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ON UNIVERSAL RELATIVES OF THE RIEMANN ZETA-FUNCTION

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Аннотация. The Riemann zeta-function ζ has the following well-known properties:

(M) It is meromorphic in $\mathbb{C}$ with a simple pole at $z=1$ with residue 1.

(SR) The symmetry relation $\zeta(z)=\overline{\zeta(\bar{z})}$ holds for $z\neq 1$.

(FE) The following functional equation holds:

$$\zeta(z)\Gamma(z/2)\pi^{-z/2}=\zeta(1-z)\Gamma((1-z)/2)\pi^{-(1-z)/2}.$$

Moreover, ζ has a universality property due to Voronin (1975). We show that arbitrarily close approximations of the Riemann zeta-function that satisfy (M), (SR), (FE) may have a different universality property. Consequently, these approximations do not satisfy the Riemann hypothesis. Moreover, we investigate the set of all "Birkhoff-universal" functions satisfying (M), (SR), (FE).

1. INTRODUCTION

The Riemann zeta-function $\zeta(z)$ has the following well-known properties, cf. [5]:

(M) It is meromorphic with at least a single pole at $z=1$ with residue 1.

(M*) It is holomorphic in the complex plane except for just one single pole at $z=1$ with residue 1.

(SR) The symmetry relation $\zeta(z)=\overline{\zeta(\bar{z})}$ holds for $z\neq 1$.

(FE) The functional equation $\zeta(z)\Gamma(z/2)\pi^{-z/2}=\zeta(1-z)\Gamma((1-z)/2)\pi^{-(1-z)/2}$ holds.

(V) It has a universality property due to Voronin (1975): For every compact set $K\subset\{z\colon \frac{1}{2}<\Re z<1\}$ with connected complement, every function

$$f\in A(K):=\{f: f \text{ is continuous on } K \text{ and holomorphic in the interior of } K\}$$

zero-free on K and every $\varepsilon>0$:

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]\colon \max_{z\in K}|\zeta(z+i\tau)-f(z)|<\varepsilon\right\}>0,$$

where *meas* stands for the Lebesgue measure.

Voronin's Theorem has many applications in physics and is very close to Riemann's famous conjecture, i. e.the real part of every non-trivial zero of ζ is $1/2$. To explain, we give the following two results, cf. [5].

Theorem 1 (Bagchi (1981))**.** *Riemann's Conjecture is* true *if and only if Voronin's result also holds in case of* $f = \zeta$.

Theorem 2. *Riemann's Conjecture is* false *if Voronin's result holds for some function* $f \not\equiv 0$ *with at least one zero in the interior* K° *of* K.

Proof: If Voronin's result holds for such a function f, we obtain a sequence $\{\tau_n\} \subset \mathbb{R}$ such that $\{\zeta(z + i\tau_n)\}$ converges compactly in K° to that function f. Assuming Riemann's hypothesis to be true, $\{\zeta(z + i\tau_n)\}$ is a sequence of holomorphic and non-vanishing functions in K°. By Hurwitz' Theorem the limit function f is either equivalently 0 or also non-vanishing. We have a contradiction.

2. NOTATIONS

Let $b = \{b_n\} \subset \mathbb{C}$ be a sequence without finite accumulation point and $a = \{a_n\}$ with $a_n \to 0$. A compact set K is called a *Mergelian set* if its complement $K^c := \mathbb{C} \setminus K$ is connected. We call a compact set K a *Vitushkin set*, if for all open disks $D \subset \mathbb{C}$ we have

$$\alpha(D \setminus K) = \alpha(D \setminus K^\circ),$$

where α denotes the continuous analytic capacity of the considered set. The family of all Mergelian sets we denote by $\mathcal{M}$ and the family of all Vitushkin sets by $\mathcal{V}$. The famous theorems of Mergelian and Vitushkin state that each function $f \in A(K)$ is uniformly approximable on K by polynomials or rational functions respectively, if and only if $K \in \mathcal{M}$ or $K \in \mathcal{V}$ respectively.

A function φ is said to satisfy one of the following universality properties if

$(B_b, \mathcal{M})$ For every $K \in \mathcal{M}$, every function $f \in A(K)$, there exists a subsequence $\{n_k\} \subset \mathbb{N}$ with $\varphi(z + b_{n_k}) \to f(z)$ uniformly on K.

$(L_{a,b}, \mathcal{M})$ For every $K \in \mathcal{M}$, every function $f \in A(K)$, there exists a subsequence $\{n_k\} \subset \mathbb{N}$ with $\varphi(a_{n_k} z + b_{n_k}) \to f(z)$ uniformly on K.

If these universality properties should hold for every $K \in \mathcal{V}$, instead of $K \in \mathcal{M}$, we denote them by $(B_b, \mathcal{V})$ and $(L_{a,b}, \mathcal{V})$ respectively. Moreover, we say that b satisfies

(A) if $b = \{b_n\} \subset \{z : \Re z \notin [0,1], \Im z \neq 0\}$,

(B) if $\operatorname{dist}(b_n; \{z : \Re z \in [0,1] \text{ or } \Im z = 0\}) \to \infty$, as $n \to \infty$.

3. MAIN RESULTS

Our aim is to construct functions that satisfy several of these properties and are "close"to ζ. The history of this problem started in 2003 with a result from Pustyl'nikov [4].

Theorem 3 (Pustyl'nikov (2003)). *There are functions satisfying (M*), (SR), (FE), but Riemann's Conjecture does not apply to them. Simultaneously, these functions approximate the Riemann zeta-function uniformly on compact sets $K \in \mathcal{M}$ with $1 \notin K$.*

In 2004, this result was improved by Gauthier and Zeron [1]. They were able to place the non-trivial zeros and the approximation of ζ was made tangentially on closed sets with arbitrarily small complement. This means that the approximants and the zeta-function itself are "almost"indistinguishable. Our results are as follows:

Theorem 4. *If b has (A) and (B), then there exists a meromorphic function ζ_1 with (M),(SR),(FE),(V) and $(B_b, \mathcal{V})$. The functions ζ_1 and ζ have the same zeros in the critical strip.*

Theorem 5. *Let $\Lambda > 0$ and b with (A). First, there exists a closed set S with area of the complement of S less than Λ. Second, for every positive function $\varepsilon \in C(S)$ there exists a meromorphic function ζ_2 with (M),(SR),(FE),$(L_{a,b}, \mathcal{V})$ for some sequence a and*

$$|\zeta(z) - \zeta_2(z)| < \varepsilon(z) \text{ for all } z \in S. \tag{3.1}$$

Finally, we obtain a generic result. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the extended complex plane and $d(z, w)$ the chordal between $z, w \in \mathbb{C}_\infty$. If f is a meromorphic function on $\mathbb{C}$, and if $f(z) = \infty$ whenever z is a pole of f in $\mathbb{C}$, then $f : \mathbb{C} \to \mathbb{C}_\infty$ is a continuous function, for short $f \in C(\mathbb{C}, \mathbb{C}_\infty)$. Setting $\mathbb{D}_n := \{z : |z| \leq n\}$, $\varrho_n(f, g) := \max\{d(f(z), g(z)) : z \in \mathbb{D}_n\}$ $(f, g \in C(\mathbb{C}, \mathbb{C}_\infty))$ and

$$\varrho(f, g) := \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\varrho_n(f, g)}{1 + \varrho_n(f, g)},$$

the metric space $(C(\mathbb{C}, \mathbb{C}_\infty), \varrho)$ is complete. This metric also generates the topology of uniform convergence on compact subsets of $\mathbb{C}$ on the space $C(\mathbb{C}, \mathbb{C}_\infty)$.

We consider the set A of all meromorphic functions with (M), (SR), (FE). Then $A \cup \{\infty\}$ is a closed subset of $(C(\mathbb{C}, \mathbb{C}_\infty), \varrho)$. Hence, $A \cup \{\infty\}$ becomes a complete metric space under the induced metric.

Theorem 6. *Let b satisfies (A) and (B). We consider the operators*

$$T_n : A \cup \{\infty\} \to C(\mathbb{C}, \mathbb{C}_\infty), \quad f(z) \mapsto f(z + b_n).$$

The set of universal functions

$$\mathcal{U} := \left\{ \varphi \in A \cup \{\infty\} \colon (M(\mathbb{C}) \cup \{f \colon f \equiv \infty\}) \subset \overline{\{T_n \varphi \colon n \in \mathbb{N}\}} \right\}$$

is a dense G_δ-set in $A \cup \{\infty\}$.

Remark 1. (i) *Let $\varphi \in \mathcal{U}$. It follows from Vitushkin's Theorem that φ satisfies $(B_b, \mathcal{V})$.*

(ii) *Although being an element of A, the zeta-function ζ is definitely not an element of $\mathcal{U}$ and it neither satisfies $(B_b, \mathcal{V})$ nor $(L_{a,b}, \mathcal{V})$. For instance, we consider $K = \{z \colon |z| = 1\} \in \mathcal{V}$ and $f \in A(K)$ with $f(z) = 1/z$. This function is not uniformly approximable by polynomials or entire functions. By the same argumentation, ζ does not satisfy a stronger Voronin-property (V^*), where "Mergelian set" is replaced by "Vitushkin set".*

(iii) *Replacing "Vitushkin set" by "Mergelian set" in the Theorems 4, 5 and 6, similar results hold with (M*) instead of (M), cf.* [3].

(iv) *Any function φ satisfying $(B_b, \mathcal{M})$ or $(L_{a,b}, \mathcal{M})$ respectively does not satisfy an analogue of the Riemann hypothesis. This can be seen by an application of Hurwitz' Theorem, as in the proof of Theorem 2.*

In a topological sense, the analogue of Theorem 6 shows that the majority of functions in A^*, meaning all meromorphic functions with (M), (SR), (FE), fails to satisfy an analogue of Riemann's Conjecture.*

(v) *The proofs techniques of these theorems combine universality with certain types of functional equations.*

4. SKETCH OF THE PROOF OF THEOREM 5

(i) We call a subset M of the complex plane *doubly symmetric* if the points $\bar{z}, 1 - z$ and $1 - \bar{z}$ are all in M for each element z of M. Without loss of generality, we assume that for each $b_n \in b$ the points $\bar{b}_n$, $1 - b_n$ and $1 - \bar{b}_n$ are not elements of b.

The closed set S will be chosen as the disjoint union of compact annuli, whose centers are always $1/2$. Therefore, S is automatically doubly symmetric. Moreover, we determine a sequence $\{r_n\}$ of radii with $r_n \in (0, \frac{1}{n})$ such that the disks

$$\begin{aligned} B_n &:= \{z : |z - b_n| \le r_n\} \\ \overline{B_n} &:= \{z : |z - \overline{b_n}| \le r_n\} \\ (1 - B_n) &:= \{z : |z - 1 + b_n| \le r_n\} \\ (1 - \overline{B_n}) &:= \{z : |z - 1 + \overline{b_n}| \le r_n\} \end{aligned}$$

are pairwise disjoint, do not intersect S and belong to $\{z\colon \Re z \notin [0,1], \Im z \neq 0\}$. The radii $\{r_n\}$ and the set S can be arranged in such a way that the area of S^c is less than Λ. Finally, we define $a_n := r_n/n$ and we record that

$$F := S \cup \bigcup_{n\in\mathbb{N}} B_n \cup \bigcup_{n\in\mathbb{N}} \overline{B_n} \cup \bigcup_{n\in\mathbb{N}} (1 - B_n) \cup \bigcup_{n\in\mathbb{N}} (1 - \overline{B_n}) \tag{4.1}$$

is also a closed, doubly symmetric set.

(ii) We use the approach

$$\zeta_2(z) := \zeta(z) \cdot \nu(z),$$

where

$$\nu(z) := m(z) \cdot m(1 - z) \cdot \overline{m(\bar{z})} \cdot \overline{m(1 - \bar{z})},$$

using an "appropriate"meromorphic function m satisfying $m(0) = m(1) = 1$. Consequently, ζ_2 satisfies (M). Also, we have $\nu(z) = \overline{\nu(\bar{z})}$ and $\nu(z) = \nu(1-z)$. Hence, ζ_2 satisfies (SR) and (FE) respectively.

(iii) The property (3.1) follows if the function ν is "close"to 1 on S. In order to get the "appropriate"function m that guarantees this closeness, we use the following lemma.

Lemma 1 (Luh, Meyrath, Nieß (2008)). *Suppose that $G \subset \mathbb{C}$ is a domain and let F be a (relatively) closed subset of G. Let be given a non-vanishing function $\varepsilon \in H(F)$. Then for every function $f \in M(F)$ there exists a function $m \in M(G)$ with*

$$|f(z) - m(z)| < |\varepsilon(z)| \quad \textit{for all} \quad z \in F.$$

This lemma is based on results by Alice Roth. To get the property $(L_{a,b}, \mathcal{V})$ for the function ζ_2, we need a countable set of functions out of which every function in $A(K)$ for $K \in \mathcal{V}$ approximable. In the case of holomorphic approximation the countable set $\mathcal{P}$ of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ is dense in $A(K)$ for all $K \in \mathcal{M}$. In the case of meromorphic approximation the following lemma holds.

Lemma 2. *The countable set $\mathcal{R}$ of all rational functions $r := p/q$ with $p, q \in \mathcal{P}$ is dense in $A(K)$ for all $K \in \mathcal{V}$.*

Both lemmas are proved in [2]. The rest of the proof is very technical, but the idea is to construct the "right"function to be approximated on F given in (4.1) with the "right"error.

Список литературы

[1] P. M. Gauthier and E. S. Zeron, "Small Perturbations of the Riemann Zeta Function and Their Zeros", Comput. Methods Funct. Theory **4**, 143-150, (2004).

[2] W. Luh, T. Meyrath and M. Nieß, "Universal Meromorphic Approximation on Vitushkin Sets", Izv. NAN Armenii, Matematika [Journal of Contemporary Mathematical Analysis (Armenian Academy of Sciences)] **43**, no. 6, 365-371 (2008).

[3] M. Nieß, "Universal Approximants of the Riemann Zeta-Function", Comput. Methods Funct. Theory **9** (no 1), 145-159, (2009).

[4] L. D. Pustyl'nikov, "Rejection of an Analogue of the Riemann Hypothesis on Zeros for an Arbitrarily Exact Approximation of the Zeta Function Satisfying the Same Functional Equation", Uspekhi Mat. Nauk, **58**, 175-176 (2003) (in Russian); English translation: Russ. Math. Surv. **58**, 193–194, (2003).

[5] J. Steuding, *Value-Distribution of L-Functions*, Lecture Notes in Mathematics **1877** (Springer, 2007).

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