed in a way of the following kind:

By help of a sequence of independent, identically distributed random lines $\gamma_1, \gamma_2, \ldots$ in the plane that intersect W, a sequence of random tessellations $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2, \ldots$ in W will be generated. It is supposed that γ_1, γ_2, γ_3 cannot meet in one point a. s. The number ζ_k of cells in $\mathcal{T}_k$ will be less or equal to $k+1$.
The initial tessellation $\mathcal{T}_0$ has only the single cell W, it is called the empty tessellation. Hence, $\zeta_0 = 1$.

The tessellation $\mathcal{T}_1$ consists of two cells, namely of the two parts into which W is divided by the line γ_1. The intersection $\gamma_1 \cap W$ of the line γ_1 with the window is called an I-segment. We have $\zeta_1 = 2$.

The line γ_2 chooses randomly with probability $1/2$ one of the two cells of $\mathcal{T}_1$, say C. If the intersection of γ_2 with C is empty, the line γ_2 will be discarded. Otherwise, C is divided into two cells. The unaffected cells of $\mathcal{T}_1$ together with the possibly new cells form the tessellation $\mathcal{T}_2$. If the line γ_2 is not rejected, the intersection of that line with C is said to be an I-segment. Note $\zeta_2 \le 3$.

In the n-th step, the line γ_n intersects some cells of $\mathcal{T}_{n-1}$, say $D_1, \ldots, D_m$. It is clear that $1 \leq m \leq n$ a. s. Independently of the past, now some random decisions will be made. With probability $1 - m/n$, the line γ_n leaves the tessellation $\mathcal{T}_{n-1}$ unchanged; we would say that γ_n is rejected. With the complementary probability m/n, a member of the set of cells $\{D_1, \ldots, D_m\}$ is selected for further treatment, where each of them has equal chance of being chosen. If the cell D_k is selected, it is divided by γ_n into two parts which are considered as cells of $\mathcal{T}_n$ as well as the unaffected cells of $\mathcal{T}_{n-1}$. The intersection $\gamma_n \cap D_k$ is called an I-segment of $\mathcal{T}_n, \mathcal{T}_{n+1}, \ldots$. The number ζ_n of cells of $\mathcal{T}_n$ fulfills $\zeta_n = \zeta_{n-1}$ with probability $(n-m)/n$, and $\zeta_n = \zeta_{n-1} + 1$ with probability m/n.

The sequence $(\zeta_k)_{k=0,1,\ldots}$ is non-decreasing with $\zeta_k \leq k+1$.

A more formalized description of the construction is presented in section 2.

A large variety of similar models for creating random tessellations by cell division is treated by Cowan in [3]. Our selection of cells is "perimeter-weighted" in the sense of Cowan, where "perimeter" means a pseudo-perimeter according to Ambartzumian [1, 2], cf. subsection 3.2 and [9].

Note that in our terminology cells and edges are non-empty *open* sets. The union of all cells and their boundaries is equal to the closure of W.

The random tessellations $\mathcal{T}_k$ are addressed as "random line-generated tessellations"; $k = 0, 1, \ldots$.

Some mean values for the random line-generated tessellations $\mathcal{T}_k$ are calculated in Section 3.

The sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots$ may be considered as a Markov chain with discrete time and the set of tessellations in W as state space. Note that then the transition probabilities are *not* time-homogeneous. The Markov chain $\mathcal{T}_0, \mathcal{T}_1, \ldots$ can be interpreted as a process of cell division. In contrast to the Markovian process of cell division treated in [7, 8], it lives on a *discrete* time axis, and in this way, many technical difficulties can be avoided. Cowan also prefers a discrete time axis for the Markovian processes in [3] producing random cells and random tessellations.

Not only the random states $\mathcal{T}_n$ of the chain at fixed time instants n can be investigated, but also the states $\mathcal{T}_\nu$ at a random time instant ν.

We are only interested in the special case that the random non-negative integer ν is independent of the sequence $\mathcal{T}_1, \mathcal{T}_2, \ldots$ and has a geometric distribution. Such random tessellations T_ν could be called mixed random line-generated tessellations; we simply speak of "mixed line-generated tessellations". For a strong definition and treatment, see Section 4.

If the distribution of the random lines $\gamma_1, \gamma_2, \ldots$ can be obtained by the restriction of a *shift-invariant* line measure to the set of lines intersecting W, then we get an interesting subclass of mixed line-generated tessellations. We speak of the *homogeneous case.* The mean values evaluated in Section 5 coincide in the homogeneous case with that for corresponding random STIT tessellations in W of NAGEL AND WEISS [4, 5, 6]. Relying on unpublished results, the author expects that in the homogeneous case, the mixed line-generated tessellation $\mathcal{T}_\nu$ has the same distribution as a corresponding random STIT tessellation in W. A related conjecture is formulated in Section 5.

One of the main achievements of NAGEL AND WEISS [5], the construction of a spatially homogeneous random tessellation in the *whole* plane, is beyond the scope of this paper. But for many characteristics of such unbounded random STIT tessellations, there exist unbiased estimators depending on bounded regions only. Hence, if conjecture 3 of Section 5 is true, on principle, all those characteristics can be calculated by the methods presented in the following. These procedures are not very elegant, but are almost elementary.

2. A sequence of random tessellations

As in the introduction, let W be a non-empty, bounded, convex, polygonal open subset of the plane, called *window.*

Denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{N}_0$ the set of non-negative integers, by $\mathcal{G}$ the set of all lines in the plane, by $\mathfrak{G}$ the common $\sigma-$algebra over $\mathcal{G}$ [10, 9], by $\mathcal{G}_x$ the set of lines containing $x \in \mathbb{R}^2$, and given any subset $B \subset \mathbb{R}^2$, by $\mathcal{G}_B$ the set of all lines having points in B,

$$\mathcal{G}_B = \{g \in \mathcal{G} : g \cap B \neq \varnothing\}.$$

The $\sigma-$algebra $\mathfrak{G} \cap \mathcal{G}_B$ over $\mathcal{G}_B$ is denoted by $\mathfrak{G}_B$.

Let Q be a probability measure on the measurable space $[\mathcal{G}, \mathfrak{G}]$ with $Q(\mathcal{G}_W) = 1$, i.e. $Q(\mathcal{G} \setminus \mathcal{G}_W) = 0$. Furthermore, it is assumed that $Q(\mathcal{G}_x) = 0$ for all $x \in \mathbb{R}^2$. Such measures are said to be *bundleless*; they are in particular atomless.

Furthermore, let

$$(\alpha_n)_{n\in\mathbb{N}}, \quad (\gamma_n)_{n\in\mathbb{N}}$$

be a system of random variables with the property of maximal independence, defined on a probability space $[\Omega, \mathfrak{F}, \mathcal{P}]$, where α_n is a random non-negative integer uniformly distributed in the finite set $\{0, \ldots, n-1\}$ and γ_n is a random line distributed according to Q; $n \in \mathbb{N}$.

If γ is a line not containing the origin O, denote by $\overline{\gamma}$ the open halfplane bounded by γ and containing O and by $\underline{\gamma}$ the other open halfplane bounded by γ. It doesn't matter how the halfplanes are denoted when the line γ goes through O, because our line measure Q is assumed to be bundleless.

We define recursively a sequence of $(n+1)$- tuples

$$(C_{00}),(C_{10},C_{11}),\dots(C_{n0},\dots C_{nn}),\dots$$

of so-called *quasi-cells* by

$$C_{00}=W,\quad C_{10}=\underline{\gamma}_1\cap W,\quad C_{11}=\overline{\gamma}_1\cap W$$

and for $n=2,3,\dots$:

$$C_{nj} \quad = \quad \begin{cases} C_{n-1,j} & \text{if } j\in\{0,\dots,n-1\}\,; \quad j\neq\alpha_n, \\ C_{n-1,\alpha_n}\cap\underline{\gamma}_n & \text{if } j=\alpha_n, \\ C_{n-1,\alpha_n}\cap\overline{\gamma}_n & \text{if } j=n. \end{cases}$$

Many of the quasi-cells are empty. The lines by which they are produced do not contribute in a visible way to the system of boundary lines of the random tessellation. In the language of simulation theory, one would say that these lines are rejected. The non-empty quasi-cells are called *cells*.

The *cells* belonging to the $(n+1)-$tuple $(C_{n0},C_{n1},\dots,C_{nn})$ form the tessellation $\mathcal{T}_n$ of the plane we are interested in; $n\in\mathbb{N}_0$.

By a *cell* C in W we mean a non-empty convex polygonal *open* subset of W, and by a *tessellation* $\mathcal{T}$ in W a finite set of non-overlapping cells in W with the property that the union of the closures of these cells is equal to the closure of W.

Remark 1. *By a suitable definition of the state space, the random sequence*

$$(C_{00}),\ (C_{10},C_{11}),\ (C_{20},C_{21},C_{22}),\ (C_{30},C_{31},C_{32},C_{33}),\cdots$$

can be interpreted as a Markov chain on the discrete time-axis $\mathbb{N}_0$ *with time-homogeneous transition probabilities.*

A sequence J_1, J_2, $J_3,\dots$ of so-called quasi-segments is given by $J_1=\gamma_1\cap W$, and in general

$$J_n=\gamma_n\cap C_{n-1,\alpha_n};\quad n\in\mathbb{N}.$$

Many of the quasi-segments are empty. If J_n is non-empty, it is called an *I-segment* of $\mathcal{T}_n,\mathcal{T}_{n+1},\dots$. The non-empty members of $\{J_1,\dots,J_n\}$ are the I-segments of $\mathcal{T}_n$; $n\in\mathbb{N}$.

Remark 2. *Analogous random tessellations on the sphere can be generated by random great circles.*

3. Some mean values

3.1. Mean total edge length . Let E_k be the mean total length of all edges in the random tessellation $\mathcal{T}_k$; $k = 0, 1, \ldots$. Then it is equal to the mean total length of all I-segments in $\mathcal{T}_k$.

Obviously, $E_0 = 0$ and

$$E_1 = \int Q(dg)|g \cap W|,$$

where $|g \cap W|$ denotes the length of the segment $g \cap W$.

Relying on the construction of $\mathcal{T}_k$ described in Section 2, we obtain

$$E_2 = \int Q(dg_1)\left(|g_1 \cap W| + \frac{1}{2}\int Q(dg_2)\left(|g_2 \cap \overline{g}_1 \cap W| + |g_2 \cap \underline{g}_1 \cap W|\right)\right)$$

or

$$E_2 = \left(1 + \frac{1}{2}\right) E_1.$$

Analogously,

$$E_n = \left(1 + \frac{1}{2} + \ldots \frac{1}{n}\right) E_1; \quad n \in \mathbb{N}.$$

The generating function $G_E : [0, 1) \to [0, \infty)$ for the sequence $E_0, E_1, \ldots$ is defined by

$$G_E(z) = \sum_{k=0}^{\infty} z^k E_k.$$

We obtain

$$G_E(z) = E_1 \sum_{k=1}^{\infty} z^k \sum_{m=1}^{k} \frac{1}{m} = E_1 \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m}^{\infty} z^k = \frac{E_1}{1-z} \sum_{m=1}^{\infty} \frac{z^m}{m}.$$

Finally,

$$G_E(z) = -E_1 \frac{\ln(1-z)}{1-z}. \tag{3.1}$$

Remark 3. *Analogously, the entire intensity measure of total edge length can be evaluated for $\mathcal{T}_k$. The total mass of this finite measure is then the mean total edge length E_k; $k \in \mathbb{N}_0$, cf. proposition 1.*

3.2. Mean total Ambartzumian length of edges . The results in this subsection are not only interesting for its own; they are also helpful for calculating the mean values in subsection 3.3.

Let R be a locally finite bundleless measure on $[\mathcal{G}, \mathfrak{G}]$. The $R-$pseudo-length $|\mathfrak{s}|_R$ in the sense of Ambartzumian [1, 2] of a segment $\mathfrak{s}$ in the plane is defined by

$$|\mathfrak{s}|_R = \frac{1}{2} R(\mathcal{G}_{\mathfrak{s}}).$$

Although for certain R, a non-empty segment can have the $R-$pseudo-length 0, we simply speak of "R-length" for short.

It is important for us that the R-length as a function on the set of segments is additive on lines.

If R is shift- and rotation-invariant, the $R-$length is proportional to the Euclidean length, and the results in this subsection reduce to that in subsection 3.1.

In the case $R = Q$, we speak of the *intrinsic* length belonging to the random tessellations $\mathcal{T}_k$; $k \in \mathbb{N}_0$.

Denote the mean total R-length of edges of $\mathcal{T}_k$ by $A_k(R)$.

Analogously to subsection 3.1, we obtain

$$\begin{aligned}
A_0(R) &= 0, \\
A_1(R) &= \int Q(dg)|g \cap W|_R \\
&= \frac{1}{2}\int Q(dg)R(\mathcal{G}_{g\cap W}), \\
\dots &= \dots, \\
A_n(R) &= \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) A_1(R); \quad n \in \mathbb{N}
\end{aligned}$$

The result for $n = 1$ can be written in the form

$$A_1(R) = \frac{1}{2}\int Q(dg)\int R(dh)M(g,h),$$

where $M(g,h) = 1$ if $g, h \in \mathcal{G}$ meet in W and $M(g,h) = 0$ otherwise.

Analogously to formula (3.1), the generating function $G_R : [0,1) \to [0,\infty)$ for the sequence $A_0(R), A_1(R), A_2(R), \dots$ is given by

$$G_R(z) = -A_1(R)\frac{\ln(1-z)}{1-z}. \tag{3.2}$$

3.3. **Mean number of nodes .** Denote by N_k the mean number of nodes of $\mathcal{T}_k$ in W; $k \in \mathbb{N}_0$. It is easily seen that

$$N_0 = 0,\ N_1 = 0,\ N_2 = 2A_1(Q),$$

where $A_1(Q)$ is the mean intrinsic length, i.e. the mean $Q-$length, of the single I-segment in $\mathcal{T}_1$. Generally, we find

$$N_{n+1} = N_n + \frac{4}{n+1}A_n; \quad n \in \mathbb{N},$$

where $A_n = A_n(Q)$ is the mean total intrinsic length of all edges in $\mathcal{T}_n$, which was evaluated in subsection 3.2. This recursion formula leads to

$$N_{n+1} = 4\sum_{m=1}^{n}\frac{A_m}{m+1}; \quad n \in \mathbb{N}. \tag{3.3}$$

The generating function $G_N : [0,1) \to [0,\infty)$ for the sequence $(N_k)_{k=0,1,\ldots}$ is defined by

$$G_N(z) = \sum_{k=0}^{\infty} z^k N_k = \sum_{k=2}^{\infty} z^k N_k.$$

Formula (3.3) implies

$$G_N(z) = \sum_{n=1}^{\infty} N_{n+1} z^{n+1} = 4 \sum_{n=1}^{\infty} z^{n+1} \sum_{m=1}^{n} \frac{A_m}{m+1} =$$

$$= 4 \sum_{m=1}^{\infty} \frac{A_m}{m+1} \sum_{n=m}^{\infty} z^{n+1} = \frac{4}{1-z} \sum_{m=1}^{\infty} \frac{A_m}{m+1} z^{m+1}$$

or

$$(1-z) G_N(z) = 4 \sum_{m=1}^{\infty} \frac{A_m}{m+1} z^{m+1}.$$

Differentiation leads to

$$\frac{d}{dz}\Big((1-z) G_N(z)\Big) = 4 \sum_{m=1}^{\infty} A_m z^m.$$

The power series on the right hand side is equal to $G_Q(z)$ provided by formula (3.2), hence

$$(1-z) G_N(z) = -4 A_1(Q) \int_0^z du \frac{\ln(1-u)}{1-u}$$

and finally

$$G_N(z) = \frac{2}{1-z} A_1(Q) \ln^2(1-z); \quad 0 \le z < 1, \tag{3.4}$$

where

$$A_1(Q) = \frac{1}{2}(Q \times Q)\left\{(g,h) \in \mathcal{G}^2 : g \cap h \cap W \neq \varnothing\right\}$$

The numbers N_k itself can be derived from (3.4) by representing the analytical function in z on the right-hand side as a power series.

Remark 4. *Even the entire intensity measure for nodes can be evaluated by similar methods. The total mass of such an intensity measure is then equal to the corresponding mean value calculated above, cf. proposition 2.*

3.4. **A restricted capacity functional.** Let K be a compact convex subset of W and S_k the probability that K is contained in exactly one cell of the random tessellation $\mathcal{T}_k$, i. e. that no I-segment of $\mathcal{T}_k$ intersects K; $\quad k \in \mathbb{N}_0$. Obviously,

$$S_0 = 1,\ S_1 = Q(\mathcal{G} \setminus \mathcal{G}_K)$$

and

$$S_2 = \int_{\mathcal{G}\setminus\mathcal{G}_K} Q(dg_1)\left(\frac{1}{2}+\frac{1}{2}\int_{\mathcal{G}\setminus\mathcal{G}_K} Q(dg_2)\right) = S_1\left(\frac{1}{2}+\frac{1}{2}S_1\right).$$

Analogously,

$$S_n = S_{n-1}\left(\frac{n-1}{n}+\frac{1}{n}S_1\right)$$

or

$$S_n = \left(\frac{S_1+n-1}{n}\right)S_{n-1}; \quad n \in \mathbb{N}.$$

Finally,

$$S_n = \left(\frac{S_1+n-1}{n}\right)\left(\frac{S_1+n-2}{n-1}\right)\cdots\left(\frac{S_1+1}{2}\right)S_1; \quad n \in \mathbb{N}$$

or

$$S_k = (-1)^k\binom{-S_1}{k}; \quad k = 0, 1, \ldots.$$

Let $G_S : [0,1) \to [0,1]$ be the generating function for the sequence $S_0, S_1, \ldots$, i.e.

$$G_S(z) = \sum_{k=0}^{\infty} z^k S_k.$$

We obtain

$$G_S(z) = \sum_{k=0}^{\infty} z^k (-1)^k \binom{-S_1}{k}.$$

The right hand side is the well-known binomial series, and hence

$$G_S(z) = (1-z)^{-S_1}; \quad 0 < z < 1. \tag{3.5}$$

4. Mixed line-generated tessellations

4.1. **Definition.** Let ν be a random non-negative integer independent of the sequences

$$(\alpha_n)_{n\in\mathbb{N}}, (\gamma_n)_{n\in\mathbb{N}}$$

and geometrically distributed with parameter $0 < p < 1$:

$$\mathcal{P}(\nu = k) = p(1-p)^k; \quad k \in \mathbb{N}_0.$$

It is convenient for our purposes to introduce the new parameter $t = -\ln p$ for the geometric distribution, i.e.

$$\mathcal{P}(\nu = k) = e^{-t}(1-e^{-t})^k; \quad k \in \mathbb{N}_0.$$

We are interested in the random tessellation $\mathcal{T}_\nu$. Let $\mathcal{T}^t$ be a random tessellation distributed as $\mathcal{T}_\nu$, where ν has a geometric distribution with the new parameter t; $0 < t < \infty$.

Formally, the random tessellations $\mathcal{T}_k$ are mappings from the probability space $[\Omega, \mathfrak{F}, \mathcal{P}]$ into the space of tessellations; $k \in \mathbb{N}_0$. These mappings induce a $\sigma-$algebra over the space of tessellations. The distributions P_k (laws) of the $\mathcal{T}_k$ are probability measures on that $\sigma-$algebra. The law P^t of $\mathcal{T}^t$ is given by

$$P^t = \sum_{k=0}^{\infty} e^{-t}(1-e^{-t})^k P_k.$$

The random tessellation $\mathcal{T}^t$ is addressed as a *mixed line generated tessellation* with characteristics Q, t.

In addition to theorem 1 in Subsection 4.2.4 and corollary 6 in Section 5, the following observation may be regarded as a motivation for investigating such mixtures.

Conjecture 1. *Let $\mathcal{T}$ be a mixed line-generated tessellation in W with characteristics Q, t and $\widehat{W}$ a window in the sense of section 2 with $\widehat{W} \subset W$ and $Q(\mathcal{G}_{\widehat{W}}) > 0$. Then the cutout of $\mathcal{T}$ in $\widehat{W}$ can be interpreted as a mixed line-generated tessellation in $\widehat{W}$ with characteristics*

$$\widehat{Q} = \frac{1}{Q(\mathcal{G}_{\widehat{W}})} Q(\centerdot \cap \mathcal{G}_{\widehat{W}}), \quad \widehat{t} = tQ(\mathcal{G}_{\widehat{W}}).$$

4.2. **Mean values .**

4.2.1. *General formula.* If M_k is one of the mean values for $\mathcal{T}_k$ treated in section 3, the corresponding mean value of $\mathcal{T}^t$ is denoted by M^t, and we get

$$M^t = \sum_{k=0}^{\infty} e^{-t}(1-e^{-t})^k M_k$$

or

$$M^t = e^{-t} G_M(1-e^{-t}), \tag{4.1}$$

where $G_M : (0,1) \to [0,\infty)$ means the generating function for the sequence $(M_k)_{k\in\mathbb{N}_0}$:

$$G_M(z) = \sum_{k=0}^{\infty} z^k M_k.$$

Now, the characteristics can be easily deduced from that in Section 3.

4.2.2. *Edge length intensity measure.* According to formula (4.1), the mean total edge length E^t of $\mathcal{T}^t$ is equal to $e^{-t}G_E(1-e^{-t})$. Combining this with (3.1), we find

$$E^t = t \int Q(dg)|g \cap W|.$$

Obviously, in an analogous manner the entire edge length intensity measure can be deduced.

For easy formulation, the intensity measures are described as measures on $\mathbb{R}^2$ having zero mass at $\mathbb{R}^2 \setminus W$. A measure on $\mathbb{R}^2$ in our sense is a measure on the measurable space $[\mathbb{R}^2, \mathfrak{R}_2]$, where $\mathfrak{R}_2$ denotes the Borel $\sigma-$algebra over $\mathbb{R}^2$. In this manner, we often do not mention the $\sigma-$algebra involved if there is no danger of confusion.

Proposition 1. *The edge length intensity measure of* $\mathcal{T}^t$ *is given by*

$$t \int Q(dg)\mu(g; \centerdot \cap W),$$

where $\mu(g; \centerdot)$ *is a special measure on* $\mathbb{R}^2$*, namely the 1-dimensional Hausdorff-measure concentrated at* g.

Note that the total mass of this intensity measure is equal to E^t.

Corollary 1. *The edge length intensity measure of* $\mathcal{T}^t$ *is equal to* $\beta_t(\centerdot \cap W)$*, where* β_t *denotes the edge length intensity measure of a Poisson line field in* $\mathbb{R}^2$ *with intensity measure* tQ *(on* $\mathcal{G}$*).*

A Poisson line field in $\mathbb{R}^2$ is a Poisson hyperplane process (mosaic) in the case of dimension $d = 2$ in the sense of Schneider and Weil [10].

4.2.3. *Node intensity measure.* According to formula (4.1), the mean number of nodes N^t of $\mathcal{T}^t$ is equal to $e^{-t}G_N(1-e^{-t})$. Combining this with (3.4), we find

$$N^t = t^2 c(Q),$$

where

$$c(Q) = (Q \times Q)\left\{(g,h) \in \mathcal{G}^2 : g \cap h \cap W \neq \varnothing\right\}. \tag{4.2}$$

Analogously, the complete node intensity measure can be calculated.

Proposition 2. *The node intensity measure of* $\mathcal{T}^t$ *is given by*

$$t^2 \int Q(dg) \int Q(dh)\delta(g,h; \centerdot),$$

where $\delta(g,h;\centerdot)$ *denotes the Dirac measure on* $\mathbb{R}^2$ *concentrated at the intersection point* $g \cap h$ *of the lines* g, h*, if this intersection point exists and is contained in* W*, and* $\delta(g,h;\centerdot)$ *is equal to the zero-measure otherwise.*

Note that the total mass of this node intensity measure is equal to N^t.

Corollary 2. *The node intensity measure of* $\mathcal{T}^t$ *is equal to* $2\nu_t(\centerdot \cap W)$*, where* ν_t *denotes the node intensity measure of a Poisson line field in* $\mathbb{R}^2$ *with intensity measure* tQ *(on* $\mathcal{G}$*).*

4.2.4. *Restricted capacity functional.* As a consequence of formulas (3.5), (4.1) we have

$$S^t(K) = \exp\left(-tQ(\mathcal{G}_K)\right), \tag{4.3}$$

where $K \subset W$ is convex and compact.

Proposition 3. *The restricted capacity functional for $\mathcal{T}^t$ is equal to that for the cutout in W of a Poisson line tessellation with intensity measure tQ (on $\mathcal{G}$).*

A remarkable consequence should be pointed out.

Theorem 1. *The cell of $\mathcal{T}^t$ containing a fixed point $x \in W$ has the same distribution as the intersection with W of the cell containing x of a Poisson line tessellation with intensity measure tQ (on $\mathcal{G}$).*

4.3. **Remark on iterations.** The following considerations are devoted to specialists already familiar with the notions of iteration (nesting) of random tessellations and stability under iteration [5], [7] .

The leading normalized line measure Q is fixed in this subsection.
Given the mixed line-generated tessellations $\mathcal{T}^t$ with law P^t and $\mathcal{T}^s$ with law P^s, let $\mathcal{Y}_1, \mathcal{Y}_2, \ldots$ be a sequence of i. i. d. copies of $\mathcal{T}^s$, independent of $\mathcal{T}^t$. If $\{Z_1, \ldots, Z_\kappa\}$ is the set of cells of $\mathcal{T}^t$ and $\mathfrak{Y}_n$ the set of cells of $\mathcal{Y}_n$ $(n = 1, 2, \ldots)$, then the set of cells

$$\bigcup_{n=1}^{\kappa} (\mathfrak{Y}_n \cap Z_n)$$

forms a new tessellation, the law of which is denoted by $P^t \boxplus P^s$.

Conjecture 2. *The class of all mixed line-generated tessellations (related to Q) as a whole is stable under iteration in the following sense: Every operation of iteration maps the mentioned class into itself, i. e. an iterated mixed line-generated tessellation is again a mixed line-generated tessellation. If the mixed line-generated tessellation $\mathcal{T}^t$ is iterated according to the law P^s of $\mathcal{T}^s$, then the law $P^t \boxplus P^s$ of the outcome fulfils*

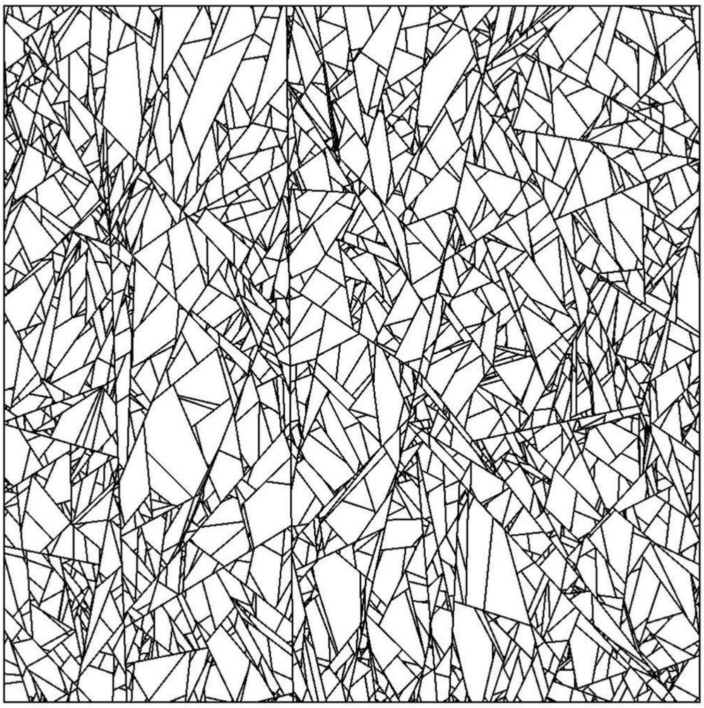
$$P^t \boxplus P^s = P^{t+s}.$$

5. Homogeneous case

Let Λ be a *shift-invariant*, locally finite measure on the space of lines $[\mathcal{G}, \mathfrak{G}]$ with $\Lambda(\mathcal{G}_W) = 1$, not concentrated on a set of parallel lines.

Denote by $\mathcal{T}^t$ the mixed line-generated tessellation in W in the sense of Section 4, now related to the line measure $Q = \Lambda(. \cap \mathcal{G}_W)$.

As a consequence of proposition 1, we obtain the following result.



Proposition 4. *The edge length intensity measure of* $\mathcal{T}^t$ *is equal to*

$$\frac{t}{2}\Lambda(\mathcal{G}_B)\mu_2(\centerdot \cap W),$$

where B *is the unit disk, and* μ_2 *denotes the Lebesgue measure on* $\mathbb{R}^2$.

Let $\mathcal{H}^t$ be a random STIT tessellation [5] in the whole plane related to the line measure $t\Lambda$.

Corollary 3. *According to* [6] *and proposition 4, the cutout in* W *of the random STIT tessellation* $\mathcal{H}^t$ *has the same edge length intensity measure as* $\mathcal{T}^t$.

Now, proposition 2 is applied to the homogeneous case.

Proposition 5. *The node intensity measure of* $\mathcal{T}^t$ *is equal to*

$$\frac{t^2}{4}\int_{\mathcal{G}_B}\Lambda(dg)\int_{\mathcal{G}_B}\Lambda(dh)\,|\sin\measuredangle(g,h)|\,\mu_2(\centerdot \cap W).$$

Corollary 4. *According to* [6] *and proposition 5, the cutout in* W *of the random STIT tessellation* $\mathcal{H}^t$ *has the same node intensity measure as* $\mathcal{T}^t$.

Proposition 6. *The restricted capacity functional of $\mathcal{T}^t$ in formula (4.3) can be written in the form*

$$S^t(K) = \exp(-t\Lambda(\mathcal{G}_K)); \quad K \subset W \text{ compact, convex.}$$

Corollary 5. *According to* [6] *and proposition 6, the cutout in W of the random STIT tessellation $\mathcal{H}^t$ has the same restricted capacity functional as $\mathcal{T}^t$.*

Note also the following consequence.

Corollary 6. *The cell of $\mathcal{T}^t$ containing the point $x \in W$ as well as the intersection with W of the cell in $\mathcal{H}^t$ containing x are distributed as the intersection with W of the Crofton cell related to Λ, x.*

If conjecture 2 could be verified, also the following conjecture is true.

Conjecture 3. *The random tessellations $\mathcal{H}^t$ restricted to W and $\mathcal{T}^t$ are identically distributed.*

The following two additional ways are proposed for proving the conjecture:

(1) Comparing the algorithms for producing the tessellations.
(2) Evaluating the capacity functional.

In the case of random STIT tessellations, the capacity functional is already known as a recursion formula [5]; the stability under iteration even was the starting point for all investigations.

The above Figure is related to the homogeneous case. It shows a simulation of an anisotropic random STIT tessellation according to an algorithm of Nagel and Weiss [4, 5, 10] and was provided by Joachim Ohser, Hochschule Darmstadt.

Note added after submission

The present paper was submitted on December 31, 2009. On February 16, 2010, the author received the profound and very important preprint of Tomasz Schreiber and Christoph Thaele: "Typical Geometry, Second-Order Properties and Central Limit Theory for Iteration Stable Tessellations" from Werner Nagel.

In the article of Schreiber and Thaele, the process of cell division living on the *continuous* time axis that was introduced in [7, 8] and the related STIT tessellations are treated with the efficient and very powerful methods of martingale theory.

Because also inhomogeneous counterparts are considered, probably the conjectures above can be easily verified by the methods of Schreiber and Thaele.

In the present paper, the focus is on a tessellation-valued Markov chain living on a *discrete* time axis and producing STIT tessellations and inhomogeneous counterparts. Furthermore, the aim of the present paper is to provide an easy access to the theory.

Notes added in proof

1. Remark 2 in the present paper is influenced by an idea of Matthias Reitzner.

2. The argument in the exponential function in formula (4.3) may be addressed as the Ambartzumian perimeter related to the line measure tQ of the compact convex set K.

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