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UNIVERSAL MEROMORPHIC APPROXIMATION ON VITUSHKIN SETS

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Аннотация. The paper proves the following result on universal meromorphic approximation: Given any unbounded sequence $\{\lambda_n\} \subset \mathbb{C}$, there exists a function ϕ, meromorphic on $\mathbb{C}$, with the following property. For every compact set K of rational approximation (i.e. Vitushkin set), and every function f, continuous on K and holomorphic in the interior of K, there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\phi(z + \lambda_{n_k})\}$ converges to $f(z)$ uniformly on K.
A similar result is obtained for arbitrary domains $G \neq \mathbb{C}$. Moreover, in case $\{\lambda_n\} = \{n\}$ the function ϕ is frequently universal in terms of Bayart/Grivaux [3].

Dedicated to the memory of academician S. N. Mergelyan

1. INTRODUCTION

1.1. **Notations.** For a set S in the complex plane we denote by
- $C(S)$: the family of all continuous functions on S,
- $H(S)$: the family of all holomorphic functions on S,
- $M(S)$: the family of all meromorphic functions on S.

For a compact set $K \subset \mathbb{C}$ we introduce the following spaces of functions (each of which is endowed with the uniform norm):
- $A(K) :=$ $C(K) \cap H(\overset{\circ}{K})$, where $\overset{\circ}{K}$ stands for the (possibly empty) interior of K,
- $P(K)$: all functions which are uniformly approximable on K by polynomials,
- $R(K)$: all functions which are uniformly approximable on K by rational functions (with poles not in K).

We obviously have

$$P(K) \subset R(K) \subset A(K) \subset C(K).$$

The two main problems: Which topological assumptions on K guarantee that $P(K) = A(K)$ or $R(K) = A(K)$, respectively, were solved in the celebrated theorems of S. N. Mergelian [11] and A. G. Vitushkin [14, 15].

A compact set K is called a Mergelian set if its complement $K^c := \mathbb{C} \setminus K$ is connected. The set of all Mergelian sets will be denoted by $\mathcal{M}$. Then Mergelian's theorem states that $P(K) = A(K)$ if and only if $K \in \mathcal{M}$.

A compact set K is called a Vitushkin set if for all open disks $D \subset \mathbb{C}$ the following property is satisfied

$$\alpha(D \setminus K) = \alpha(D \setminus \overset{\circ}{K})$$

(where α denotes the continuous analytic capacity of the considered set; see for instance D. Gaier [6]). The family of all Vitushkin sets will be denoted by $\mathcal{V}$. Vitushkin's theorem states that $R(K) = A(K)$ if and only if $K \in \mathcal{V}$.

1.2. **Universal holomorphic approximation.** In 1929 Birkhoff [4] proved the following remarkable result (which we state in a slightly modified version).

Theorem B. Given any unbounded sequence $\{\lambda_n\} \subset \mathbb{C}$. Then there exists a function $\phi \in H(\mathbb{C})$ with the following property. For every set $K \in \mathcal{M}$ and every function $f \in A(K)$ there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\phi(z + \lambda_{n_k})\}$ converges to $f(z)$ uniformly on K.

This result, usually considered as the first example of a so-called universal entire function, has been the starting point and motivation for extended investigations dealing with different types of universalities. In the course of time a great number of universal functions has been discovered and there exists an extensive literature on this subject. For details and bibliographical information we refer to the excellent survey article of K.-G. Grosse-Erdmann [7], where the results on universalities of the relevant literature are completely collected and classified.

If instead of the whole complex plane an arbitrary open set $O \neq \emptyset$ is considered, then in the paper [8] by Luh and Martirosian a very elementary proof (using only standard methods of complex analyis) for the existence of a universal holomorphic function on O was given.

Theorem LM. Suppose that $O \neq \emptyset$ is an open set in the complex plane. Then there exists a function $\phi \in H(O)$ with the following properties: For every $K \in \mathcal{M}$, for every $f \in A(K)$ and for every $\zeta \in \partial O$ there exists a sequence $\{(a_n, b_n)\} \subset \mathbb{C}^2$ such that

$t_n(z) := a_n z + b_n \in O$ for all $z \in K$ and $n \in \mathbb{N}$,
$\{t_n(z)\}$ converges to ζ for all $z \in K$,
$\{\phi \circ t_n(z)\}$ converges to $f(z)$ uniformly on K.

1.3. **Universal meromorphic approximation.** The first result on universal meromorphic approximation is due to Luh and Martirosian [9]. There the existence of a meromorphic function ϕ on $\mathbb{C}$ was proved which is universal under prescribed translates and has the same properties as Birkhoff's entire function; in addition it was shown that ϕ has slow transcendental growth: Its Nevanlinna characteristic satisfies $T(r,\phi) = O(q(r)\log^2 r)$ for $r \to \infty$, where $q : \mathbb{R}^+ \to \mathbb{R}^+$ is any increasing function with $\lim\limits_{r\to\infty} q(r) = \infty$.

Using the techniques of the hypercyclicity criterion K. C. Chan constructed in the recent paper [5] a meromorphic function ϕ on $\mathbb{C}$ with the property that the sequence $\{\phi(z+n)\}_{n\in\mathbb{N}}$ is dense in the metric space $S(G) := M(G)\cup\{f \equiv \infty\}$ for every domain $G \subset \mathbb{C}$ (where $S(G)$ is endowed with a metric defined by the chordal distance).

By A. Roth's nice result [13] and Vitushkin's famous theorem it seems to be more natural to investigate universal meromorphic approximation not on domains or on compacta in $\mathcal{M}$ but on Vitushkin sets.

It also seems to be interesting to obtain the existence of universal meromorphic functions not by a hypercyclicity criterion but by a constructive procedure.

2. MAIN RESULTS

It is the object of this article to prove the following two results on universal meromorphic approximation.

Theorem 1. Given any unbounded sequence $\{\lambda_n\} \subset \mathbb{C}$. Then there exists a function $\phi \in M(\mathbb{C})$ with the following property.

For every set $K \in \mathcal{V}$ and every function $f \in A(K)$ there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\phi(z + \lambda_{n_k})\}$ converges to $f(z)$ uniformly on K.

Theorem 2. Let $G \subset \mathbb{C}$, $G \neq \mathbb{C}$ be a domain and suppose that there are given sequences $\{a_n\}$ with $a_n \to 0$ for $n \to \infty$ and $\{b_n\} \subset G$ such that ∂G is exactly the set of accumulation points of $\{b_n\}$.

Then there exists a function $\phi \in M(G)$ with the following property.

For every $K \in \mathcal{V}$, for every $f \in A(K)$ and every $\zeta \in \partial G$ there exist subsequences $\{m_k\}$ and $\{n_k\}$ of $\mathbb{N}$ with

$$\begin{aligned} t_k(z) := \quad & a_{m_k} z + b_{n_k} \in G && \text{for all } z \in K \text{ and all } k \in \mathbb{N}, \\ & b_{n_k} \to \zeta && \text{for } k \to \infty \end{aligned}$$

such that $\{\phi \circ t_k(z)\}$ converges to $f(z)$ uniformly on K.

3. AUXILIARY RESULTS

For the proof of our main results a couple of auxiliary tools are necessary which we compile in this section. The first result is a combination of two results due to A. Roth, see [12] and [13, pp. 110-111].

Theorem R. Suppose that $G \subset \mathbb{C}$ is a domain and let F be a (relatively) closed subset of G.

Any function $f \in M(F)$ is uniformly approximable by functions from $M(G)$.

Any function $f \in H(F)$ is uniformly approximable by functions from $M(G)$ with poles only in $G \setminus F$.

Using this result we can prove the following Lemma (approximation with prescribed error).

Lemma 1. Suppose that $G \subset \mathbb{C}$ is a domain and let F be a closed subset of G. Let be given a function $\varepsilon \in H(F)$ with $|\varepsilon(z)| > 0$ for all $z \in F$. Then the following holds:

For every function $f \in M(F)$ there exists a function $m \in M(G)$ with

$$|f(z) - m(z)| < |\varepsilon(z)| \text{ for all } z \in F.$$

Proof. We may assume that $0 < |\varepsilon(z)| < 1$ for all $z \in F$. By Theorem R we find a function $m_1 \in M(G)$ with poles off F and

$$\left| \frac{2}{\varepsilon(z)} - m_1(z) \right| < 1 \text{ for all } z \in F.$$

It follows that m_1 must be zero-free on F. We consider the function $g \in M(F)$ with $g(z) := m_1(z)f(z)$ and find again by Theorem R a function $m_2 \in M(G)$ with

$$|g(z) - m_2(z)| = |m_1(z)f(z) - m_2(z)| < 1 \text{ for all } z \in F.$$

The function $\dfrac{m_2}{m_1}$ belongs to $M(G)$ and satisfies

$$\left| f(z) - \frac{m_2(z)}{m_1(z)} \right| < \frac{1}{|m_1(z)|} < |\varepsilon(z)| \text{ for all } z \in F.$$

$\square$

Lemma 2. There exists a countable set $\mathcal{R}$ of rational functions which is dense in $A(K)$ for every $K \in \mathcal{V}$.

Proof. We consider the countable set $\mathcal{R}$ of all rational functions $\dfrac{p(z)}{q(z)}$, where p and q are polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$.

Let be given $K \in \mathcal{V}$, $f \in A(K)$ and $\varepsilon > 0$. Then by Vitushkin's theorem we find a rational function $r^*(z) = \dfrac{p^*(z)}{q^*(z)}$ with poles off K and

$$\max_K |f(z) - r^*(z)| < \frac{\varepsilon}{2}.$$

Since q^* is zero-free on K we have $a := \min_K |q^*(z)| > 0$. We define

$$m(p^*) := \max_K |p^*(z)|, \qquad m(q^*) := \max_K |q^*(z)|$$

and fix $N \in \mathbb{N}$, with $K \subset \overline{\mathbb{D}}_N := \{z : |z| \leq N\}$. By Runge's approximation theorem there exist polynomials p and q with coefficients in $\mathbb{Q} + i\mathbb{Q}$ with

$$\begin{aligned} \max_{\overline{\mathbb{D}}_N} |p(z) - p^*(z)| &< \frac{a^2 \cdot \varepsilon}{8m(q^*)}, \\ \max_{\overline{\mathbb{D}}_N} |q(z) - q^*(z)| &< \min \left\{ \frac{a}{2}, \frac{a^2\varepsilon}{8m(p^*)} \right\}. \end{aligned}$$

We first obtain $\min_K |q(z)| \geq \frac{a}{2}$ and the function $r := \frac{p}{q} \in \mathcal{R}$ satisfies for all $z \in K$:

$$\begin{aligned} |r^*(z) - r(z)| &= \left| \frac{p^*(z)q(z) - p(z)q^*(z)}{q^*(z)q(z)} \right| \leq \\ &\leq \frac{2}{a^2}\{|p^*(z)|\,|q(z) - q^*(z)| + |q^*(z)|\,|p(z) - p^*(z)|\} < \frac{\varepsilon}{2} \end{aligned}$$

and hence $\max_K |f(z) - r(z)| < \varepsilon$. □

4. PROOFS OF THE MAIN RESULTS

4.1. **Proof of Theorem 1.** We consider for all $j \in \mathbb{N}$ closed disks $D_j := \{z : |z - \lambda_j| \leq \rho_j\}$ and suppose without loss of generality that $D_j \cap D_k = \emptyset$ for $j \neq k$ and $\rho_j < \rho_{j+1} \to \infty$ for $j \to \infty$ (if necessary we choose a subsequence of $\{\lambda_n\}$ with these properties).

Let $\{r_n\}$ be an enumeration of the set $\mathcal{R}$ from Lemma 2. On the closed set $F := \bigcup_{j=1}^{\infty} D_j$ we consider the functions g and ε which are defined by

$$\begin{aligned} g(w) &:= r_j(w - \lambda_j) \quad &&\text{if } w \in D_j, \\ \varepsilon(w) &:= \frac{1}{j} &&\text{if } w \in D_j. \end{aligned}$$

We have $g \in M(F)$ and $\varepsilon \in H(F)$. By Lemma 1 we find a function $\phi \in M(\mathbb{C})$ and

$$|g(w) - \phi(w)| < \varepsilon(w) \text{ for all } w \in F.$$

Consequently, we have

$$\max_{D_j} |r_j(w - \lambda_j) - \phi(w)| < \frac{1}{j}$$

or

$$\max_{|z|\leq\rho_j} |r_j(z) - \phi(z+\lambda_j)| < \frac{1}{j}.$$

Now, let be given a set $K \in \mathcal{V}$ and a function $f \in A(K)$. According to Lemma 2 we find a sequence $\{n_k\}$ with $n_k \to \infty$ and

$$\max_K |f(z) - r_{n_k}(z)| < \frac{1}{k}.$$

For all sufficiently great k the set K is contained in $\{z : |z| \leq \rho_{n_k}\}$ and we obtain for those k

$$\begin{aligned}
&\max_K |\phi(z+\lambda_{n_k}) - f(z)| \leq \\
&\leq \max_{|z|\leq\rho_{n_k}} |\phi(z+\lambda_{n_k}) - r_{n_k}(z)| + \max_K |r_{n_k}(z) - f(z)| < \\
&< \frac{1}{n_k} + \frac{1}{k},
\end{aligned}$$

which proves Theorem 1. □

4.2. **Proof of Theorem 2.** There exists a sequence $\{H_n\}$ of compact sets with the properties:

- $H_n \subset \overset{\circ}{H}_{n+1} \subset G$ for all $n \in \mathbb{N}$.
- For every compact set $K \subset G$ there exists an $n_0 \in \mathbb{N}$ with $K \subset H_{n_0}$.

Suppose that $\{\zeta^{(k)}\}$ is a sequence of points in ∂G which is dense in ∂G. For each $k \in \mathbb{N}$ we choose a subsequence $\{z_\nu^{(k)}\}_{\nu\in\mathbb{N}}$ of $\{b_n\}$ with $\lim\limits_{\nu\to\infty} z_\nu^{(k)} = \zeta^{(k)}$ such that for each $\nu \in \mathbb{N}$ the points $z_\nu^{(1)}, \cdots, z_\nu^{(\nu)}$ are pairwise distinct and such that for a sequence $\{H_{n_\nu}\}$ of $\{H_n\}$ we have $z_\nu^{(k)} \in \overset{\circ}{G}_{\nu+1} \setminus G_\nu$ for $k = 1, \ldots, \nu$, where $G_\nu := H_{n_\nu}$.

Next, we choose an increasing subsequence $\{l_\nu\}$ of $\mathbb{N}$ and radii $\rho_\nu := \sqrt{|a_{l_\nu}|}$ with the property that the closed disks

$$D_{\nu,k} := \{z : |z - z_\nu^{(k)}| \leq \rho_\nu\}$$

are pairwise disjoint for $k = 1, \ldots, \nu$ and that

$$\Omega_\nu := \bigcup_{k=1}^{\nu} D_{\nu,k} \subset \overset{\circ}{G}_{\nu+1} \setminus G_\nu.$$

Let $\{r_n\}$ be again an enumeration of the set $\mathcal{R}$ from Lemma 2. On the set $F := \bigcup\limits_{\nu=1}^{\infty} \Omega_\nu$, which is closed in G, we consider the functions g and ε which are defined by

$$\begin{aligned}
g(w) &:= r_\nu\left(\frac{w - z_\nu^{(k)}}{a_{l_\nu}}\right) && \text{if } w \in D_{\nu,k}, \\
\varepsilon(w) &:= \frac{1}{\nu} && \text{if } w \in \Omega_\nu.
\end{aligned}$$

We have $g \in M(F)$ and $\varepsilon \in H(F)$. By Lemma 1 we find a function $\phi \in M(G)$ with

$$|\phi(w) - g(w)| < \varepsilon(w) \text{ for all } w \in F.$$

Consequently, we have

$$\max_{D_{\nu,k}} \left| \phi(w) - r_\nu \left(\frac{w - z_\nu^{(k)}}{a_{l_\nu}} \right) \right| < \frac{1}{\nu},$$

or

$$\max_{|z| \le \frac{1}{\rho_\nu}} \left| \phi(a_{l_\nu} z + z_\nu^{(k)}) - r_\nu(z) \right| < \frac{1}{\nu}.$$

Now, let be given a set $K \in \mathcal{V}$, a function $f \in A(K)$ and a point $\zeta \in \partial G$. Obviously, ζ is an accumulation point of $\{\zeta^{(k)}\}$. According to Lemma 2 there exists a sequence $\{\nu_s\} \subset \mathbb{N}$ with $\nu_s \to \infty$ such that

$$\max_K |r_{\nu_s}(z) - f(z)| < \frac{1}{s}.$$

For all sufficiently great s, say $s > s_0$, the set K is contained in

$$\left\{ z : |z| \le \frac{1}{\rho_{\nu_s}} \right\},$$

and we obtain for all $s > s_0$ and all $k = 1, \ldots, \nu_s$:

$$\begin{aligned} \max_K & \left| \phi(a_{l_{\nu_s}} z + z_{\nu_s}^{(k)}) - f(z) \right| \le \\ & \le \max_{|z| \le 1/\rho_{\nu_s}} |\phi(a_{l_{\nu_s}} z + z_{\nu_s}^{(k)}) - r_{\nu_s}(z)| + \max_K |r_{\nu_s}(z) - f(z)| < \\ & < \frac{1}{\nu_s} + \frac{1}{s}. \end{aligned}$$

The point ζ is an accumulation point of the set

$$\{z : z = z_{\nu_s}^{(k)};\ k = 1, \ldots, \nu_s;\ s > s_0\}$$

and therefore we find $j_s \in \{1, \ldots, \nu_s\}$ such that $z_{\nu_s}^{(j_s)} \to \zeta$ for $s \to \infty$. For all $s > s_0$ and $z \in K$, we have $a_{l_{\nu_s}} z + z_{\nu_s}^{(j_s)} \in D_{\nu_s, j_s} \subset G$ for all $s > s_0$. If we now define for $k \in \mathbb{N}$

$$a_{m_k} := a_{l_{\nu_{s_0+k}}},\ b_{n_k} := z_{\nu_{s_0+k}}^{(j_{s_0+k})},$$

then $\phi(a_{m_k} z + b_{n_k})$ converges to $f(z)$ uniformly on K.

5. REMARKS ON A DENSITY PROPERTY

In a recent paper [3] F. Bayart and S. Grivaux proved the following nice version of Birkhoff's theorem:

Theorem BG. There exists a function $\phi \in H(\mathbb{C})$ with the following property. For every set $K \in \mathcal{M}$, every function $f \in A(K)$ and every $\varepsilon > 0$ there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ with positive lower density such that

$$|\phi(z+n_k)+f(z)| < \varepsilon \text{ for all } z \in K.$$

(The lower density of $\{n_k\}$ in the sense of G. Pylya is defined by

$$\underline{d}(\{n_k\}) := \liminf_{t\to\infty} \frac{N(\{n_k\},t)}{t},$$

where $N(\{n_k\},t)$ denotes the number of elements of $\{n_k\}$ in the interval $[1,t]$.)

The authors prove this result by a technical lemma (Lemma 2.2 in [3]) and the usual application of Arakelian's theorem [1, 2]. Using this lemma and combining the techniques from the proofs of Theorem 1 and Theorem BG we easily obtain

Theorem 3. There exists a function $\phi \in M(\mathbb{C})$ with the following property. For every set $K \in \mathcal{V}$, every function $f \in A(K)$ and every $\varepsilon > 0$ there exists a sequence $\{n_k\}$ with positive lower density such that

$$|\phi(z+n_k)-f(z)| < \varepsilon \text{ for all } z \in K.$$

Of course, the sequence $\{n_k\}$ is depending on K, f and ε. It seems to be an interesting problem, whether $\{n_k\}$ with $\underline{d}(\{n_k\}) > 0$ can be constructed in such a way that it depends on K and f only, so that $\{\phi(z+n_k)\}$ converges to $f(z)$ uniformly on K.

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