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SHARP FUNCTION BOUNDEDNESS FOR VECTOR-VALUED MULTILINEAR SINGULAR INTEGRAL OPERATORS WITH NON-SMOOTH KERNELS

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Аннотация. Sharp function inequalities for several vector-valued, multilinear singular integral operators with non-smooth kernels are obtained. As an application, some weighted $L^p(p > 1)$ norm inequalities for the vector-valued multilinear operators are derived.

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1. Preliminaries and Theorems

As the development of the theory of singular integral operators, their commutators and multilinear operators have been well studied (see [1] - [5], [7], [12] - [15]). In [8], some singular integral operators with non-smooth kernels are introduced, whose kernels satisfy some requirements which are weaker than those for the Calderón-Zygmund singular integral operators. In [6] and [11], the boundedness of the singular integral operators with non-smooth kernels and their commutators is proved.

The main purpose of this paper is to study the vector-valued, multilinear, singular integral operators with non-smooth kernels defined as follows (see [8], [11]).

Definition 1.1. *A family of operators D_t, $t > 0$ is said to be an approximation to the identity, if for every $t > 0$ the family D_t can be represented by the kernel $a_t(x, y)$ as follows:*

$$D_t(f)(x) = \int_{R^n} a_t(x, y)f(y)dy$$

for every $f \in L^p(R^n)$ with $p \geq 1$, and $a_t(x, y)$ satisfies the inequality

$$|a_t(x, y)| \leq h_t(x, y) = Ct^{-n/2}s(|x - y|^2/t),$$

where s is a positive, bounded and decreasing function such for some $\epsilon > 0$

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0.$$

Definition 1.2. *A linear operator T is called singular integral operator with non-smooth kernels, if T is bounded on $L^2(R^n)$ and is associated with some kernel $K(x,y)$ such that*

$$T(f)(x) = \int_{R^n} K(x,y)f(y)dy$$

for every continuous function f with a compact support, and for almost all x, which do not belong to that support. In addition, it is required that:
(1) There exists an approximations to the identity $\{B_t, t>0\}$ such that TB_t possesses an associated kernel $k_t(x,y)$ and there exist constants c_1, $c_2>0$ such that

$$\int_{|x-y|>c_1t^{1/2}} |K(x,y)-k_t(x,y)|dx \le c_2 \quad \textit{for all} \quad y\in R^n.$$

(2) There exists an approximations to the identity $\{A_t, t>0\}$ such that A_tT possesses an associated kernel $K_t(x,y)$ which satisfies the inequalities

$$|K_t(x,y)| \le c_4t^{-n/2} \quad \textit{if} \quad |x-y|\le c_3t^{1/2},$$

and

$$|K(x,y)-K_t(x,y)| \le c_4t^{\delta/2}|x-y|^{-n-\delta} \quad \textit{if} \quad |x-y|\ge c_3t^{1/2},$$

for some constants c_3, $c_4>0$, $\delta>0$.

Let m_j $(j=1,\dots,l)$ be positive integers $m_1+\dots+m_l=m$ and b_j $(j=1,\dots,l)$ be functions on R^n. Set

$$R_{m_j+1}(b_j;x,y) = b_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!}D^\alpha b_j(y)(x-y)^\alpha, \quad 1\le j\le m.$$

Given functions f_i $(i=1,2,\dots)$ defined on R^n, for any $1<r<\infty$ the the vector-valued multilinear operator associated to T is defined by the formula

$$|T_b(f)(x)|_r = \left(\sum_{i=1}^{\infty}(T_b(f_i)(x))^r\right)^{1/r},$$

where

$$T_b(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(b_j;x,y)}{|x-y|^m}K(x,y)f_i(y)dy.$$

Set

$$|T(f)(x)|_r = \left(\sum_{i=1}^{\infty}|T(f_i)(x)|^r\right)^{1/r} \quad \text{and} \quad |f(x)|_r = \left(\sum_{i=1}^{\infty}|f_i(x)|^r\right)^{1/r}.$$

Note that $|T_b(f)|_r$ is just the vector-valued multilinear commutator of T and b_j when $m=0$ (see [14]), while $|T_b(f)|_r$ is a nontrivial generalizations of the commutator when $m>0$. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors(see [2] - [5]). Hu and Yang (see [10]) proved a variant sharp estimate for multilinear singular integral operators. In [14], Pérez and Trujillo-Gonzalez proved a sharp estimate for multilinear commutators when $b_j \in Osc_{expL^{r_j}}(R^n)$.

The main purpose of this paper is to prove a sharp function inequality for the vector-valued, multilinear singular integral operators with non-smooth kernels when $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$. As the application, some $L^p (p > 1)$ norm inequality for the vector-valued, multilinear operators are obtained.

Throughout the paper, $Q = Q(x, d)$ denotes a cube in R^n with sides parallel to the coordinate axes, the center of which is a point x and the length of sides is d. If b is a locally integrable function, then its sharp function is defined as

$$b^{\#}(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy,$$

where, and in what follows,

$$b_Q = |Q|^{-1} \int_Q b(x) dx.$$

It is well-known that(see [9], [16]) that

$$b^{\#}(x) \approx \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |b(y) - c| dy$$

and

$$||b - b_{2^k Q}||_{BMO} \leq Ck||b||_{BMO} \text{ for } k \geq 1.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and $||b||_{BMO} = ||b^{\#}||_{L^\infty}$.

Assuming that M is the Hardy-Littlewood maximal operator

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

we set $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. Further, the sharp maximal function $M_A^{\#}$ associated with approximation to the identity $\{A_t, t > 0\}$ is defined as

$$M_A^{\#}(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)| dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

The below two theorems are the main result of this paper.

Theorem 1.3. *Let $1 < r < \infty$ and let $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ $(j = 1, \ldots, l)$. Then, there exists a constant $C > 0$ such that*

$$M_A^{\#}(|T_b(f)|_r)(\tilde{x}) \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

for any function $f \in C_0^\infty(R^n)$, any $1 < s < \infty$ and any point $\tilde{x} \in R^n$,

Theorem 1.4. *Let $1 < r < \infty$ and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ $(j = 1, \dots, l)$. Then, $|T_b|_r$ is bounded on $L^p(R^n)$ for any $1 < p < \infty$, that is*

$$|||T_b(f)|_r||_{L^p} \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) |||f|_r||_{L^p}.$$

2. Some Lemmas

We present some preliminary lemmas.

Lemma 2.1. *([4]) Let b be a function on R^n and let $D^\alpha b \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(b; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x, with the side length $5\sqrt{n}|x - y|$.

Lemma 2.2. ([8], [11]*) Let $1 < r < \infty$ and let T be a singular integral operators with non-smooth kernel as in Definition 1.2. Then, for every $f \in L^p(R^n)$, $(1 < p < \infty)$,*

$$|||T(f)|_r||_{L^p} \leq C|||f|_r||_{L^p}.$$

Lemma 2.3. *([6]) Let $\{A_t, t > 0\}$ be an approximation to the identity and $b \in BMO(R^n)$. Then, for every $f \in L^p(R^n), p > 1$, $1 < r < \infty$ and $x \in R^n$*

$$\sup_{x \in Q} \frac{1}{|Q|} \int_Q |A_{t_Q}((b - b_Q)f)(y)| dy \leq C||b||_{BMO} M_r(f)(x),$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

Lemma 2.4. *([8], [11]) There exists a constant $C > 0$ such that for any $\gamma > 0$ and $\lambda > 0$*

$$|\{x \in R^n : M(f)(x) > D\lambda, M_A^{\#}(f)(x) \leq \gamma\lambda\}| \leq C\gamma|\{x \in R^n : M(f)(x) > \lambda\}|,$$

where D is a fixed constant depending only on n. So that

$$||M(f)||_{L^p} \leq C||M_A^{\#}(f)||_{L^p}$$

for every $f \in L^p(R^n), 1 < p < \infty$.

3. Proofs of Theorems

Proof of Theorem 1.3. It suffices to prove that for $f \in C_0^\infty(R^n)$

$$\frac{1}{|Q|}\int_Q \left||T_b(f)(x)|_r - |A_{t_Q}T_b(f)(x)|_r\right| dx \le$$

$$\le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_s(|f|_r)(x).$$

Without loss of generality, we can assume $l = 2$. We fix a cube $Q = Q(x_0, d)$ and a point $\tilde{x} \in Q$ and suppose that $\tilde{Q} = 5\sqrt{n}Q$ and

$$\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}}x^\alpha.$$

Then

$$R_m(A_j; x, y) = R_m(\tilde{A}_j; x, y) \quad \text{and} \quad D^\alpha\tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}} \quad \text{for} \quad |\alpha| = m_j.$$

Now, we decompose $f = g + h = \{g_i\} + \{h_i\}$ in $g_i = f_i\chi_{\tilde{Q}}$ and $h_i = f_i\chi_{R^n\setminus\tilde{Q}}$. Then

$$T_b(f_i)(x) = \int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m}K(x,y)f_i(y)dy =$$

$$= \int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j; x, y)}{|x-y|^m}K(x,y)g_i(y)dy -$$

$$-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m}K(x,y)g_i(y)dy$$

$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x,y)g_i(y)dy$$

$$+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x,y)g_i(y)dy$$

$$+\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j; x, y)}{|x-y|^m}K(x,y)h_i(y)dy = T\left(\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j; x, \cdot)}{|x-\cdot|^m}g_i\right)$$

$$-T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2; x, \cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}g_i\right)$$

$$-T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1; x, \cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)$$

$$+T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{\alpha_1!\alpha_2!|x-\cdot|^m}g_i\right)+T\left(\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}h_i\right)$$

and

$$A_{t_Q}T_b(f_i)(x)=\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}K_t(x,y)f_i(y)dy$$

$$=\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,y)g_i(y)dy$$

$$-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m}K_t(x,y)g_i(y)dy$$

$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K_t(x,y)g_i(y)dy$$

$$+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K_t(x,y)g_i(y)dy$$

$$+\int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}K_t(x,y)h_i(y)dy$$

$$=A_{t_Q}T\left(\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}g_i\right)$$

$$-A_{t_Q}T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}g_i\right)$$

$$-A_{t_Q}T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)$$

$$+A_{t_Q}T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)$$

$$+A_{t_Q}T\left(\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}h_i\right).$$

Further, by Minkowski's inequality,

$$\big||T_b(f)(x)|_r-|A_{t_Q}T_b(f)(x)|_r\big|=\left|\left(\sum_{i=1}^{\infty}(T_b(f_i)(x))^r\right)^{1/r}-\left(\sum_{i=1}^{\infty}(A_{t_Q}T_b(f_i)(x))^r\right)^{1/r}\right|$$

$$\le \left(\sum_{i=1}^{\infty}|T_b(f_i)(x)-A_{t_Q}T_b(f_i)(x)|^r\right)^{1/r}$$

$$\le \left(\sum_{i=1}^{\infty}\left|T\left(\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}g_i\right)\right|^r\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\left|T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}g_i\right)\right|^r\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\left|T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)\right|^r\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\left|T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)\right|^r\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\left|A_{t_Q}T\left(\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}g_i\right)\right|^r\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\left|A_{t_Q}T\left(\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\frac{R_{m_2}(\tilde{b}_2;x,\cdot)(x-\cdot)^{\alpha_1}D^{\alpha_1}\tilde{b}_1}{|x-\cdot|^m}g_i\right)\right|^r\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\left|A_{t_Q}T\left(\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\frac{R_{m_1}(\tilde{b}_1;x,\cdot)(x-\cdot)^{\alpha_2}D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)\right|^r\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\left|T\left(\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\frac{(x-\cdot)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1D^{\alpha_2}\tilde{b}_2}{|x-\cdot|^m}g_i\right)\right|^r\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\left|(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}h_i\right)\right|^r\right)^{1/r}$$

$$:=I_1(x)+I_2(x)+I_3(x)+I_4(x)+I_5(x)+I_6(x)+I_7(x)+I_8(x)+I_9(x).$$

Thus

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q\left||T_b(f)(x)|_r-|A_{t_Q}T_b(f)(x)|_r\right|dx\\
\le\ &\frac{1}{|Q|}\int_Q I_1(x)dx+\frac{1}{|Q|}\int_Q I_2(x)dx+\frac{1}{|Q|}\int_Q I_3(x)dx+\frac{1}{|Q|}\int_Q I_4(x)dx\\
+\ &\frac{1}{|Q|}\int_Q I_5(x)dx+\frac{1}{|Q|}\int_Q I_6(x)dx+\frac{1}{|Q|}\int_Q I_7(x)dx+\frac{1}{|Q|}\int_Q I_8(x)dx\\
+\ &\frac{1}{|Q|}\int_Q I_9(x)dx:=I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9.
\end{aligned}$$

Let us estimate I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8 and I_9 separately. First, by lemma 2.1 we obtain that for any $x \in Q$ and $y \in \tilde{Q}$

$$R_m(\tilde{b}_j; x, y) \leq C|x-y|^m \sum_{|\alpha_j|=m} ||D^{\alpha_j} b_j||_{BMO}.$$

Consequently, by the L^s-boundedness of T (lemma 2.2), we get

$$\begin{aligned}
I_1 &\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) \frac{1}{|Q|} \int_Q |T(g)(x)|_r dx \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) \left(\frac{1}{|Q|} \int_{R^n} |T(g)(x)|_r^s dx \right)^{1/s} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) \left(\frac{1}{|Q|} \int_{R^n} |g(x)|_r^s dx \right)^{1/s} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|_r^s dx \right)^{1/s} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

To estimate I_2, we denote $s = pq$ and using lemma 2.2 and Hölder's inequality for $1 < p < \infty$, $q > 1$ and $1/q + 1/q' = 1$, we get

$$\begin{aligned}
I_2 &\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1} \tilde{b}_1 g)(x)|_r dx \\
&\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1} \tilde{b}_1 g)(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |D^{\alpha_1} \tilde{b}_1(x)||g(x)|_r^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1} b_1(x) - (D^{\alpha} b_j)_{\tilde{Q}}|^{pq'} dx \right)^{1/pq'} \\
&\times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|_r^{pq} dx \right)^{1/pq} \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^{\alpha} b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

In the same way, we obtain

$$I_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^{\alpha} b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

Similarly, denoting $s = pq_3$ for $1 < p < \infty$, $q_1, q_2, q_3 > 1$ and $1/q_1 + 1/q_2 + 1/q_3 = 1$, for I_4 we obtain

$$\begin{aligned} I_4 &\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 g)(x)|_r dx \\ &\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{R^n} |T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 g)(x)|_r^p dx\right)^{1/p} \\ &\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{R^n} |D^{\alpha_1}\tilde{b}_1(x) D^{\alpha_2}\tilde{b}_2(x)||g(x)|_r^p dx\right)^{1/p} \\ &\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{b}_1(x)|^{pq_1} dx\right)^{1/pq_1} \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |D^{\alpha_2}\tilde{b}_2(x)|^{pq_2} dx\right)^{1/pq_2} \\ &\quad \times \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f(x)|_r^{pq_3} dx\right)^{1/pq_3} \leq C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) M_s(|f|_r)(\tilde{x}). \end{aligned}$$

For I_5, I_6, I_7 and I_8, we use lemma 2.3 and similar to I_1, I_2, I_3, I_4, we get

$$\begin{aligned} I_5 &\leq C \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n} |T(g)(x)|_r^s dx\right)^{1/s} \\ &\leq C \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f(x)|_r^s dx\right)^{1/s} \\ &\leq C \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO}\right) M_s(|f|_r)(\tilde{x}), \end{aligned}$$

$$\begin{aligned} I_6 &\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{R^n} |T(D^{\alpha_1}\tilde{b}_1 g)(x)|_r^p dx\right)^{1/p} \\ &\leq C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2} b_2||_{BMO} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|}\int_{R^n} |D^{\alpha_1}\tilde{b}_1(x)||g(x)|_r^p dx\right)^{1/p} \\ &\leq C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^{\alpha} b_j||_{BMO}\right) M_s(|f|_r)(\tilde{x}), \end{aligned}$$

$$I_7 \leq C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^{\alpha} b_j||_{BMO}\right) M_s(|f|_r)(\tilde{x}),$$

$$I_8 \leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{R^n} |T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 g)(x)|_r^p dx\right)^{1/p}$$

$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{b}_1(x)D^{\alpha_2}\tilde{b}_2(x)||g(x)|_r^p dx\right)^{1/p}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)M_s(|f|_r)(\tilde{x}).$$

To estimate I_9, we write

$$\left(\sum_{i=1}^{\infty}\left|(T-A_{t_Q}T)\left(\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,\cdot)}{|x-\cdot|^m}h_i\right)\right|^r\right)^{1/r}$$

$$\le\left(\sum_{i=1}^{\infty}\int_{R^n}\left|\frac{\prod_{j=1}^{2}R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m}(K(x,y)-K_t(x,y))h_i(y)\right|^r dy\right)^{1/r}$$

$$\le\left(\sum_{i=1}^{\infty}\int_{R^n}\left|\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m}(K(x,y)-K_t(x,y))h_i(y)\right|^r dy\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left|\frac{D^{\alpha_1}\tilde{b}_1(y)(x-y)^{\alpha_1}R_{m_2}(\tilde{b}_2;x,y)}{|x-y|^m}(K(x,y)-K_t(x,y))h_i(y)\right|^r dy\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left|\frac{D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_2}R_{m_1}(\tilde{b}_1;x,y)}{|x-y|^m}(K(x,y)-K_t(x,y))h_i(y)\right|^r dy\right)^{1/r}$$

$$+\left(\sum_{i=1}^{\infty}\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\times\right.$$

$$\left.\times\int_{R^n}\left|\frac{D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}(K(x,y)-K_t(x,y))h_i(y)\right|^r dy\right)^{1/r}$$

$$=I_9^{(1)}+I_9^{(2)}+I_9^{(3)}+I_9^{(4)}.$$

Then, we observe that lemma 2.1 and the inequality

$$|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)||b||_{BMO}\text{ for }Q_1\subset Q_2,$$

(see [16]) imply that for any $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$\begin{aligned}|R_m(\tilde{b};x,y)| &\le C|x-y|^m\sum_{|\alpha|=m}(||D^\alpha b||_{BMO}+|(D^\alpha b)_{\tilde{Q}(x,y)}-(D^\alpha b)_{\tilde{Q}}|)\\ &\le Ck|x-y|^m\sum_{|\alpha|=m}||D^\alpha b||_{BMO}.\end{aligned}$$

Note that $|x-y| \geq d = t^{1/2}$ and $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$. Therefore, by the conditions on K and K_t and Minkowski's inequality

$$\begin{aligned} I_9^{(1)} &= \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{\prod_{j=1}^{2} |R_{m_j}(\tilde{b}_j;x,y)|}{|x-y|^m} |K(x,y)-K_t(x,y)||f(y)|_r dy \\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2 \frac{d^\delta}{|x_0-y|^{n+\delta}} |f(y)|_r dy \\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) \sum_{k=1}^{\infty} k^2 2^{-\delta k} \frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |f(y)|_r dy \\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) M_s(|f|_r)(\tilde{x}). \end{aligned}$$

For $I_9^{(2)}$, we get

$$\begin{aligned} I_9^{(2)} &\leq C\left(\sum_{|\alpha|=m_2} ||D^\alpha b_2||_{BMO}\right) \sum_{|\alpha|=m_1} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{k\, d^\delta}{|x_0-y|^{n+\delta}} |D^{\alpha_1}\tilde{b}_1(y)||f(y)|_r dy \\ &\leq C\left(\sum_{|\alpha|=m_2} ||D^\alpha b_2||_{BMO}\right) \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k 2^{-\delta k} \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |D^{\alpha_1}\tilde{b}_1(y)|^{r'} dy\right)^{1/r'} \\ &\quad \times \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |f(y)|_r^s dy\right)^{1/s} \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) M_s(|f|_r)(\tilde{x}). \end{aligned}$$

Similarly,

$$I_9^{(3)} \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) M_s(|f|_r)(\tilde{x}).$$

For $I_9^{(4)}$, taking $q_1, q_2 > 1$ such that $1/s + 1/q_1 + 1/q_2 = 1$, we obtain

$$\begin{aligned} I_9^{(4)} &\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{d^\delta}{|x_0-y|^{n+\delta}} |D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f(y)|_r dy \\ &\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \sum_{k=1}^{\infty} 2^{-\delta k} \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |f(y)|_r^s dy\right)^{1/s} \\ &\quad \times \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |D^{\alpha_1}\tilde{b}_1(y)|^{q_1} dy\right)^{1/q_1} \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |D^{\alpha_2}\tilde{b}_2(y)|^{q_2} dy\right)^{1/q_2} \\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO}\right) M_s(|f|_r)(\tilde{x}). \end{aligned}$$

Thus,

$$I_9 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x}).$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. In Theorem 1, we choose $1 < s < p$ and using lemma 2.2, we get

$$|||T_b(f)|_r||_{L^p} \leq ||M(|T_b(f)|_r)||_{L^p} \leq C||M_A^\#(|T_b(f)|_r)||_{L^p}$$
$$\leq C \prod_{j=1}^{l} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) ||M_s(|f|_r)||_{L^p} \leq$$
$$\leq C \prod_{j=1}^{l} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) |||f|_r||_{L^p}.$$

This finishes the proof. □

4. Applications

In this section, Theorems 1.3 and 1.4 are applied to holomorphic functional calculus of linear elliptic operators. First, we review some definitions of holomorphic functional calculus (see [11]). Given $0 \leq \theta < \pi$, introduce the domain

$$S_\theta = \{z \in C : |\arg(z)| \leq \theta\} \bigcup \{0\}$$

and denote its interior by S_θ^0. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta|||(\eta I - L)^{-1}|| \leq C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, we set

$$H_\infty(S_\mu^0) = \{f : S_\theta^0 \to C : \ f \ \text{ is holomorphic and } ||f||_{L^\infty} < \infty\},$$

where $||f||_{L^\infty} = \sup\{|f(z)| : z \in S_\mu^0\}$. Further, set

$$\Psi(S_\mu^0) = \left\{ g \in H_\infty(S_\mu^0) : \exists s > 0, \exists c > 0 \text{ such that } |g(z)| \leq c \frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

For L of type θ and $g \in H_\infty(S_\mu^0)$, define $g(L) \in L(E)$ as

$$g(L) = -(2\pi i)^{-1} \int_\Gamma (\eta I - L)^{-1} g(\eta) d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \geq 0\}$ parameterized clockwise around S_θ by $\theta < \phi < \mu$. If, in addition, L is one-one and its range is dense, then for any $f \in H_\infty(S_\mu^0)$

$$f(L) = [h(L)]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. Besides, L is said to have a bounded holomorphic functional calculus on the sector S_μ, if

$$||g(L)|| \leq N||g||_{L^\infty}$$

for some $N > 0$ and all $g \in H_\infty(S_\mu^0)$.
Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ such that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$. Then, applying theorem 6 of [8] and theorem 1.4, we arrive at the following statement.

Theorem 4.1. *Given $1 < r < \infty$ let the following conditions be satisfied:*
(i) The holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2-\theta$ is represented by kernels $a_z(x,y)$ which satisfy the upper bound

$$|a_z(x,y)| \leq c_\nu h_{|z|}(x,y), \quad \nu > \theta$$

for $x, y \in R^n$, and $0 \leq |\arg(z)| < \pi/2 - \theta$, where $h_t(x,y) = Ct^{-n/2}s(|x-y|^2/t)$ and s is a positive, bounded and decreasing function such that

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0.$$

(ii) The operator L has a bounded holomorphic functional calculus in $L^2(R^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S_\mu^0)$, the operator $g(L)$ is such that

$$||g(L)(|f|_r)||_{L^2} \leq c_\nu ||g||_{L^\infty} |||f|_r||_{L^2}.$$

Then, for $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$, the multilinear operator $g(L)_b$ associated to $g(L)$ and b_j satisfies the conditions:

(a) For $1 < s < \infty$ and $\tilde{x} \in R^n$,

$$M_A^{\#}(|g(L)_b(f)|_r)(\tilde{x}) \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) M_s(|f|_r)(\tilde{x});$$

(b) For any $1 < p < \infty$, $|g(L)_b|_r$ is bounded on $L^p(R^n)$, that is

$$|||g(L)_b(f)|_r||_{L^p} \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} b_j||_{BMO} \right) |||f|_r||_{L^p}.$$

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