Известия НАН Армении. Математика, том 45, н. 3, 2010, стр. 57-72.

SHARP AND WEIGHTED BOUNDEDNESS FOR MULTILINEAR OPERATORS OF PSEUDO-DIFFERENTIAL OPERATORS ON MORREY SPACE

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Abstract. The paper proves boundedness of the multilinear operators related to some pseudo-differential operators on the generalized weighted Morrey spaces using the sharp estimate of the multilinear operators.

MSC2000 number: 42B20, 42B25

Keywords: multilinear operator; pseudo-differential operator; Morrey space; BMO; A_1-weight.

1. Preliminaries and statements of main results

Throughout this paper, φ denotes a positive, increasing function on R^+ and it is assumed that there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \quad \text{for} \quad t \ge 0.$$

Let w be a weight function on $\mathbb{R}^n$, that is a nonnegative locally integrable function, and f be a locally integrable function on $\mathbb{R}^n$. Define that, for $1 \le p < \infty$,

$$||f||_{L^{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n,\ d>0} \left(\frac{1}{\varphi(d)} \int_{B(x,d)} |f(y)|^p w(y)dy \right)^{1/p},$$

where $B(x,d) = \{y \in R^n : |x-y| < d\}$. The generalized weighted Morrey spaces are defined by

$$L^{p,\varphi}(\mathbb{R}^n, w) = \{f \in L^1_{loc}(\mathbb{R}^n) : ||f||_{L^{p,\varphi}(w)} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(\mathbb{R}^n, w) = L^{p,\delta}(\mathbb{R}^n, w)$, which is the classical Morrey space (see [16], [17]).

As the development of the Calderón-Zygmund singular integral operators, their commutators and multilinear operators have been well studied (see [3] - [6], [9]). In [14], Hu and Yang proved a version sharp estimate for the multilinear singular integral operators. In [18], [19], C. Pérez, G. Pradolini and R. Trujillo-Gonzalez obtained a sharp weighted estimates for the singular integral operators and their commutators. The boundedness of the pseudo-differential operators was studied by many authors

(see [1], [7], [12], [15], [20] - [21]). In [20], the boundedness of the commutators associated to the pseudo-differential operators are obtained. The main purpose of this paper is to study the multilinear pseudo-differential operators as follows.
We say a symbol $\sigma(x,\xi)$ belongs to the class $S^m_{\rho,\delta}$, if

$$\left|\frac{\partial^\mu}{\partial x^\mu}\frac{\partial^\nu}{\partial \xi^\nu}\sigma(x,\xi)\right| \le C_{\mu,\nu}(1+|\xi|)^{m-\rho|\nu|+\delta|\mu|}, \qquad x,\xi\in\mathbb{R}^n,$$

where μ,ν are multi-indices and $|\mu|=|\mu_1|+\ldots+|\mu_n|$. A pseudo-differential operator with symbol $\sigma(x,\xi)\in S^m_{\rho,\delta}$ is defined by

$$T(f)(x)=\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi,$$

where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f. It is known (see [1]) that there exists a kernel $K(x,y)$ such that

$$T(f)(x)=\int_{\mathbb{R}^n} K(x,x-y)f(y)dy,$$

where

$$K(x,y)=\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi}\sigma(x,\xi)d\xi.$$

In [12] the boundedness of the pseudo-differential operators with symbol $\sigma\in S^{-\beta}_{1-\theta,\delta}$ $(\beta<n\theta/2, 0\le\delta<1-\theta)$ are obtained. In [15] the boundedness of the pseudo-differential operators with symbol of orders 0 and $-\infty$ is proved. In [1] some sharp estimate of the pseudo-differential operators with symbol $\sigma\in S^{-n\theta/2}_{1-\theta,\delta}$ $(0<\theta<1, 0\le\delta<1-\theta)$ are obtained. In [20] the boundedness of the pseudo-differential operators and their commutators with symbol $\sigma\in S^{-n\theta/2}_{1-\theta,\delta}$ $(0<\theta<1, 0\le\delta<1-\theta)$ are obtained. Our study are motivated by these papers.
Assuming that T is a pseudo-differential operator with symbol $\sigma(x,\xi)\in S^m_{\rho,\delta}$ that m_j $(j=1,\ldots,l)$ are some positive integers such that $m_1+\ldots+m_l=m$ and b_j are functions given on $\mathbb{R}^n$, we set

$$R_{m_j+1}(b_j;x,y)=b_j(x)-\sum_{|\alpha|\le m_j}\frac{1}{\alpha!}D^\alpha b_j(y)(x-y)^\alpha, \quad 1\le j\le m.$$

The multilinear operator associated to T is defined by

$$T_b(f)(x)=\int_{R^n}\frac{\prod_{j=1}^l R_{m_j+1}(b_j;x,y)}{|x-y|^m}K(x,x-y)f(y)\,dy.$$

Note that for $m=0$, T_b is just the multilinear commutator generated by T and b (see [18], [19]), while for $m>0$, T_b is nontrivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3] - [6]). Besides, the Morrey space can be considered as an extension of the Lebesgue space, since the Morrey space $L^{p,\lambda}$ becomes Lebesgue space L^p for $\lambda = 0$). Hence, it is natural and important to study the boundedness of multilinear singular integral operators on the Morrey spaces $L^{p,\lambda}$ with $\lambda > 0$ (see [2], [10], [11]). The purpose of this paper is twofold. First, we establish a sharp inequality for multilinear pseudo-differential operator T_b with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}$ $(0 < \theta < 1, 0 \leq \delta < 1-\theta)$. Then, we use this sharp inequality to prove the boundedness for the multilinear operators on the generalized weighted Morrey spaces.

Now, we introduce some notations. Denote by Q a cube in R^n with sides parallel to the coordinate axes. For any locally integrable function f, its sharp function is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x)\, dx$. It is well-known (see [13]) that

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c|\, dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#} \in L^\infty(R^n)$ and denote $||f||_{BMO} = ||f^{\#}||_{L^\infty}$.

Let M be the Hardy-Littlewood maximal operator

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)|\, dy, \quad 0 < p < \infty.$$

We set $M_p(f) = (M(f^p))^{1/p}$ and denote by A_1 the class of Muckenhoupt weights (see [13]):

$$A_1 = \{0 < w \in L^1_{loc}(R^n) : M(w)(x) \leq Cw(x), a.e.\}.$$

The following theorem is the main result of this paper.

Theorem. *Let T be a pseudo-differential operator with a symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}$ $(0 < \theta < 1,\ 0 \leq \delta < 1-\theta)$ and let $2 < p < \infty$, $0 < D < 2^n$, $w \in A_1$, and $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Then*

$$||T_b(f)||_{L^{p,\varphi}(w)} \leq C \prod_{j=1}^{l} \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} b_j||_{BMO} \right) ||f||_{L^{p,\varphi}(w)}.$$

2. Proof of the Theorem

To prove the theorem, we need the following lemmas.

Lemma 1. *([3]) Let b be a function on R^n and $D^\alpha b \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then, for any $x \neq y$,*

$$|R_m(b;x,y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x with the side length $5\sqrt{n}|x-y|$.

Lemma 2. *([1]) Let T be a pseudo-differential operator with a symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}$ $(0 < \theta < 1,\ 0 \le \delta < 1-\theta)$. Then, for every $p,\ 1 < p < \infty$,*

$$||T(f)||_{L^p} \le C_p ||f||_{L^p}, \qquad f \in L^p(R^n).$$

Lemma 3. *([1]) Let $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \le \delta < 1-\theta)$ and K be the kernel of a pseudo-differential operator T with a symbol σ. Then, for $|x_0 - x| \le d < 1$ and $k \ge 1$,*

$$\left(\int_{(2^k d)^{1-\theta} \le |y-x_0| < (2^{k+1}d)^{1-\theta}} |K(x,x-y) - K(x_0,x_0-y)|^2 dy \right)^{1/2} \le$$

$$\le C \frac{|x_0 - x|^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}},$$

provided m is an integer such that $n/2 < m < n/2 + 1/(1-\theta)$.

Lemma 4. *([1]) Let $\sigma \in S^0_{\rho,\delta}(0 < \rho < 1)$ and*

$$K(x,w) = \int_{R^n} e^{2\pi i w\cdot\xi} \sigma(x,\xi)\, d\xi.$$

Then, for $|w| \ge 1/4$ and any integer $N \ge 1$,

$$|K(x,w)| \le C_N |w|^{-2N}.$$

Lemma 5. *Let $1 < p < \infty$, $0 < D < 2^n$, $w \in A_1$. Then, for any function $f \in L^{p,\varphi}(R^n, w)$*

(a) $||M(f)||_{L^{p,\varphi}(w)} \le C||f^\#||_{L^{p,\varphi}(w)}$;

(b) $||M_q(f)||_{L^{p,\varphi}(w)} \le C||f||_{L^{p,\varphi}(w)}$ for $1 < q < p$.

Proof. (a) Let $f \in L^{p,\varphi}(R^n, w)$. Then $M(w\chi_B) \in A_1$ for any ball $B = B(x,d) \subset R^n$ (see [8]). Therefore, using the inequality (see [13])

$$\int_{R^n} |M(f)(y)|^p u(y) dy \le C \int_{R^n} |f^\#(y)|^p u(y) dy,$$

which is true for any $u \in A_1$, we get

$$\int_B |M(f)(y)|^p w(y) dy \le \int_{R^n} |M(f)(y)|^p M(w\chi_B)(y) dy \le$$

$$\le C \int_{R^n} |f^\#(y)|^p M(w\chi_B)(y) dy \le$$

$$\leq C\left[\int_B |f^{\#}(y)|^p M(w)(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^k B}|f^{\#}(y)|^p\left(\sup_{Q\ni y}\frac{1}{|Q|}\int_B w(z)dz\right)dy\right] \leq$$

$$\leq C\left[\int_B |f^{\#}(y)|^p M(w)(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B\setminus 2^k B}|f^{\#}(y)|^p\left(\frac{1}{|2^{k+1}B|}\int_B w(z)dz\right)dy\right] \leq$$

$$\leq C\left[\int_B |f^{\#}(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B}|f^{\#}(y)|^p\frac{M(w)(y)}{2^{n(k+1)}}dy\right] \leq$$

$$\leq C\left[\int_B |f^{\#}(y)|^p w(y)dy + \sum_{k=0}^{\infty}\int_{2^{k+1}B}|f^{\#}(y)|^p\frac{w(y)}{2^{nk}}dy\right] \leq$$

$$\leq C||f^{\#}||^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty}2^{-nk}\varphi(2^{k+1}d) \leq C||f^{\#}||^p_{L^{p,\varphi}(w)}\sum_{k=0}^{\infty}(2^{-n}D)^k\varphi(d) \leq$$

$$\leq C||f^{\#}||^p_{L^{p,\varphi}(w)}\varphi(d).$$

Thus,

$$||M(f)||_{L^{p,\varphi}(w)} \leq C||f^{\#}||_{L^{p,\varphi}(w)}.$$

The inequality (b) is proved by an argument similar to that in the proof of (a), and we omit the details. □

Key Lemma. *Let T be a pseudo-differential operator with a symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}$ $(0<\theta<1,\ 0\leq\delta<1-\theta)$ and let $D^\alpha b_j \in BMO(R^n)$ for all α with $|\alpha| = m_j$ $(j = 1,\cdots,l)$. Then there exists a constant $C>0$ such that for any $f\in C_0^\infty(R^n)$, $2<r<\infty$ and $\tilde{x}\in R^n$,*

$$(T_b(f))^{\#}(\tilde{x}) \leq C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

Proof. It suffices to prove that for any $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_b(f)(x)-C_0|dx \leq C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

Without loss of generality, we can assume $l=2$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. We consider two cases.

Case 1. $d\leq 1$. Let Q^* be the concentric with Q cube with side length $d^{1-\theta}$, $\tilde{Q} = 5\sqrt{n}Q^*$ and

$$\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha b_j)_{\tilde{Q}}x^\alpha.$$

Then $R_{m_j}(b_j;x,y) = R_{m_j}(\tilde{b}_j;x,y)$ and $D^\alpha \tilde{b}_j = D^\alpha b_j - (D^\alpha b_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. Consequently, for $f = f\chi_{\tilde{Q}} + f\chi_{R^n\setminus\tilde{Q}} = f_1 + f_2$ we obtain

$$T_b(f)(x) = \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,x-y)f(y)dy =$$

$$= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,x-y)f_1(y)dy -$$

$$- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1} D^{\alpha_1}\tilde{b}_1(y)}{|x-y|^m} K(x,x-y)f_1(y)dy -$$

$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2} D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,x-y)f_1(y)dy +$$

$$+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,x-y)f_1(y)dy +$$

$$+ \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,x-y)f_2(y)\,dy.$$

Therefore,

$$\frac{1}{|Q|}\int_Q |T_b(f)(x) - T_{\tilde{b}}(f_2)(x_0)|dx \le$$

$$\le \frac{1}{|Q|}\int_Q \left| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,x-y)f_1(y)dy \right| dx +$$

$$+ \frac{C}{|Q|}\int_Q \left| \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{b}_1(y) K(x,x-y)f_1(y)dy \right| dx +$$

$$+ \frac{C}{|Q|}\int_Q \left| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2}\tilde{b}_2(y) K(x,x-y)f_1(y)dy \right| dx +$$

$$+ \frac{C}{|Q|}\int_Q \left| \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m} K(x,x-y)f_1(y)dy \right| dx +$$

$$+ \frac{1}{|Q|}\int_Q \left| T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(x_0) \right| dx =: I_1 + I_2 + I_3 + I_4 + I_5.$$

To estimate the quantities I_1, I_2, I_3, I_4 and I_5, first, for $x \in Q$ and $y \in \tilde{Q}$, we use Lemma 1 and obtain

$$R_m(\tilde{b}_j;x,y) \le C|x-y|^m \sum_{|\alpha_j|=m} ||D^{\alpha_j} b_j||_{BMO}.$$

Now, we suppose $\sigma(x,\xi) = \sigma(x,\xi)|\xi|^{n\theta/2}|\xi|^{-n\theta/2} = q(x,\xi)|\xi|^{-n\theta/2}$. Then $q(x,\xi) \in S^0_{1-\theta,\delta}$. Therefore, denoting the pseudo-differential operator with symbol $q(x,\xi)$ by S

and applying the Hardy-Littlewood-Soboleve fractional integration theorem and the L^2-boundedness of S (see [1]), we obtain that for $1/p = 1/2 - \theta/2$,

$$\begin{aligned}
I_1 &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\frac{1}{|Q|}\int_Q |T(f_1)(x)|dx \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_Q |T(f_1)(x)|^p dx\right)^{1/p} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)|Q|^{-1/p}\left(\int_{R^n} |S(f_1)(x)|^2 dx\right)^{1/2} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)|Q|^{-1/p}\left(\int_{R^n} |f_1(x)|^2 dx\right)^{1/2} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\frac{|\tilde{Q}|^{1/2}}{|Q|^{1/p}}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f(x)|^2 dx\right)^{1/2} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f(x)|^r dx\right)^{1/r} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_r(f)(\tilde{x}).
\end{aligned}$$

For I_2, we use Lemma 1 and Hölder's inequality and obtain that for $1/r + 1/r' = 1/2$,

$$I_2 \leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|^p dx\right)^{1/p} \leq$$

$$\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1/p}\left(\int_{R^n} |S(D^{\alpha_1}\tilde{b}_1 f_1)(x)|^2 dx\right)^{1/2} \leq$$

$$\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1/p}\left(\int_{R^n} |D^{\alpha_1}\tilde{b}_1(x) f_1(x)|^2 dx\right)^{1/2} \leq$$

$$\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\frac{|\tilde{Q}|^{1/2}}{|\tilde{Q}|^{1/p}}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |D^{\alpha_1}b_1(x) - (D^{\alpha}b_j)_{\tilde{Q}}|^{r'} dx\right)^{1/r'}\times$$

$$\times\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f(x)|^r dx\right)^{1/r} \leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

Similarly, for I_3, we get

$$I_3 \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M_r(f)(\tilde{x}).$$

For I_4, taking $r_1, r_2 > 1$ such that $1/r + 1/r_1 + 1/r_2 = 1/2$, we obtain

$$I_4 \le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_Q |T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)(x)|^p dx \right)^{1/p} \le$$

$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/p} \left(\int_{R^n} |S(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)(x)|^2 dx \right)^{1/2} \le$$

$$\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/p} \left(\int_{R^n} |D^{\alpha_1}\tilde{b}_1(x) D^{\alpha_2}\tilde{b}_2(x) f_1(x)|^2 dx \right)^{1/2} \le$$

$$\le C \frac{|\tilde{Q}|^{1/2}}{|\tilde{Q}|^{1/p}} \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \prod_{j=1}^{2} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_j}\tilde{b}_j(x)|^{r_j} dx \right)^{1/r_j} \times$$

$$\times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|^r dx \right)^{1/r} \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M_r(f)(\tilde{x}).$$

To estimate I_5, observe that

$$T_{\tilde{b}}(f_2)(x) - T_{\tilde{b}}(f_2)(x_0) = \int_{R^n} \left(\frac{K(x, x-y)}{|x-y|^m} - \frac{K(x_0, x_0-y)}{|x_0-y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{b}_j; x, y) f_2(y) dy$$

$$+ \int_{R^n} \left(R_{m_1}(\tilde{b}_1; x, y) - R_{m_1}(\tilde{b}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{b}_2; x, y)}{|x_0-y|^m} K(x_0, x_0-y) f_2(y) dy$$

$$+ \int_{R^n} \left(R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{b}_1; x_0, y)}{|x_0-y|^m} K(x_0, x_0-y) f_2(y) dy$$

$$- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{b}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x, x-y) - \frac{R_{m_2}(\tilde{b}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0, x_0-y) \right]$$

$$\times D^{\alpha_1}\tilde{b}_1(y) f_2(y) dy$$

$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{b}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x, x-y) - \frac{R_{m_1}(\tilde{b}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0, x_0-y) \right]$$

$$\times D^{\alpha_2}\tilde{b}_2(y) f_2(y) dy$$

$$+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x, x-y) - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0, x_0-y) \right]$$

$$\times D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y) f_2(y) dy$$

$$=: \quad I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)}.$$

By Lemma 1 and the following inequality (see [13])

$$|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|)||b||_{BMO},$$

which is true when $Q_1 \subset Q_2$, imply

$$\begin{aligned} |R_m(\tilde{b}; x, y)| &\le C|x-y|^m \sum_{|\alpha|=m} (||D^\alpha b||_{BMO} + |(D^\alpha b)_{\tilde{Q}(x,y)} - (D^\alpha b)_{\tilde{Q}}|) \\ &\le Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha b||_{BMO}, \end{aligned}$$

for $x \in Q$ and $y \in Q(x_0, (2^{k+1}d)^{1-\theta}) \backslash Q(x_0, (2^k d)^{1-\theta})$. Therefore, noting that $|x-y| \sim |x_0 - y|$ for $x \in \tilde{Q}$ and $y \in R^n \setminus \tilde{Q}$, we obtain

$$\begin{aligned} |I_5^{(1)}| \le\ & \sum_{k=0}^\infty k^2 \int_{(2^k d)^{1-\theta} \le |y-x_0| < (2^{k+1}d)^{1-\theta}} |K(x, x-y) - K(x_0, x_0 - y)| \\ & \times \frac{1}{|x-y|^m} \prod_{j=1}^2 |R_{m_j}(\tilde{b}_j; x, y)||f(y)|dy \\ & + \sum_{k=0}^\infty k^2 \int_{(2^k d)^{1-\theta} \le |y-x_0| < (2^{k+1}d)^{1-\theta}} \left| \frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right| \\ & \times |K(x_0, x_0-y)| \prod_{j=1}^2 |R_{m_j}(\tilde{b}_j; x, y)||f(y)|dy \\ \le\ & C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) \sum_{k=0}^\infty k^2 \left(\int_{|y-x_0|<(2^{k+1}d)^{1-\theta}} |f(y)|^2 dy \right)^{1/2} \\ & \times \left(\int_{(2^k d)^{1-\theta} \le |y-x_0| < (2^{k+1}d)^{1-\theta}} |K(x, x-y) - K(x_0, x_0-y)|^2 dy \right)^{1/2} \\ & + C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) \sum_{k=0}^\infty k^2 \left(\int_{|y-x_0|<(2^{k+1}d)^{1-\theta}} |f(y)|^2 dy \right)^{1/2} \\ & \times \left(\int_{(2^k d)^{1-\theta} \le |y-x_0| < (2^{k+1}d)^{1-\theta}} \frac{|x_0-x|^2}{|x_0-y|^2} |K(x_0, x_0-y)|^2 dy \right)^{1/2}, \end{aligned}$$

for the second term above, arguing as in the proof of Lemma 2.1 of [1], we obtain

$$\left(\int_{(2^k d)^{1-\theta} \le |y-x_0| < (2^{k+1}d)^{1-\theta}} \frac{|x_0-x|^2}{|x_0-y|^2} |K(x_0, x_0-y)|^2 dy \right)^{1/2} \le C \frac{|x_0-x|^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}},$$

thus, by Lemma 3 and for $n/2 < m$, we get

$$\begin{aligned}
|I_5^{(1)}| &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\times \\
&\quad \times\sum_{k=0}^{\infty}k^2\frac{d^{(1-\theta)(m-n/2)}}{(2^kd)^{m(1-\theta)}}\left(\int_{|y-x_0|<(2^{k+1}d)^{1-\theta}}|f(y)|^2dy\right)^{1/2}\le \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{k(1-\theta)(n/2-m)}\times \\
&\quad \times\left(\frac{1}{|Q(x_0,(2^kd)^{1-\theta})|}\int_{Q(x_0,(2^kd)^{1-\theta})}|f(y)|^rdy\right)^{1/r} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{k(1-\theta)(n/2-m)}M_r(f)(\tilde{x})\le \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)M_r(f)(\tilde{x}).
\end{aligned}$$

To estimate $I_5^{(2)}$, by the equality (see [3]):

$$R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{b};x,x_0)(x-y)^\beta$$

and Lemma 1, we get

$$|R_m(\tilde{b};x,y)-R_m(\tilde{b};x_0,y)|\le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}||D^\alpha b||_{BMO},$$

thus

$$\begin{aligned}
&|I_5^{(2)}|\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\times \\
&\times\sum_{k=0}^{\infty}\int_{(2^kd)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}}k\frac{|x-x_0|}{|x_0-y|}|K(x_0,x_0-y)||f(y)|dy\le \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)\sum_{k=1}^{\infty}k2^{k(1-\theta)(n/2-m)}\times \\
&\times\left(\frac{1}{|Q(x_0,(2^kd)^{1-\theta})|}\int_{Q(x_0,(2^kd)^{1-\theta})}|f(y)|^rdy\right)^{1/r}\le \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha b_j||_{BMO}\right)M_r(f)(\tilde{x}).
\end{aligned}$$

Similarly, we obtain

$$|I_5^{(3)}| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M_r(f)(\tilde{x}).$$

For $I_5^{(4)}$, as for $I_5^{(1)}$ and $I_5^{(2)}$, we get that for $1/r + 1/r' = 1/2$

$$|I_5^{(4)}| \le C \sum_{|\alpha_1|=m_1} \int_{R^n} \left| \frac{(x-y)^{\alpha_1}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1}}{|x_0-y|^m} \right| |R_{m_2}(\tilde{b}_2; x, y)||K(x, x-y)||D^{\alpha_1}\tilde{b}_1(y)||f_2(y)|dy +$$

$$+C \sum_{|\alpha_1|=m_1} \int_{R^n} |R_{m_2}(\tilde{b}_2; x, y) - R_{m_2}(\tilde{b}_2; x_0, y)| \frac{|(x_0-y)^{\alpha_1}|}{|x_0-y|^m} |K(x, x-y)||D^{\alpha_1}\tilde{b}_1(y)||f_2(y)|dy +$$

$$+C \sum_{|\alpha_1|=m_1} \int_{R^n} |K(x, x-y) - K(x_0, x_0-y)| \frac{|(x_0-y)^{\alpha_1}|}{|x_0-y|^m} |R_{m_2}(\tilde{b}_2; x_0, y)||D^{\alpha_1}\tilde{b}_1(y)||f_2(y)|dy \le$$

$$\le C \sum_{|\alpha|=m_2} ||D^\alpha b_2||_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=1}^{\infty} k 2^{k(1-\theta)(n/2-m)} \times$$

$$\times \left(\frac{1}{|Q(x_0, (2^k d)^{1-\theta})|} \int_{Q(x_0,(2^k d)^{1-\theta})} |f(y) D^{\alpha_1}\tilde{b}_1(y)|^2 dy \right)^{1/2} \le$$

$$\le C \sum_{|\alpha|=m_2} ||D^\alpha b_2||_{BMO} \sum_{k=1}^{\infty} k 2^{k(1-\theta)(n/2-m)} \left(\frac{1}{|Q(x_0, (2^k d)^{1-\theta})|} \int_{Q(x_0,(2^k d)^{1-\theta})} |f(y)|^r dy \right)^{1/r} \times$$

$$\times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q(x_0, (2^k d)^{1-\theta})|} \int_{Q(x_0,(2^k d)^{1-\theta})} |(D^{\alpha_1} b_1(y) - (D^{\alpha_1} b_1)_{\tilde{Q}})|^{r'} dy \right)^{1/r'} \le$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) \sum_{k=1}^{\infty} k^2 2^{k(1-\theta)(n/2-m)} M_r(f)(\tilde{x}) \le$$

$$\le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M_r(f)(\tilde{x}).$$

Similarly,

$$|I_5^{(5)}| \le C \prod_{j=1}^{2} \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) M_r(f)(\tilde{x}).$$

For $I_5^{(6)}$, as for $I_5^{(1)}$, we get, that for $1/r + 1/r_1 + 1/r_2 = 1/2$,

$$|I_5^{(6)}| \le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} \left| \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} \right| \times$$

$$\times |K(x, x-y)||D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f_2(y)|dy +$$

$$+C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \int_{R^n} |K(x, x-y) - K(x_0, x_0-y)| \frac{|(x_0-y)^{\alpha_1+\alpha_2}|}{|x_0-y|^m} \times$$

$$\times|D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f_2(y)|dy \le$$

$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}2^{k(1-\theta)(n/2-m)}\times$$

$$\times\left(\frac{1}{|Q(x_0,(2^kd)^{1-\theta})|}\int_{Q(x_0,(2^kd)^{1-\theta})}|f(y)D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)|^2dy\right)^{1/2} \le$$

$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}2^{k(1-\theta)(n/2-m)}\left(\frac{1}{|Q(x_0,(2^kd)^{1-\theta})|}\int_{Q(x_0,(2^kd)^{1-\theta})}|f(y)|^rdy\right)^{1/r}\times$$

$$\times\prod_{j=1}^{2}\left(\frac{1}{|Q(x_0,(2^kd)^{1-\theta})|}\int_{Q(x_0,(2^kd)^{1-\theta})}|D^{\alpha_j}b_j(y)-(D^{\alpha_j}b_j)_{\tilde{Q}}|^{r_j}dy\right)^{1/r_j} \le$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

Thus

$$|I_5| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

Case 2. $d > 1$. In this case, let $\tilde{Q} = 5\sqrt{n}Q$ and

$$\tilde{b}_j(x) = b_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^{\alpha}b_j)_{\tilde{Q}}x^{\alpha}.$$

Then $R_{m_j}(b_j;x,y) = R_{m_j}(\tilde{b}_j;x,y)$ and

$$D^{\alpha}\tilde{b}_j = D^{\alpha}b_j - (D^{\alpha}b_j)_{\tilde{Q}}, \quad |\alpha| = m_j.$$

Hence, for $f = f\chi_{\tilde{Q}} + f\chi_{R^n\setminus\tilde{Q}} = f_1 + f_2$, we have

$$\frac{1}{|Q|}\int_Q|T_b(f)(x)|dx$$

$$\le \frac{1}{|Q|}\int_Q\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m}K(x,x-y)f_1(y)dy\right|dx$$

$$+\frac{C}{|Q|}\int_Q\left|\sum_{|\alpha_1|=m_1}\int_{R^n}\frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{b}_1(y)K(x,x-y)f_1(y)dy\right|dx$$

$$+\frac{C}{|Q|}\int_Q\left|\sum_{|\alpha_2|=m_2}\int_{R^n}\frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{b}_2(y)K(x,x-y)f_1(y)dy\right|dx$$

$$+\frac{C}{|Q|}\int_Q\left|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{b}_1(y)D^{\alpha_2}\tilde{b}_2(y)}{|x-y|^m}K(x,x-y)f_1(y)dy\right|dx$$

$$+\frac{1}{|Q|}\int_Q |T_{\tilde{b}}(f_2)(x)|dx =: J_1 + J_2 + J_3 + J_4 + J_5.$$

As for I_1, I_2, I_3 and I_4, by the $L^p(1 < p < \infty)$-boundedness of T (see Lemma 2), we get

$$J_1 \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|Q|}\int_{R^n}|T(f_1)(x)|^r dx\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)|Q|^{-1/r}\left(\int_{R^n}|f_1(x)|^r dx\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r dx\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}b_j||_{BMO}\right)M_r(f)(\tilde{x});$$

$$J_2 \le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{b}_1 f_1)(x)|^p dx\right)^{1/p}$$

$$\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}|Q|^{-1/r}\left(\int_{R^n}|D^{\alpha_1}\tilde{b}_1(x)f_1(x)|^p dx\right)^{1/p}$$

$$\le C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}b_2||_{BMO}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_1}b_1(x)-(D^{\alpha}b_1)_{\tilde{Q}}|^{r'}dx\right)^{1/r'}\times$$

$$\times\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r dx\right)^{1/r} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_r(f)(\tilde{x});$$

$$J_3 \le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_r(f)(\tilde{x});$$

$$J_4 \le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1/r}\left(\int_{R^n}|T(D^{\alpha_1}\tilde{b}_1 D^{\alpha_2}\tilde{b}_2 f_1)(x)|^r dx\right)^{1/r}$$

$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}|Q|^{-1/r}\left(\int_{R^n}|D^{\alpha_1}\tilde{b}_1(x)D^{\alpha_2}\tilde{b}_2(x)f_1(x)|^r dx\right)^{1/r}$$

$$\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\prod_{j=1}^{2}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_j}\tilde{b}_j(x)|^{r_j}dx\right)^{1/r_j}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^r dx\right)^{1/r}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

To estimate J_5, observe that

$$T_{\tilde{b}}(f_2)(x) = \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{b}_j;x,y)}{|x-y|^m} K(x,x-y) f_2(y)dy$$
$$- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{b}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m} K(x,x-y) D^{\alpha_1}\tilde{b}_1(y) f_2(y)dy$$
$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{b}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m} K(x,x-y) D^{\alpha_2}\tilde{b}_2(y) f_2(y)dy$$
$$+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x,x-y) D^{\alpha_1}\tilde{b}_1(y) D^{\alpha_2}\tilde{b}_2(y) f_2(y)dy.$$

Hence, we use Lemma 4 and similar to I_5, we get

$$|T_{\tilde{b}}(f_2)(x)| \le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) \sum_{k=0}^\infty k^2 \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |x-y|^{-2n} |f(y)|dy$$
$$+C \sum_{|\alpha|=m_2} ||D^\alpha b_2||_{BMO} \sum_{|\alpha_1|=m_1} \sum_{k=0}^\infty k \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |x-y|^{-2n} |D^{\alpha_1}\tilde{b}_1(y)||f(y)|dy$$
$$+C \sum_{|\alpha|=m_1} ||D^\alpha b_1||_{BMO} \sum_{|\alpha_2|=m_2} \sum_{k=0}^\infty k \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |x-y|^{-2n} |D^{\alpha_2}\tilde{b}_2(y)||f(y)|dy$$
$$+C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |x-y|^{-2n} |D^{\alpha_1}\tilde{b}_1(y)||D^{\alpha_2}\tilde{b}_2(y)||f(y)|dy$$
$$\le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} ||D^\alpha b_j||_{BMO} \right) d^{-n} \sum_{k=1}^\infty k^2 2^{-kn} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|^r dy \right)^{1/r}$$
$$+C \sum_{|\alpha|=m_2} ||D^\alpha b_2||_{BMO} d^{-n} \sum_{k=1}^\infty k 2^{-kn} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|^r dy \right)^{1/r}$$
$$\times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |(D^{\alpha_1}b_1(y) - (D^{\alpha_1}b_1)_{\tilde{Q}})|^{r'} dy \right)^{1/r'}$$
$$+C \sum_{|\alpha|=m_1} ||D^\alpha b_1||_{BMO} d^{-n} \sum_{k=1}^\infty k 2^{-kn} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|^r dy \right)^{1/r}$$
$$\times \sum_{|\alpha_2|=m_2} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |(D^{\alpha_2}b_2(y) - (D^{\alpha_1}b_2)_{\tilde{Q}})|^{r'} dy \right)^{1/r'}$$
$$+C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} d^{-n} \sum_{k=1}^\infty 2^{-kn} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|^r dy \right)^{1/r}$$

$$\times \prod_{j=1}^{2}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_j}b_j(y)-(D^{\alpha_j}b_j)_{\tilde{Q}}|^{r_j}dy\right)^{1/r_j}$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)\sum_{k=1}^{\infty}k^2 2^{-kn}M_r(f)(\tilde{x})$$
$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

Thus,

$$|J_5|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)M_r(f)(\tilde{x}).$$

This completes the proof of Key Lemma. □

Proof of Theorem. Taking $2<r<p$ in Key Lemma, by Lemma 5, we obtain

$$\begin{aligned}||T_b(f)||_{L^{p,\varphi}(w)} &\leq ||M(T_b(f))||_{L^{p,\varphi}(w)}\leq C||(T_b(f))^{\#}||_{L^{p,\varphi}(w)}\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)||M_r(f)||_{L^{p,\varphi}(w)}\\ &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^{\alpha}b_j||_{BMO}\right)||f||_{L^{p,\varphi}(w)}.\end{aligned}$$

This finishes the proof. □

Acknowledgement. The author would like to express gratitude to the referee for his comments and suggestions.

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Поступила 7 апреля 2009