Известия НАН Армении. Математика, том 45, н. 3, 2010, стр. 41-46.

ON GROUPS ACTING BY COHOMOGENEITY ONE ON THE EUCLIDEAN SPACE $\mathbb{R}^n$

PARVIZ AHMADI

Department of Mathematics, Faculty of Sciences, Zanjan University Zanjan, Iran
E-mail: *p.ahmadi@znu.ac.ir*

Abstract. In the present paper we study closed Lie subgroups $G \subset Iso(\mathbb{R}^n)$ acting by cohomogeneity one on $\mathbb{R}^n$ and prove that when there is no singular orbit, then there is a simply connected, solvable and closed Lie subgroup $F \subset G$ which acts by cohomogeneity one on $\mathbb{R}^n$ and the two actions are orbit equivalent.

MSC2000 number: 53C30, 57S25
Keywords: Cohomogeneity one, Euclidean space.

1. Introduction

The study of nontransitive actions of isometry groups of Riemannain manifolds is an interesting direction in the group theory. The first and most natural case is the case when the action has an orbit of codimension one, the so called cohomogeneity one action. Many mathematicians have studied this subject and obtained nice results. The subject is still an active one, see [1, 2, 7, 8, 10, 11]. In the article we study the closed Lie subgroups $G \subset Iso(\mathbb{R}^n)$ acting by cohomogeneity one on $\mathbb{R}^n$ and prove that if there is no singular orbit, then there is a simply connected, solvable and closed Lie subgroup $F \subset G$ which acts by cohomogeneity one and the two actions are orbit equivalent (see Theorem 3.1 and Corollary 3.5).

2. Preliminaries

Let M be a complete Riemannian manifold of dimension n and G be a connected closed Lie subgroup of isometries of M. We say that M is of cohomogeneity one under the action of G, if G has an orbit of codimension one. The results by Mostert (see [8]), for the compact case (G is compact), and Berard Bergery (see [2]), for the general case, state that the orbit space M/G, equipped with the quotient topology, is a topological Hausdorff space homeomorphic to $\mathbb{R}$, S^1, $[0, +\infty)$ or $[0, 1]$.
Consider the projection map $M \to M/G$ to the orbit space. Given a point $x \in M$, we say that the orbit $G(x)$ is *principal* (resp. *singular*) if the corresponding image

in the orbit space M/G is an internal (resp. boundary) point. A point x whose orbit is principal (resp. singular) will be called *regular* (resp. *singular*).
Denote by $\mathbb{R}^n$ the n-dimensional real vector space with the usual Euclidean inner product. By $Iso(\mathbb{R}^n)$ we denote the group of isometries of $\mathbb{R}^n$, that is $O(n) \ltimes \mathbb{R}^n$ (see [9, p. 240]). We write the action of an isometry $\gamma \in Iso(\mathbb{R}^n)$ as

$$\gamma(x) = g(x) + v, \quad x \in \mathbb{R}^3_1,$$

where $g \in O(n)$ is called the *linear part* and $v \in \mathbb{R}^n$ is called the *translational part* of γ. Denote by

$$L : G \longrightarrow O(n)$$

the projection on the linear part of $O(n) \ltimes \mathbb{R}^n$. If $L(G)$ is trivial then G is called a pure translation group.

Let $M = \mathbb{R}^n$ and G be a connected, closed Lie subgroup of $Iso(\mathbb{R}^n)$, which acts isometrically on M. We recall some facts from the theory of Lie groups.

Theorem 2.1. ([7]) *Let $M = \mathbb{R}^n$ be of cohomogeneity one under the action of a connected, closed Lie subgroup $G \subset Iso(M)$. Then either each principal orbit is isometric to $\mathbb{R}^{n-1}$, and there exists no singular orbit or each principal orbit is isometric to $S^m(c) \times \mathbb{R}^{n-m-1}$, $1 \leqslant m \leqslant n-1$, where m is fixed for all orbits, and the unique singular orbit is isometric to $\mathbb{R}^{n-m-1}$.*

Lemma 2.2. ([3, p.51]) *A simply connected solvable Lie group is diffeomorphic to $\mathbb{R}^n$, $n = dimG$.*

Lemma 2.3. ([3, p.52]) *Let G be a connected Lie group. Then the following conditions are equivalent:*
(i) The Lie group G is diffeomorphic to $\mathbb{R}^n$, $n = dimG$.
(ii) The maximal compact subgroup of G is trivial.

Lemma 2.4. *If G is a compact solvable Lie group, then it is isomorphic to a torus $\mathbb{T}^k$ for some $k \geqslant 0$.*

Proof. Since G is compact, it is reductive by Proposition 1.4 of [3, p.131], hence $\mathfrak{z}(\mathfrak{g}) = \mathfrak{rad}(\mathfrak{g}) = \mathfrak{g}$. Thus G is an Abelian compact group. $\square$

Lemma 2.5. ([6]) *If the Lie group G is compact, or connected and semisimple, then any smooth representation of G by affine transformations of $\mathbb{R}^n$ admits a fixed point.*

3. The Main Result

Two isometric actions on a Riemannian manifold M are said to be orbit equivalent if there exists an isometry of M mapping the orbits of one of these actions onto the orbits of the other. Suppose that $M = \mathbb{R}^n$ is of cohomogeneity one under the action of a connected, closed Lie subgroup $G \subset Iso(M)$. By Theorem 2.1, if there is no singular orbit, then each orbit is isometric to $\mathbb{R}^{n-1}$ and the action of G is orbit equivalent to the action of the pure translation Lie group $H = \mathbb{R}^{n-1}$ on $\mathbb{R}^n$, with $H(0) = G(0)$. What we can say about the existence of a simply connected solvable closed Lie subgroup F of G such that the action of F is orbit equivalent to the action of G on $\mathbb{R}^n$.

Theorem 3.1. *Let $\mathbb{R}^n$ be of cohomogeneity one under the action of a connected, closed Lie subgroup $G \subset Iso(\mathbb{R}^n)$. If there is no singular orbit, then there exists a simply connected, solvable, closed Lie subgroup F of G such that acts freely and by cohomogeneity one on $\mathbb{R}^n$. In particular, the action of F on $\mathbb{R}^n$ is orbit equivalent to the action of G. Furthermore, F has a pure translation normal Lie subgroup T with*

$$dim(T) \geqslant n - [n/2] - 1$$

and

$$\overline{L(F)} = \mathbb{T}^k,$$

where $k \geqslant n - dim(T) - 1$.

Proof. Let $G = S \ltimes R$ be a Levi decomposition of G. By Lemma 2.5 each semisimple subgroup of G fixes a point $x_\circ \in \mathbb{R}^n$, hence $S \subset G_{x_\circ}$ which shows that R acts on $G(x_\circ)$ transitively. Therefore R acts on $\mathbb{R}^n$ by cohomogeneity one and by Theorem 2.1 $R(x)$ is not singular orbit for each $x \in \mathbb{R}^n$. If K is the maximal compact subgroup of R then by Lemma 2.4 it is isomorphic to a torus $\mathbb{T}^k$ for some $k \geqslant 0$. By Theorem 7.1 of [3, p.66] there exists a simply-connected solvable normal Lie subgroup F of R such that $R = \mathbb{T}^k \ltimes F$. Since $\mathbb{T}^k$ is a compact Lie subgroup of G by Lemma 2.5 it fixes some point $y \in \mathbb{R}^n$, i.e. $\mathbb{T}^k = R_y$, hence F acts on $R(y)$ transitively. Since $R(y)$ is not singular orbit, F acts on $\mathbb{R}^n$ by cohomogeneity one . Because F is simply connected and solvable, the maximal compact Lie subgroup of F is trivial by Lemmas 2.2 and 2.3, and each isotropy subgroup is $F_x = \{I\}$ which shows that the action of F is free.

Now we show that F has a pure translation normal Lie subgroup T with the mentioned conditions. Consider the homomorphism $L : F \to SO(n)$. Since $ker(L)$ is a pure translation normal Lie subgroup of G, then $F/ker(L)$ is solvable. Thus $L(F)$ (so $\overline{L(F)}$) is solvable (see [4, p.56]) and $\overline{L(F)}$ is a compact solvable Lie subgroup of $SO(n)$. Therefore, by Lemma 2.4 it is isomorphic to $\mathbb{T}^k$ for some $k \geqslant 0$. Each maximal torus in $SO(n)$ is conjugate to $\mathbb{T}^{[n/2]}$ (see [5, p.252]), so $dim(L(F)) \leqslant k \leqslant [n/2]$. Since F acts by cohomogeneity one and freely on $\mathbb{R}^n$, $dim(F) = n-1$. Thus, by the following relations

$$dim(L(F)) \leqslant [n/2],$$
$$dim(ker(L)) + dim(L(F)) = n-1$$

we have

$$dim(ker(L)) \geqslant n - [n/2] - 1.$$

Thus $ker(L)$ is the pure translation normal Lie subgroup of F, which we were looking for. □

Example 3.2. We give an example, that shows that $L(F)$ may be not closed in $SO(n)$. Let α be an irrational number and

$$G = \left\{ \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos t & \sin t & 0 & 0 & 0 \\ 0 & -\sin t & \cos t & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \alpha t & \sin \alpha t & 0 \\ 0 & 0 & 0 & -\sin \alpha t & \cos \alpha t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right], \left[\begin{array}{c} 0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ t \end{array} \right] \;\middle|\; t, x_i \in \mathbb{R} \right\}.$$

Then G is a closed, simply connected and solvable subgroup of $Iso(\mathbb{R}^6)$ acting by cohomogeneity one on $\mathbb{R}^6$ (hence $F = G$), but $L(G)$ is not closed in $SO(6)$.

The following example shows that the simply connected, closed and solvable Lie subgroup F introduced in Theorem 3.1 is not unique up to isomorphism.

Example 3.3. Consider the usual isometric action of the Lie subgroup

$$G = \left\{ \left(\left[\begin{array}{cc} 1 & \\ & SO(n-1) \end{array} \right], \left[\begin{array}{c} 0 \\ X \end{array} \right] \right) \mid X \in \mathbb{R}^{n-1} \right\} \subset Iso(\mathbb{R}^n)$$

on $\mathbb{R}^n$. Each of the following Lie subgroups of G is simply connected, closed and solvable and its action is orbit equivalent to that of G. Further,

$$F_1 = \left\{ \left(I_{(n-1)\times(n-1)}, \left[\begin{array}{c} 0 \\ X \end{array} \right] \right) \mid X \in \mathbb{R}^{n-1} \right\}$$

$$F_2 = \left\{ \left(\left[\begin{array}{c|ccc} I_{(n-2k)\times(n-2k)} & & & \\ \hline & R_{\theta_1} & & \\ & & \ddots & \\ & & & R_{\theta_k} \end{array} \right], \left[\begin{array}{c} 0 \\ \Theta \\ \\ X \end{array} \right] \right) \;\middle|\; \Theta = \left[\begin{array}{c} \theta_1 \\ \vdots \\ \theta_k \end{array} \right] \in \mathbb{R}^k, X \in \mathbb{R}^{n-k-1} \right\}$$

$$F_3 = \left\{ \left(\left[\begin{array}{c|c} I_{(n-4)\times(n-4)} & \\ \hline & \begin{array}{cc} R_\theta & \\ & R_{\alpha\theta} \end{array} \end{array} \right], \left[\begin{array}{c} 0 \\ \theta \\ X \end{array} \right] \right) \middle| \theta \in \mathbb{R}, X \in \mathbb{R}^{n-2} \right\},$$

where α is a fixed irrational number, $k \leqslant \frac{n-1}{3}$ and

$$R_\theta = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right].$$

Each of the Lie groups F_1, F_2 and F_3 is diffeomorphic to $\mathbb{R}^{n-1}$, so they are simply connected and act freely on $\mathbb{R}^n$. By Lemma 2.5 it implies that the Levi factor of each of them is trivial so they are solvable. We also have $G(x) = F_1(x) = F_2(x) = F_3(x) \cong \mathbb{R}^{n-1}$ for each $x \in \mathbb{R}^n$, so their actions are orbit equivalent.

The proof of the following two corollaries are similar to that of Theorem 3.1 and we leave it to the reader.

Corollary 3.4. *Let $M = \mathbb{R}^n$ and G be a closed Lie subgroup of $Iso(\mathbb{R}^n)$. If the action of G on M is transitive, then there exists a simply-connected, solvable, closed Lie subgroup F of G acting freely and transitively on $\mathbb{R}^n$. Furthermore, F has a pure translation Lie subgroup T with*

$$dim(T) \geqslant n - [n/2]$$

and

$$\overline{L(F)} = \mathbb{T}^k,$$

where $k \geqslant n - dim(T)$.

Corollary 3.5. *Let $\mathbb{R}^n$ be of cohomogeneity one under the action of a connected, closed Lie subgroup $G \subset Iso(\mathbb{R}^n)$. If there is a singular orbit $B = \mathbb{R}^{n-m-1}$, then G has a simply-connected, solvable Lie subgroup F acting freely and transitively on $\mathbb{R}^{n-m-1}$. Furthermore, F has a pure translation Lie subgroup T with*

$$dim(T) \geqslant n - m - [n/2] - 1$$

and

$$\overline{L(F)} = \mathbb{T}^k,$$

where $k \geqslant n - m - dim(T)$.

Список литературы

[1] A. V. Alekseevsky and D. V. Alekseevsky, "G-manifolds with one dimensional orbit space", Adv. Sov. Math., **8**, 1 - 31 (1992).

[2] L. Berard-Bergery, "Sur de nouvells *variété* riemanniennes d'Einstein", Inst. Élie Cartan, **6**, 1 - 60 (1982).

[3] V. V. Gorbatsevich, A. L. Onishik and E. B. Vinberg, Lie Groups and Lie Algebras III, *springer* (1993).

[4] V. V. Gorbatsevich, A. L. Onishik and E. B. Vinberg, Lie Groups and Lie Algebras I, *springer* (1994).

[5] A. W. Knapp, Lie Groups Beyond an Introduction, second edition, Progress in Math., **140** (2002).

[6] J. Milnor, "On fundamental groups of complete affinely flat manifolds", Adv. in math., **25**, 178 - 187 (1977).

[7] R. Mirzaie and S. M. B. Kashani, "On cohomogeneity one flat Riemannian manifolds", Glasgow Math. J., **44** (2002).

[8] P. S. Mostert, "On a compact Lie group acting on a manifold", Ann. Math., **65**(3), 447 - 455 (1957).

[9] B. O' Neill, Semi-Riemannian Geometry With Application to Relativity, Academic Press, New York (1983).

[10] F. Podesta and A. Spiro: "Cohomogeneity one manifolds and hypersurfaces of the Euclidean spaces", Ann. Global Anal.Geom., **13** (2), 169 - 184 (1995).

[11] C. Searle, "Cohomogeneity and positive curvature in low dimension", Math. Z. **214**, 491 - 498 (1993).

Поступила 18 сентября 2008